The Tutte polynomial relations for planar and surface graphs

Slava Krushkal

March 20, 2010

The *chromatic polynomial* $\chi_\Gamma(Q)$ of a graph Γ, for $Q \in \mathbb{Z}_+$, is the number of colorings of the vertices of Γ with the colors $1, \ldots, Q$ where no two adjacent vertices have the same color.

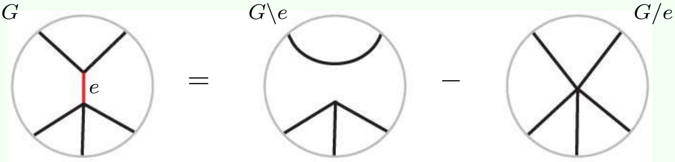
- $$\chi_\Gamma(Q) = \sum_{S \subset \{\text{edges of } \Gamma\}} (-1)^{|S|} \, Q^{k(S)}$$

where $k(S)$ is the number of connected components of the graph which has the same vertices as Γ and whose edge set is given by S.

- The contraction-deletion rule: given any edge e of Γ which is not a loop,

$$\chi_\Gamma(Q) = \chi_{\Gamma\setminus e}(Q) - \chi_{\Gamma/e}(Q)$$

If Γ contains a loop then $\chi_\Gamma \equiv 0$.
If Γ has no edges and V vertices, then $\chi_\Gamma(Q) = Q^V$.



W.T. Tutte (1969):

the "golden identity": for a planar triangulation T,

$$\chi_T(\phi+2) = (\phi+2)\,\phi^{3\,V(T)-10}\,(\chi_T(\phi+1))^2,$$

where $V(T)$ is the number of vertices of the triangulation.

ϕ denotes the golden ratio, $\phi = \frac{1+\sqrt{5}}{2}$.

Another Tutte's relation:

$$\chi_{Z_1}(\phi+1)+\chi_{Z_2}(\phi+1)=\phi^{-3}[\chi_{Y_1}(\phi+1)+\chi_{Y_2}(\phi+1)],$$

where Y_i, Z_i are planar graphs which are locally related as follows:

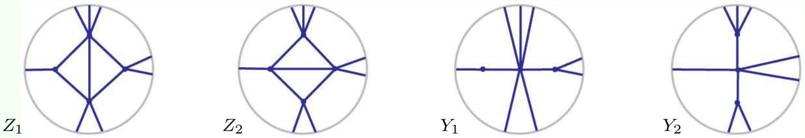


Figure:

Reference: P. Fendley and V. Krushkal, *Tutte chromatic identities from the Temperley-Lieb algebra*, Geometry and Topolology 13(2009), 709-741 [arXiv:0711.0016]

Outline:

Define the chromatic algebra $\mathcal{C}_n^Q$.

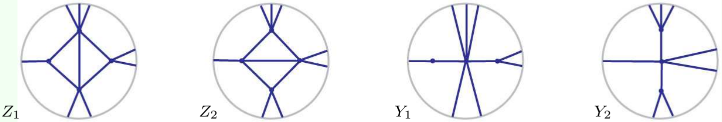
Basic idea: consider the contraction-deletion rule as a linear relation in the vector space spanned by graphs, rather than just a relation defining the chromatic polynomial.

The Markov trace of a graph is the chromatic polynomial of its dual.

Identities such as Tutte's can then be understood as finding elements of the *trace radical*: elements of the chromatic algebra which, multiplied by any other element of the algebra, are in the kernel of the Markov trace.

Tutte's polynomial relation:

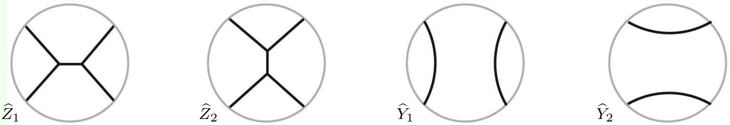
$$\chi_{Z_1}(\phi+1) + \chi_{Z_2}(\phi+1) = \phi^{-3}[\chi_{Y_1}(\phi+1) + \chi_{Y_2}(\phi+1)]$$



Prove that the relation

$$\widehat{Z}_1 + \widehat{Z}_2 = \phi^{-3}[\widehat{Y}_1 + \widehat{Y}_2]$$

holds in the chromatic algebra $\mathcal{C}_2^{\phi+1}$:



Outline:

Construct a map: chromatic algebra $\longrightarrow$ Temperley-Lieb algebra:

$$\mathcal{C}_n^Q \longrightarrow TL_{2n}^d, \quad Q = d^2$$

The trace radical in the Temerley-Lieb algebra is well-understood: the Jones-Wenzl projectors at special values of d.

