Engel groups and universal surgery models

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Abstract

We introduce a collection of $\frac{1}{2}$-π_1-null four-dimensional surgery problems. This is an intermediate notion between the classically studied universal surgery models and the π_1-null kernels which are known to admit a solution in the topological category. Using geometric applications of the group-theoretic 2-Engel relation, we show that the $\frac{1}{2}$-π_1-null surgery problems are universal, in the sense that solving them is equivalent to establishing four-dimensional topological surgery for all fundamental groups. As another application of these methods, we formulate a weaker version of the π_1-null disk lemma and show that it is sufficient for proofs of topological surgery and s-cobordism theorems for good groups.

1. Introduction

The four-dimensional topological surgery conjecture is known to hold for a class of *good* fundamental groups, and its validity for arbitrary groups remains an open problem. This conjecture, underlying geometric classification theory of topological 4-manifolds, states that a degree one normal map $(M, \partial M) \longrightarrow (X, \partial X)$ from a topological 4-manifold to a Poincaré pair, which is a $\mathbb{Z}[\pi_1 X]$-homology equivalence over ∂X, is normally bordant to a homotopy equivalence if and only if the Wall obstruction vanishes. It was proved in the simply-connected case by the first author in [**3**]; the class of good groups has since been extended to include elementary amenable groups [**4**], and more recently the groups of subexponential growth [**7, 10**]. The surgery conjecture for arbitrary groups may be reduced to a collection of *universal* problems [**2, 4**] with free fundamental groups.

The main question is whether a given surgery kernel may be 'decoupled' from the fundamental group of the ambient 4-manifold M. If the surgery kernel is represented by a π_1-null collection of hyperbolic pairs, where π_1-*null* means that all double point loops are contractible in M, then the surgery problem is known to admit a solution [**6**, Theorem 6]. (See Section 3 for a more detailed discussion of surgery problems.) We introduce an intermediate notion of $\frac{1}{2}$-π_1-*null* surgery kernels, where one side is a capped grope (as in the traditional universal models), and the other side is a π_1-null sphere; more details are given below and in Section 3. Our first result is that these problems are universal, in the sense that solving them is equivalent to establishing four-dimensional topological surgery for all fundamental groups.

THEOREM 1. *The collection of $\frac{1}{2}$-π_1-null surgery models is universal.*

In light of Theorem 1, the current status of the surgery conjecture may be summarized as in Figure 1.1. It is a natural question (the answer to which we do not presently know) whether the $\frac{1}{2}$-π_1-null problems could be improved further, for example, using the higher Engel relations discussed in [**5**, Appendix], to yield a solution to the surgery conjecture for all groups.
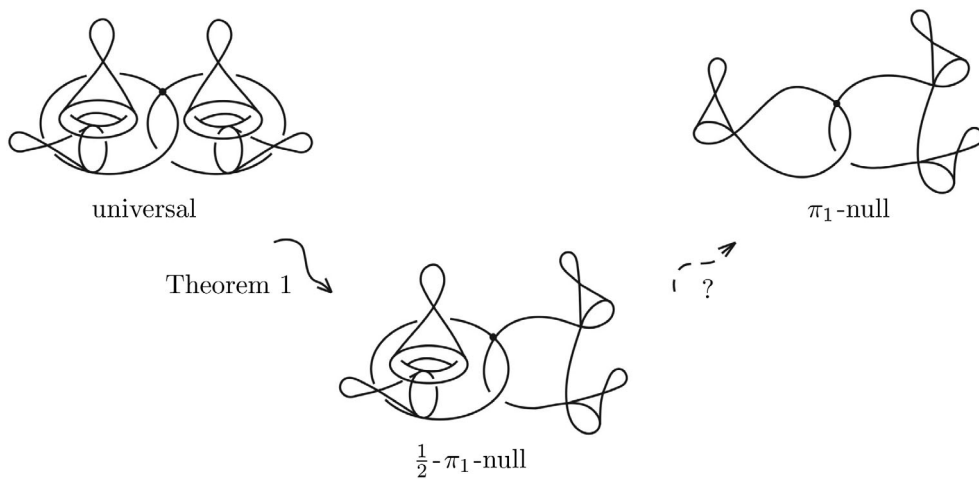
FIGURE 1.1. *A schematic outline: Theorem 1 reduces the usual universal surgery problems to $\frac{1}{2}$-π_1-null kernels. In general, the surfaces pictured as tori may have higher genus, and there are higher multiplicities of double points. It is an open question whether there is a further reduction to π_1-null problems; a positive answer would imply the topological four-dimensional surgery conjecture for all groups.*

Next we will elaborate on the construction of the old and new universal surgery models, and the associated link-slicing problems. (See Section 3 for more details.) The usual universal surgery models are given by $S^2 \vee S^2$-like capped gropes [**6**, Sections 2.4 and 5.1]. Solving these surgery problems is equivalent to the existence of free slices for links in the collection $\{\mathrm{Wh}(\mathrm{Bing}(\mathrm{Hopf}))\}$, the links obtained from the Hopf link by (a non-trivial amount of) ramified Bing doubling and then ramified Whitehead doubling. The free-slice condition is the requirement that the fundamental group of the complement of slices in the 4-ball is free, generated by meridians.

The $\frac{1}{2}$-π_1-null models are standard thickenings of 2-complexes of the form $(G^c \vee S) \cup H^c$, where G^c is a sphere-like capped grope, S is a sphere with self-intersections, and H^c consists of disk-like capped gropes attached to double-point loops of S (Figure 3.3). (Note that $S \cup H^c$ is slightly better than just a π_1-null sphere: contracting H^c along any one of its caps gives a null-homotopy for the double point loops of S.) The links corresponding to these models are of the form $\{\mathrm{Wh}(\mathrm{Bing}(\mathrm{Wh}))\}$, ramified Whitehead double of an (iterated) ramified Bing double of the Whitehead link.

It is interesting to note that some of the representatives of $\{\mathrm{Wh}(\mathrm{Bing}(\mathrm{Wh}))\}$ are Whitehead doubles of (homotopically trivial)$^+$ links which are known [**7**] to be freely slice. However, for sufficiently high genus surface stages of G^c and H^c, and for sufficiently high multiplicity of double points, these links are of the form $\mathrm{Wh}(L)$, where L is a homotopically essential link with trivial linking numbers. In the previously considered collection $\{\mathrm{Wh}(\mathrm{Bing}(\mathrm{Hopf}))\}$, all representatives are of this type.