Pull them back to get elements in the trace radical of the chromatic algebra.

Outline:

Construct a map: chromatic algebra $\longrightarrow$ Temperley-Lieb algebra:

$$\mathcal{C}_n^Q \longrightarrow TL_{2n}^d, \qquad Q = d^2$$

The trace radical in the Temerley-Lieb algebra is well-understood: the Jones-Wenzl projectors at special values of d.

Pull them back to get elements in the trace radical of the chromatic algebra.

Result: A generalization of Tutte's relation for the chromatic polynomial at $Q = 2 + 2\cos\left(\frac{2\pi j}{n+1}\right)$. Recursive formula.

These values of Q: $Q = 2 + 2\cos\left(\frac{2\pi j}{n+1}\right)$ are generalizations of

Beraha numbers: $B_n = 2 + 2\cos\left(\frac{2\pi}{n+1}\right) \quad (B_5 = \phi + 1)$.

Tutte conjectured the existence of chromatic polynomial relations for each Beraha number.

These values of Q: $Q = 2 + 2\cos\left(\frac{2\pi j}{n+1}\right)$ are generalizations of

Beraha numbers: $B_n = 2 + 2\cos\left(\frac{2\pi}{n+1}\right) \quad (B_5 = \phi + 1)$.

Tutte conjectured the existence of chromatic polynomial relations for each Beraha number.

Beraha experimentally observed (in the 1970s) that the zeros of the chromatic polynomial of large planar triangulations seem to accumulate near these numbers (B_n).

Another Tutte's result:

$$|\chi_T(\phi + 1)| \leq \phi^{5-k}$$

where T is a planar triangulation and k is the number of its vertices.

Analogue for other Beraha numbers??

The Temperley-Lieb algebra in degree n, TL_n, is an algebra over $\mathbb{C}[d]$ generated by $1, E_1, \ldots, E_{n-1}$ with the relations

$$E_i^2 = E_i, \qquad E_i E_{i\pm 1} E_i = \frac{1}{d^2}\, E_i, \qquad E_i E_j = E_j E_i \text{ for } |i-j| > 1.$$

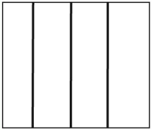
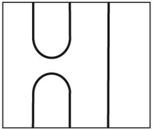
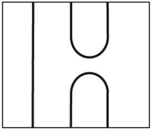
Define $TL = \cup_n TL_n$. The indeterminate d may be specialized to a complex number, and then it is denoted TL_n^d.

The Temperley-Lieb algebra in degree n, TL_n, is an algebra over $\mathbb{C}[d]$ generated by $1, E_1, \ldots, E_{n-1}$ with the relations

$$E_i^2 = E_i, \qquad E_i E_{i\pm1} E_i = \frac{1}{d^2}\, E_i, \qquad E_i E_j = E_j E_i \text{ for } |i-j| > 1.$$

Define $TL = \cup_n TL_n$. The indeterminate d may be specialized to a complex number, and then it is denoted TL_n^d.

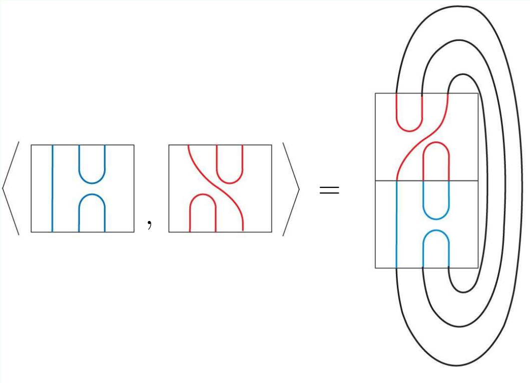
Pictorially, an element of TL_n is a linear combination of $1-$dimensional submanifolds in a rectangle R. Each submanifold meets both the top and the bottom of the rectangle in exactly n points. The multiplication corresponds to vertical stacking of rectangles. Generators:

$1 =$  $E_1 = \frac{1}{d}$  $E_2 = \frac{1}{d}$ 

Relation: Any circle in a picture may be erased and then the element in the algebra is multiplied by d

The trace tr_d: $TL_n^d \longrightarrow \mathbb{C}$ is defined on rectangular pictures by connecting the top and bottom endpoints by disjoint arcs and evaluating $d^{\#circles}$.