The proof of Theorem 1 relies on a combination of geometric applications of the group theoretic 2-Engel relation, and handle slides in a handle decomposition of surgery kernels. The main algebraic input in the proof is the fact that the free group (on any number of generators) modulo the universal 2-Engel relation $[x, [x, y]]$ is nilpotent of class 3 (see Section 2). This contrasts the situation with the Milnor group [**12**], used in the study of link homotopy: the free Milnor group on n generators is nilpotent of class n.

The 2-Engel relation was also used to formulate a collection of universal surgery models in [**5**, Section 7]. Our Theorem 1 is a sharpening of the result of [**5**], in particular the new $\frac{1}{2}$-π_1-null surgery models have a very clean description in terms of their spines and the fundamental
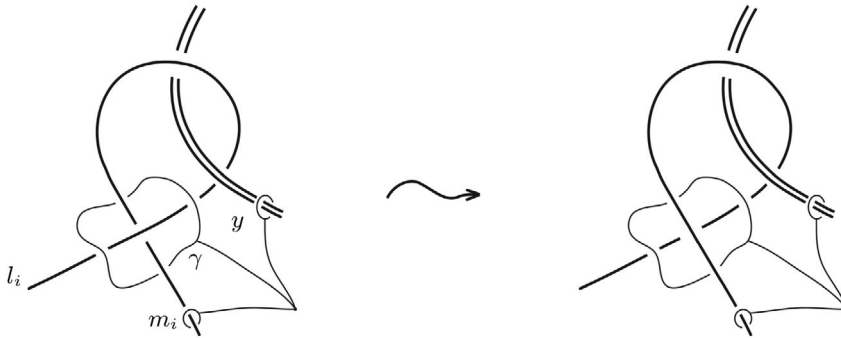
FIGURE 2.1. *A non-generic time during a link homotopy. The based curve γ in the link complement corresponds to the defining relation $[m_i, m_i^y]$ of the Milnor group.*

group. The proof of Theorem 1 is different from the argument in [**5**], although both use the 2-Engel relation.

Another application of these techniques concerns the π_1-null disk lemma (NDL), a technical statement underlying the proofs of the four-dimensional topological surgery and s-cobordism theorems for good groups. This lemma concerns immersed disks bounded by the attaching curve of a capped grope. In Section 5, we formulate a weaker version of NDL, where instead of the attaching curve one considers curves which are trivial in the Milnor group of the body of a capped grope. Theorem 2 shows that this *a priori* weaker statement is in fact equivalent to NDL.

Section 2 summarizes the background material on the Milnor group and the 2-Engel relation. In Section 3, we give a precise description of the $\frac{1}{2}$-π_1-null surgery models. The proof of Theorem 1 is given in Section 4. Applications to the π_1-NDL are discussed in Section 5.

2. *The 2-Engel relation and the Milnor group*

This section gives a brief summary of the background material on link homotopy and group theory needed for the proof of Theorem 1. The reader is referred to [**5, 12**] for a detailed exposition.

Let π be a group normally generated by elements $g_1, \ldots, g_k$. The *Milnor group* of π, defined with respect to the given normal generating set $\{g_i\}$, is defined by

$$M\pi = \pi \,/\, \langle\!\langle\, [g_i, g_i^y]\ \ i = 1, \ldots, k,\ \ y \in G \rangle\!\rangle. \tag{2.1}$$

The Milnor group ML of a link L in S^3 is set to be the Milnor group $M\pi$, where $\pi = \pi_1(S^3 \smallsetminus L)$, defined with respect to meridians to the link components.

Two links are *link-homotopic* (Figure 2.1) if they are connected by a homotopy where different components stay disjoint for all values of the time parameter. If L, L' are link-homotopic, then their Milnor groups ML, ML' are isomorphic [**13**]. Moreover, a k-component link L is link-homotopic to the k-component unlink (*h-trivial*) if and only if ML is isomorphic to the free Milnor group $MF_{m_1,\ldots,m_k}$. A k-component link L is *h-trivial$^+$* if each one of the k links obtained by adding to L an untwisted (zero linking number) parallel copy of a single component is homotopically trivial. If a link is not h-trivial, it is called *h-essential*.

To fix the notation, the lower central series of a group G is defined inductively by $G^1 = G, G^n = [G^{n-1}, G]$. We will use will the concise notation $[g_1, g_2, \ldots, g_n]$ for the commutator $[[\ldots [g_1, g_2], \ldots, g_{n-1}], g_n]$ of elements $g_1, g_2, \ldots, g_n \in G$.
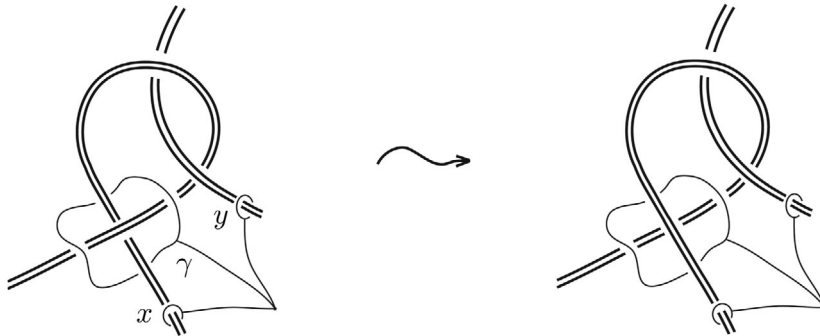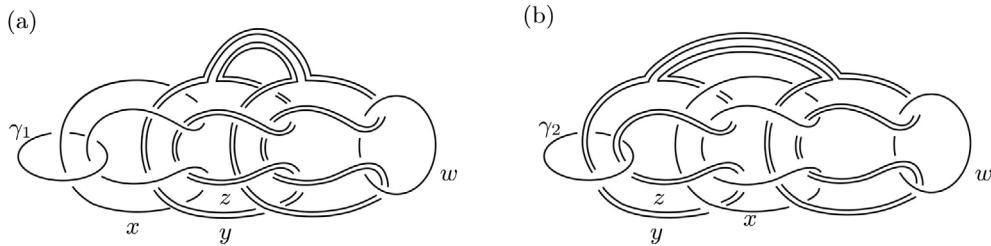
FIGURE 2.2. *An elementary weak homotopy.* (*Compare with Figure* 2.1.)

FIGURE 2.3. (a): $\gamma_1 = [x, yz, yz, w]$, (b): $\gamma_2 = [yz, x, yz, w]$.

The proof of Theorem 1 relies on geometric applications of the 2-Engel relation

$$[y, x, x] = 1, \text{or equivalently } [x, x^y] = 1. \tag{2.2}$$

A 2-*Engel group* π is a group satisfying this relation for all $x, y \in \pi$. Note that the defining Milnor relation (2.1) is the same as the 2-Engel relation, but applied only to x in a fixed set of normal generators. A geometric analogue of the 2-Engel relation is the notion of *weak homotopy*, see Figure 2.2 and [**5**, Section 6].

The free Milnor group on n generators MF_n is nilpotent of class n [**12**]. Building on earlier work of Burnside [**1**], Hopkins [**9**] (also see [**11**]) proved that the nilpotency class of 2-Engel groups is independent of the number of generators.

LEMMA 2.1. *Any 2-Engel group is nilpotent of class* $\leqslant 3$.

A proof in the setting of the Milnor group is given in [**5**]. The following corollary will be used in the proof of Theorem 1.

COROLLARY 2.2 [**5**, Corollary 2.3]. *Suppose π is a group normally generated by $g_1, \ldots, g_n$. Let $g \in \pi^k$ be an element of the kth term of the lower central series, $4 \leqslant k \leqslant n$. Then g may be represented in the Milnor group $M\pi$ as a product of (conjugates of) k-fold commutators C of the form $[h_1, \ldots, h_k]$, where two of the elements h_i are equal to each other and to a product of two generators, $h_j = h_m = g_{i_1} g_{i_2}$ for some $j \neq m$, and each other element h_i is one of the generators $g_1, \ldots, g_n$.*

The commutators $[h_1, \ldots, h_k]$ in the statement of the corollary have a geometric interpretation. Figure 2.3 shows the *elementary Engel links* where the curves γ_i read off these commutators in the complement of the other components, where $k = 4$. These links have an important property that a handle slide involving two parallel components results in a split

FIGURE 3.1. *The figure on the left shows a surgery kernel given by a height= 1 $S^2 \vee S^2$-like capped grope. The corresponding link on the right is obtained by Bing doubling both components of the Hopf link, and then Whitehead doubling each component.*
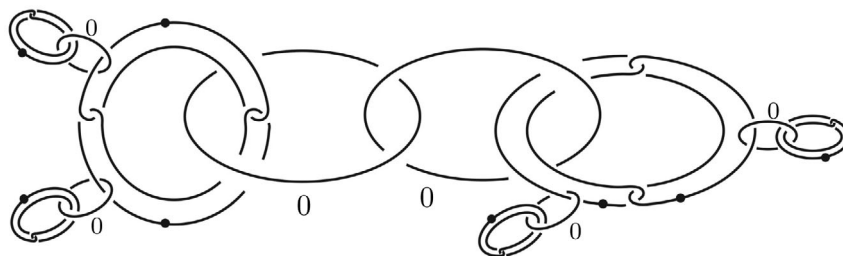

FIGURE 3.2. *A Kirby diagram for the surgery kernel in Figure 3.1.*

link consisting of the unknot and an h-trivial link. This property will be used in the proof of Theorem 1.

3. *Surgery problems*

Our setup is a degree one normal map $(M, \partial M) \longrightarrow (X, \partial X)$ from a topological 4-manifold to a Poincaré pair, which is a $\mathbb{Z}[\pi_1 X]$-homology equivalence over ∂X, with the vanishing Wall obstruction [11]. A solution is a normal cobordism to a simple homotopy equivalence.

In attempting to represent the surgery kernel by an embedded hyperbolic pair $S^2 \vee S^2$, one finds in M 2-complexes which 'approximate' $S^2 \vee S^2$. Specifically, one may represent [6, Sections 2.4 and 5.1] the surgery kernel by $S^2 \vee S^2$-like capped gropes (Figure 3.1), and the neighborhoods of these standard examples are themselves sources of surgery problems. This set constitutes a countable collection of surgery problems, called *universal*. If these are solvable, then all unobstructed problems are solvable.

There is a collection of link-slicing problems corresponding to the universal surgery problems. Starting with $S^2 \vee S^2$-like capped gropes, one obtains the collection $\{Wh(Bing(Hopf))\}$. These links are obtained from the Hopf link by a (non-zero) amount of iterated ramified Bing doubling, and then a single application of ramified $\pm$ Whitehead doubling to *each* component. See Figure 3.1 (right) for an example. These links are obtained by drawing a Kirby diagram of the surgery kernel (see Figure 3.2), eliminating the zero-framed Hopf link in the middle (a diffeomorphism on the boundary), and canceling all 2-handles. Solving a surgery problem produces a 'free' slice complement, that is, one with free π_1, freely generated by meridians, for $Wh(Bing(Hopf))$, that is, constructs a manifold $\simeq \vee S^1$ (generators = meridians) with boundary $\mathcal{S}_0(Wh(Bing(Hopf)))$.

The subject of this paper, $\frac{1}{2}$-π_1-null surgery kernels, are of the form $(G^c \vee S) \cup H^c$, where G^c is a sphere-like capped grope, and S is a sphere with self-intersections, whose double point loops bound capped gropes H^c (Figure 3.3).
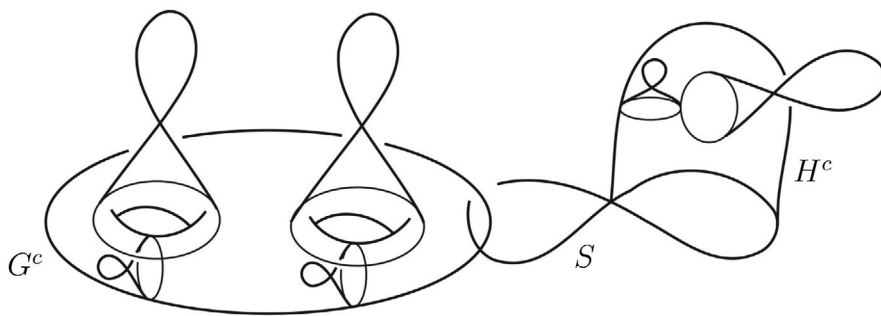
FIGURE 3.3. $\frac{1}{2}$-π_1-Null surgery kernels of the form $(G^c \vee S) \cup H^c$. This illustration shows capped gropes G^c, H^c of height 1 (capped surfaces), and a single double point for the sphere and for the caps. In general, the number of double points and the genera of surfaces (other than the sphere) are parameters in the collection of surgery models.
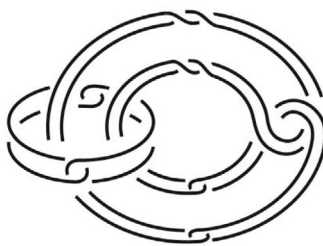

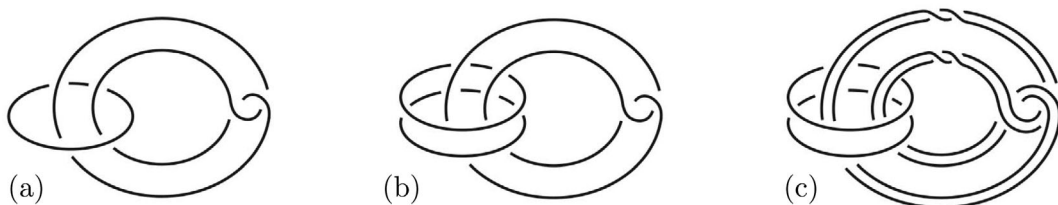FIGURE 3.4. The Bing double of the Whitehead link.

FIGURE 3.5. (a) The Whitehead link, Wh, is h-trivial$^+$. (b) Wh with a parallel copy of one of its components is h-trivial but not h-trivial$^+$. (c) Wh with parallel copies of both components is h-essential.

The link slicing problems, corresponding to these surgery problems, are of the form {Wh(Bing(Wh))}, obtained from the Whitehead link (Figure 3.5(a)) by first iterated ramified Bing doubling and then a single application of ramified Whitehead doubling. (See [6, Chapter 12] for a more detailed discussion of the link slicing formulation of surgery.) These links are constructed from the Kirby diagrams of the surgery kernel analogously to Figure 3.2. An example of such a link (corresponding to both capped surfaces G^c, H^c of genus 1) is the Whitehead double of the link in Figure 3.4. This link is freely slice [8] since it is the Whitehead double of a homotopically trivial$^+$ link. However, links in the family {Wh(Bing(Wh))} corresponding to higher genus surfaces and higher multiplicity of double points, are of the form Wh(h-essential), since ramification is used in their construction (Figure 3.5(b) and (c)). These links are not known to be slice.
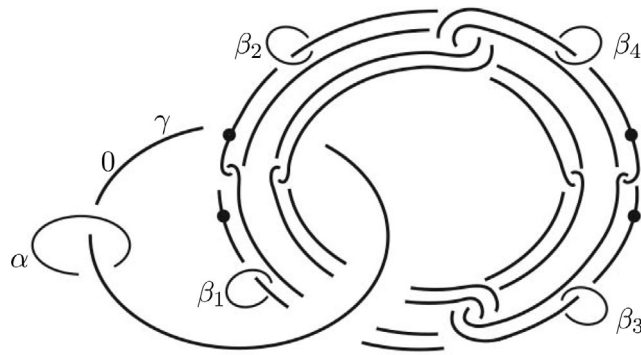
FIGURE 4.1. *The bottom two stages H of the capped grope B. The Kirby diagram for H is given by the zero-framed curve γ and the four dotted curves. (The surface stages of H in this example have genus 1.) β_i are the attaching curves of 2-stage capped gropes which form higher stages of B. α is the attaching curve of the 2-stage capped grope which (capped off with the standard disk bounded by α in D^4) forms A.*
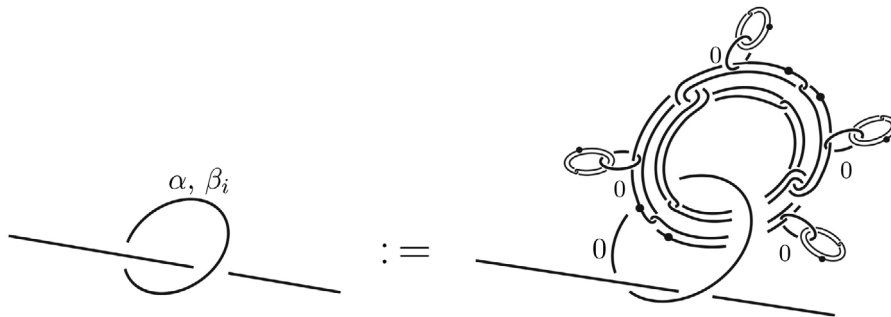

FIGURE 4.2. *Explanation of the notation in Figure 4.1.*

4. Proof Theorem 1

The starting point of the proof is one of the usual universal surgery problems X (discussed in Section 3), with a spine $A \vee B$, where A and B are capped gropes. The grope height raising procedure [6, Proposition 2.7] may be used to arrange A, B to be capped gropes of any given height. To be concrete, let A be a 2-stage capped grope, and B a 4-stage one. Solving this surgery problem amounts to finding a 4-manifold M which is a homotopy 1-complex and with the same boundary as $X^{\dagger}$.

Consider H, a disk-like symmetric grope of height 2 consisting of the bottom two stages of B (Figure 4.1). To get a complete description of the surgery kernel, one replaces the curves α, β_i in Figure 4.1 with Kirby diagrams shown in Figure 4.2. These handle diagrams correspond to the case where all surface stages have genus one, and each cap has a single self-intersection point. (This case encapsulates the main features of the unresolved surgery conjecture for free groups.) In full generality, ramified Bing and Whitehead doubles are used in the construction of the dotted curves in the figures.

The attaching curve γ of H is in the second term of the derived series $\pi_1(H)^{(2)}$, cf. [7, Lemma 1.1]. Since the nth term of the derived series of a group is contained in the 2^nth term of its lower central series, $\gamma \in \pi_1(H)^4$.

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†A collection of π_1 generators of the 1-complex is required to match up with the meridians of the link implicit in the definition of $A \vee B$ (see Section 3).
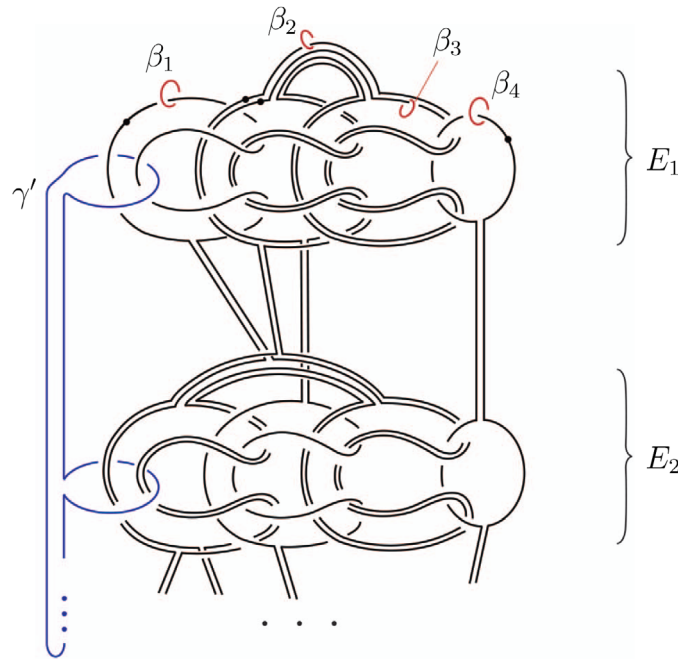
FIGURE 4.3 (colour online). *Representation as a band sum of elementary Engel links.*

Denote by U the unlink representing the 1-handles of H. In the genus 1 case in Figure 4.1, for a suitable choice of meridians m_i to the dotted components U, the curve γ represents the fourfold commutator $[[m_1, m_2], [m_3, m_4]]$ in the free fundamental group $\pi_1(S^3 \smallsetminus U)$. For brevity of notation, we continue the proof in this case. It goes through for arbitrary genera of the surface stages of H since the main algebraic fact, relevant in the proof, is that the attaching curve γ is in $\pi_1(H)^4$. We show how the argument extends to the higher genus case near the end of the proof of the theorem.

By Corollary 2.2, γ may be expressed in the free Milnor group $MF_{m_1, \ldots, m_4}$ as a product

$$\gamma = \prod_k E_k^{f_k}, \tag{4.1}$$

of conjugates of commutators of the form $E_k := [h_1, \ldots, h_4]$, where two of the elements h_i are equal to each other and to a product of two generators, $h_j = h_l = m_{i_1} m_{i_2}$ for some $j \neq l$, and each other element h_n is one of the generators m_i. Next we will construct a curve γ' in the complement to the 4-component unlink U, representing in $MU = MF_{m_1, \ldots, m_4}$ the same group element as γ, and whose geometry reflects the product structure in equation (4.1). Represent the unlink U as a band sum of the 4-component unlinks formed by the components other than γ_i in links in Figure 2.3, associated to elementary Engel commutators E_k in (4.1). This band sum is formed using straight bands, and it may be thought of as a composition of trivial string links corresponding to the elementary Engel links. The band sum representing U is illustrated by the four components on the right in Figure 4.3.

Since the band-sum decomposition of U will be important in the rest of the proof, we specify that the attaching curves β_i of the higher stages of the capped grope B, linking the components of U, are positioned in the first constituent link (corresponding to E_1), as shown in Figure 4.3.

In the unramified case which is considered now, the curve γ' is set to be the straight band sum of the curves γ_i in elementary Engel links, as shown in Figure 4.3, so it represents the product $\gamma' = \prod_k E_k$. The commutators under consideration, $\gamma = [[m_1, m_2], [m_3, m_4]]$ and all of
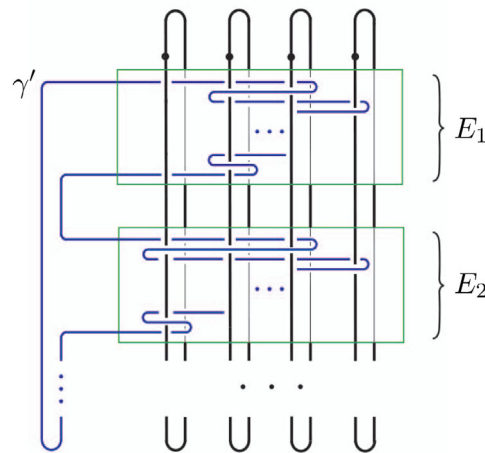
FIGURE 4.4 (colour online). *The function γ' reads off the product in equation* (4.1) *in the free Milnor group* $MF_{m_1,\ldots,m_4}$, *where m_i are meridians to the dotted components.*

the E_i, are of maximal length in $MF_4 := MF_{m_1,\ldots,m_4}$. Indeed, MF_4 is nilpotent of class 4, and all of these elements are in the 4th term of the lower central series, $(MF_4)^4$. It follows that for calculations taking place in the abelian group $(MF_4)^4$ the conjugation in equation (4.1) is immaterial (also see [**5**, Remark 4.2]), so $[\gamma] = [\gamma'] \in MF_4$. The change needed to accommodate the conjugation in (4.1) in the general ramified case is discussed at the end of the proof.

Straightening the dotted components by an isotopy (and letting γ' evolve by an isotopy in their complement), one gets a link of the type sketched in Figure 4.4. In fact, one could define γ' as a simple closed curve reading off the product of the E_k as in Figure 4.4 and carry out the rest of the proof in the group-theoretic setting, where handle slides are replaced with change of generators in the free group. We present a version using elementary Engel links, where the argument is geometrically transparent.

Since γ, γ' represent the same element in MF_4, in the free group F_4 $[\gamma] \cdot [\gamma']^{(-1)}$ equals a product of conjugates of the Milnor relations (2.1) W_j of the form $[m_i, m_i^y]$:

$$[\gamma] = [\gamma'] \cdot \prod_j W_j^{h_j}. \tag{4.2}$$

We incorporate the new factor $W := \prod W_j^{h_j}$ by adding one more block, reading off the word W in the free group, at the bottom to γ' in Figures 4.3 and 4.4. (One could also give a more precise geometric description of individual elements W_j by adding specific links, similar to the elementary Engel links in Figure 2.3. An important difference is that the links corresponding to W_j are *h-trivial*, while the elementary Engel links are h-essential. This explains why the additional factor W will play a very limited role in the proof below.) Abusing the notation, we denote the links corresponding to E_k and W by the same symbols, so U is the band-sum of the links $\{E_k\}$ and W.

To summarize, after adding the additional factor W to γ' (and keeping the same notation, γ', for the resulting curve), we have achieved $[\gamma] = [\gamma']$ in the free group F_4. Therefore, the curves γ, γ' are homotopic in the complement to the 4-component unlink U. Lemma 4.1 shows that this homotopy may be assumed to be of a special kind, useful in the conclusion of the proof of the theorem further below.
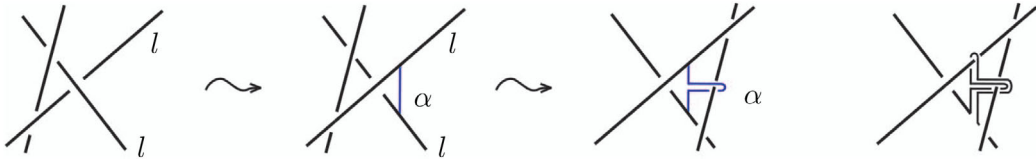
FIGURE 4.5 (colour online). *The two arrows represent a link homotopy followed by an isotopy. The curve α records the link homotopy data. The right-most figure in the panel shows the isotopy of L_0 used in Lemma* 4.1.

LEMMA 4.1. *Suppose two n-component links L_0, L_1 are link-homotopic. Then there is a level-preserving link-homotopy joining them, $H \colon \coprod^n S^1 \times [0,1] \longrightarrow S^3 \times [0,1]$, such that:*

(1) *H is an isotopy of L_0 for $t \in [0, \frac{1}{2})$;*
(2) *H is an isotopy of L_1 for $t \in (\frac{1}{2}, 1]$ which is supported in an arbitrarily small neighborhood of a 1-complex.*

In particular, all singularities (self-intersections of link components) take place at time $t = 1/2$.

Proof of Lemma 4.1. Start with a level-preserving link homotopy H' between L_0 and L_1. Each double point is formed by a transverse self-intersection of a link component. Given a double point of a link component l at time t_i, consider an arc α in $S^3 \times \{t_0 + \epsilon\}$ with both endpoints on l (and with interior disjoint from the link), such that contracting α gives the singular link at time t_i (see Figure 4.5).

At times $t > t_i + \epsilon$ let α evolve by an isotopy in the complement of the link and in the complement of other arcs. (The singular set of a link homotopy, consisting of self-intersections of components, is zero-dimensional, and α is disjoint from it by general position.) The end result at time 1 is the link L_1 with a collection of arcs $\{\alpha\}$, one for each double point of the link homotopy H'. Now a link isotopic to L_0 is obtained by taking a component l near one of the endpoints of each curve α and performing a finger move along α, as shown on the right in Figure 4.5. This describes an isotopy of L_0 for $t \in [0, \frac{1}{2})$. The self-intersections and the isotopy of L_1 for $t \in (\frac{1}{2}, 1]$ consist of the reverse of the above finger moves; they are supported in a neighborhood of the arcs α. □

Returning to the proof of the theorem, apply Lemma 4.1 to the link-homotopy from $\gamma \cup U$ to $\gamma' \cup U$. Here the components of U move by an isotopy, and only the curve γ undergoes self-intersections. Consider the curve produced in the proof of Lemma 4.1 at time $\frac{1}{2} - \epsilon$: it is isotopic to γ, and it differs from γ' by finger moves along arcs $\{\alpha\}$. Applying an isotopy to them if necessary, these arcs (and the resulting finger moves) may be assumed to be disjoint from the bands used in band sums of the links in Figure 4.3. The upshot is that γ is isotopic to a curve respecting the band sum decomposition, like γ' in Figure 4.3, except for the finger moves schematically shown in Figure 4.6.

Now change the handle structure (but keep the 4-manifold intact) in Figure 4.3 by replacing the band sums with canceling 1-, 2-handle pairs, as shown in Figure 4.6.

For the next step in the construction recall that the components of the top elementary Engel link E_1 have linking circles, the attaching curves β_i of the higher stages of the capped grope B. Slide the zero-framed 2-handles, connecting the components of E_1 and E_2, off of E_1, using the curves β_i. Now the component of E_2 have linking circles, denoted β_i^2, which bound parallel copies of the capped gropes attached to β_i. Continue this construction by sliding the zero-framed 2-handles connecting the links E_2 and E_3 using β_i^2, etc. The result is that the components of each link E_k have dual circles β_i^k, bounding a 2-stage capped grope. (Note that
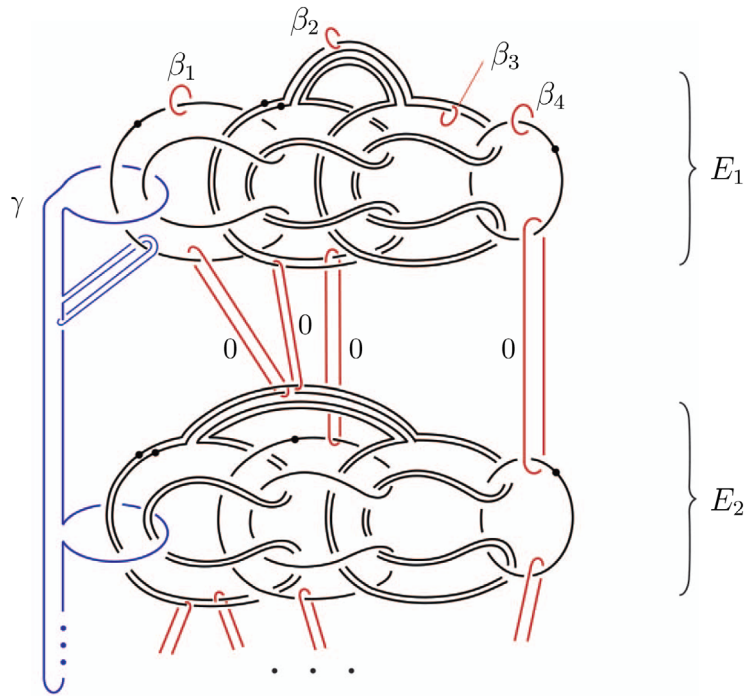
FIGURE 4.6 (colour online). *The band sums in Figure 4.3 are replaced with canceling* 1-, 2-*handle pairs.*

the bodies of all these capped gropes are disjoint, but there are cap–cap intersections.) Here, the curves β_i^1 are defined to be the original β_i. This is depicted in Figure 4.7. The last step of this construction yields dual circles β_i^W, bounding 2-stage capped gropes, for the components of the bottom link W corresponding to the factor W in equation (4.2).

Now replace all dotted components in Figure 4.7 with zero-framed 2-handles. The boundary 3-manifold stays the same, but this creates new hyperbolic pairs. To solve the original surgery problem at the start of the proof of the theorem, it would suffice to solve the new stabilized surgery problem – thus finding a 4-manifold which is a homotopy 1-complex, and with the required boundary 3-manifold.

A key feature of the elementary Engel links is that two of the components are parallel, and removing one of them yields an h-trivial link. For each link, E_k perform a handle slide of one of these components along its parallel. As shown in Figure 4.8, the resulting zero-framed components which participated in the handle slide share a dual capped grope (attached to the curve labeled β_2^k in the figure). In addition, the slid component has its own geometric dual, labeled β_3^k. The δ_{ij}-linking pairing with the dual capped gropes is restored by performing the handle slide of β_2^k over β_3^k. The result is a split link: an h-trivial link $\overline{E}_k$, obtained from an elementary Engel link E_k by omitting a component, and a trivial unlinked component.

The handle-slide step is performed on each link E_j, but not on the link W. To summarize, the entire zero-framed link now consists of

$$L := \gamma \cup \left(\cup_k \overline{E}_k \right) \cup (\text{unlinked trivial components}) \cup W, \tag{4.3}$$

and each component of L has a dual circle which is the attaching region of a 2-stage capped grope. (For γ this capped grope is given by the side A of the original surgery kernel, for all other components they are parallel copies of the top two stages of B.)
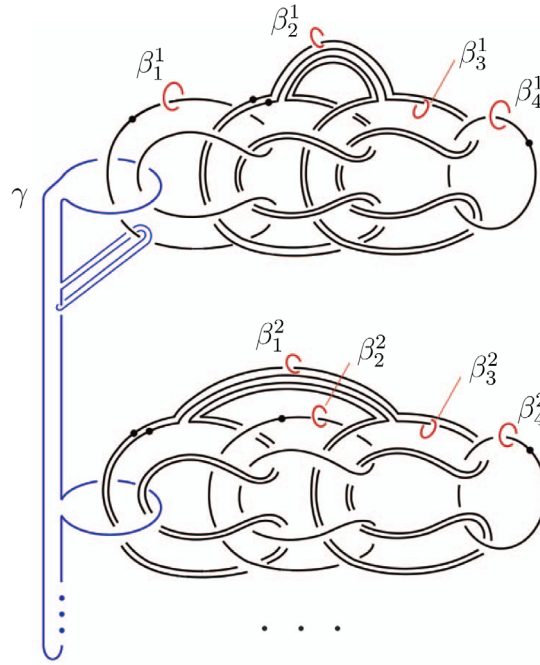
FIGURE 4.7 (colour online). *The zero-framed 2-handles in Figure 4.6, resulting from band sums, are traded for linking circles β_j^i to the components of E_i, bounding 2-stage capped gropes.*
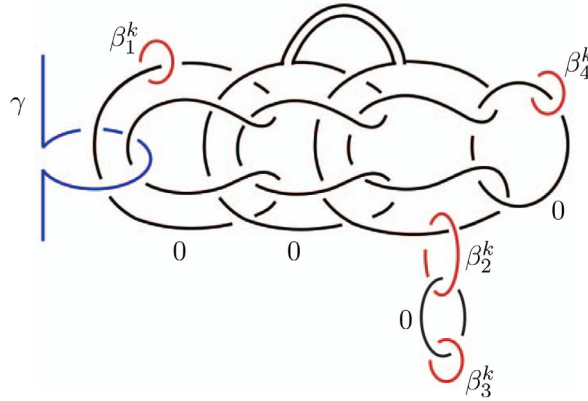

FIGURE 4.8 (colour online). *The result of a handle slide.*

The main point of the band-sum and handle-slide construction above is that the link L is h-trivial. Each link $\overline{E}_k$ is h-trivial by construction, and W is h-trivial by definition, since it corresponds to a product of Milnor's relations.

The analysis above and the resulting conclusion that the link L in (4.3) is h-trivial were achieved under the assumption that the link in Figure 4.1 is unramified (corresponding to the assumption that the two bottom surface stages of the capped grope B have genus one.) In this case, the conjugation in equation (4.1) could be disregarded. In the general case, the Engel commutators are still fourfold but the unlink U has more than 4 components, so the conjugation in the expression (4.1) for γ has to be taken into account. Note, however, that for each k in equation (4.1), a word representing the conjugating element f_k may be assumed to have none of the four constituent generators of the elementary Engel relation E_k. Indeed, one
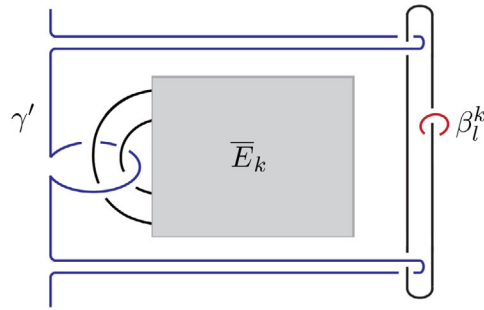
FIGURE 4.9 (colour online). *Conjugation in the ramified case.*

checks using the Magnus expansion (cf. [**6**, Section 2.1]) that conjugating $E_k = [g_{i_1}, \ldots, g_{i_4}]$ by a word containing a generator g_{i_j}, $1 \leqslant j \leqslant 4$, gives the same element in the free Milnor group MU as the conjugate of E_k by the word where g_{i_j} is omitted.

As in the unramified case, the unlink U is represented as a straight band sum of elementary Engel links as in Figure 4.3. The complexity arising from the conjugating elements f_k is carried by the curve γ': rather than being just a band sum of the curves γ_i in elementary Engel links, just above and below the link E_k paths reading off the words f_k, f_k^{-1} are incorporated into γ'. As discussed in the previous paragraph, these paths do not involve the components of E_k. The result of the proof analogous to the unramified case above is illustrated in Figure 4.9, where the schematically shown link $\overline{E}_k$ (see Figure 4.8) is h-trivial. Each factor $\overline{E}_k^{f_k}$, after null-homotoping $\overline{E}_k$, reduces to the trivial product $f_k f_k^{-1}$. Therefore, the link L in equation (4.3) is h-trivial in the ramified case as well.

To conclude the proof of Theorem 1, consider disjoint null-homotopies Δ for L in D^4. These null-homotopies, capped off with the cores of the zero-framed 2-handles attached to D^4 along L, form the π_1-null spheres in the $\frac{1}{2}$-π_1-null models (Figure 3.3). Each of these spheres has a dual capped grope which constitutes the other side of the surgery models.

The double point loops of Δ are null-homotopic in the 4-ball; these null-homotopies intersect Δ and each other. All these intersections are eliminated using the dual capped gropes; now the double point loops of the spheres bound capped gropes as required in the definition of $\frac{1}{2}$-π_1-null models. All of the gropes can be made disjoint by removing the cap–cap intersections using the usual operations of grope height raising and contraction/push-off [**6**, Sections 2.3 and 2.7].

Starting with one of the usual universal surgery problems X, the proof above showed that the surgery kernel is represented by a $\frac{1}{2}$-π_1-null model. By transitivity of surgery [**5**, Lemma 7.3], solving it would yield a solution to the original surgery problem X. □

5. The π_1-null disk lemma

In this section, we show that the methods developed in the proof of Theorem 1 also give a new formulation of the π_1-NDL. This lemma underlies the proofs of the surgery and s-cobordism conjecture for *good* groups; indeed, the class of good groups is defined precisely as the groups for which this lemma holds (see below). Recall its statement [**7**]:

π_1-NDL. *Let G^c be a Capped Grope of height 2 with the attaching curve γ, and $\phi\colon \pi_1 G^c \longrightarrow \pi$ a group homomorphism. Then γ bounds an immersed disk $\Delta\colon D^2 \longrightarrow G^c$ which is π_1-null under ϕ.*

A group π is called *good* if it satisfies NDL for all choices of capped gropes G^c and homomorphisms ϕ. In this statement, following the notation of [**7**], capital letters in the term
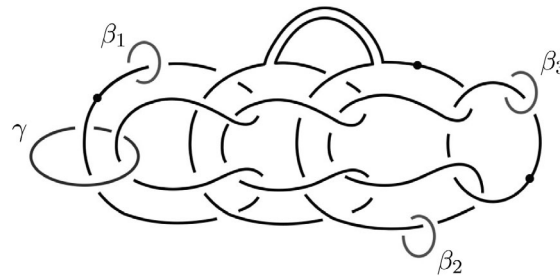
FIGURE 5.1. *An example of $\gamma \subset \partial H$ in the statement of WNDL.*

Capped Grope indicate the standard 4-dimensional thickening of the underlying 2-complex. Before stating the new version of NDL, consider the following rephrasing of its statement. Let H denote the body of a capped grope G^c of height 2. Then H is a handlebody $\natural_{i=1}^{k}(S^1 \times D^3)$, and there is a collection of framed curves in ∂H which serve as attaching regions of kinky handles (thickenings of caps of G^c). Connected to a basepoint in H, these curves represent generators of the free group $F_k \cong \pi_1(H)$. Then a basic calculation in Milnor's theory of link homotopy shows that the attaching curve γ of G^c is non-trivial in the free Milnor group $M\pi_1(H)$. In other words, representing H by the Kirby diagram U consisting of k dotted components corresponding to the chosen generators of $\pi_1(H)$, $\gamma \cup U$ is an h-essential link. The example with all surface stages of G^c of genus 1 is shown in Figure 4.1, where in the present context the curves β_i serve as the attaching curves of caps of G^c.

Our new formulation shows that it suffices to consider curves in the boundary of the handlebody H which are *trivial* in $M\pi_1(H)$. Consider the following statement:

Weak π_1-NDL (WNDL). Let $M = H \cup$ (plumbed 2$-$handles), where H is a handlebody $\natural_{i=1}^{k}(S^1 \times D^3)$, and the 2-handles are attached along a collection of standard framed curves in ∂H which represent a set of generators of the free group $\pi_1(H)$. Let γ be a curve in ∂H, trivial in $M\pi_1(H)$, where the Milnor group is defined with respect to the given generators. Let $\phi\colon \pi_1 M \longrightarrow \pi$ be a group homomorphism. Then γ bounds an immersed disk $\Delta\colon D^2 \longrightarrow M$ which is π_1-null under ϕ.

In analogy with the π_1-NDL, we say that a group π satisfies WNDL if the statement holds for all M, γ and ϕ. Arbitrary plumbings and self-plumbings are allowed among the 2-handles attached to H in the statement of WNDL, in analogy with caps of a capped grope. A representative example of $\gamma \subset \partial(\natural_{i=1}^{3}(S^1 \times D^3))$ is shown in Figure 5.1. This example is obtained by deleting a component of an elementary Engel link (Figure 2.3).

If γ were null-homotopic in the complement of the dotted unlink U, of course it would bound a π_1-null disk in M since H is π_1-null in M. However in non-trivial cases, such as the one in Figure 5.1, self-intersections of the components of U are required to make γ null-homotopic. Such self-intersections do not make sense since U represents standard slices removed from D^4.

THEOREM 2. *If a group π satisfies WNDL, then π is good.*

Proof. Consider (G^c, γ) and $\phi\colon \pi_1 G^c \longrightarrow \pi$ in the statement of NDL. The proof of Theorem 1 through the point (shown in Figure 4.6) just before stabilization applies to the curve γ within the handlebody H. Here, the circles β_i linking the components, shown in Figure 4.1 and used throughout the proof of Theorem 1, are understood to be the attaching curves of kinky handles. Instead of stabilizing (replacing dotted components with zero-framed 2-handles), perform the handle slides on 1-handles rather than on 2-handles. One gets links of the form shown in Figure 4.8, but the components are dotted rather than zero-framed. As in the proof

of Theorem 1, the link L in equation (4.3) is h-trivial. Since π is assumed to satisfy WNDL, γ bounds a π_1-null disk under ϕ. □

The main open case is $\pi = F_k$, and $\phi =$ the identity homomorphism. In other words, the question is whether γ as in the statement of WNDL bounds a π_1-null disk in M. For example, one may consider $k = 3$ and the curve γ in Figure 4.8, for various choices of plumbed and self-plumbed 2-handles attached to the curves β_i.

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