The scalar product on TL_n is defined by $\langle a, b \rangle = tr(a\,\bar{b})$.



$= d^2$

The chromatic algebra is defined as isotopy classes of graphs in a rectangle modulo local relations:

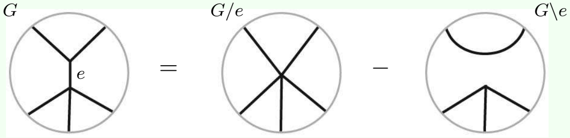


Figure: Relation (1) in the chromatic algebra

Figure: Relations (2), (3) in the chromatic algebra

Consider the set $\mathcal{G}_n$ of the isotopy classes of planar graphs G embedded in the rectangle R with n endpoints at the top and n endpoints at the bottom of the rectangle.

Let $\mathcal{F}_n$ denote the free algebra over $\mathbb{C}[Q]$ with free additive generators given by the elements of $\mathcal{G}_n$. The multiplication is given by vertical stacking. Define $\mathcal{F} = \cup_n \mathcal{F}_n$.

The *chromatic algebra* in degree n, $\overline{\mathcal{C}}_n$, is an algebra over $\mathbb{C}[Q]$ which is defined as the quotient of the free algebra $\mathcal{F}_n$ by the ideal I_n generated by the relations (1), (2), (3).

(1) If e is an inner edge of a graph G which is not a loop, then $G = G/e - G\backslash e$.
(2) If G contains an inner edge e which is a loop, then $G = (Q-1)\, G\backslash e$.
(3) If G contains a $1-$valent vertex (in the interior of the rectangle), then $G = 0$.

The trace, $tr_\chi\colon \bar{\mathcal{C}}^Q \longrightarrow \mathbb{C}$ is defined on the additive generators (graphs) G by connecting the endpoints of G by arcs in the plane (denote the result by $\overline{G}$) and evaluating

$$Q^{-1} \cdot \chi_{\widehat{\overline{G}}}(Q).$$

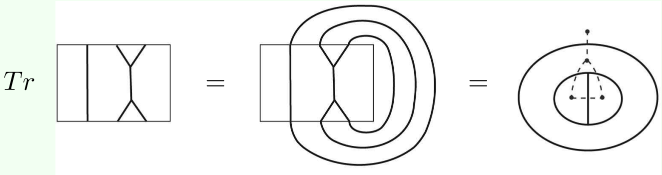


Figure: An example of the evaluation of the trace: The trace $= (Q-1)^2(Q-2)$.

There is a presentation of the chromatic algebra in terms of trivalent graphs:

$\mathcal{C}_n^Q$ is isomorphic to the algebra generated by trivalent graph in a rectangle, modulo local relations:

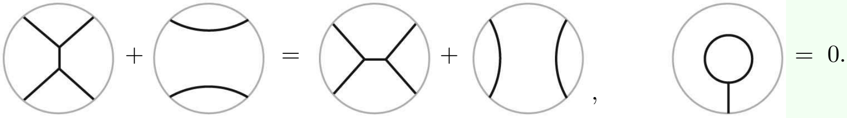


Figure: Relations in the trivalent presentation of the chromatic algebra.

Consider the algebra homomorphism $\Phi\colon\, \mathcal{C}_n^{d^2} \longrightarrow TL_{2n}^d$:

The factor in the definition of Φ corresponding to a $k-$valent vertex is $d^{(k-2)/2}$. The overall factor for a graph G is the product of the factors $d^{(k(V)-2)/2}$ over all vertices V of G.

Let G be a planar graph. Then

$$Q^{-1}\,\chi_Q(\widehat{G}) \;=\; \Phi(G)$$

Here $Q = d^2$. Therefore, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}_n^Q & \xrightarrow{\ \Phi\ } & TL_{2n}^d \\ \big\downarrow{\scriptstyle tr_\chi} & & \big\downarrow{\scriptstyle tr_d} \\ \mathbb{C} & \xrightarrow{\ =\ } & \mathbb{C} \end{array}$$

For example, for the theta-graph G,

$$Q^{-1}\chi_Q(\widehat{G}) \;=\; (Q-1)(Q-2) \;=\; d^4 - 3d^2 - 4 \;=\; \Phi(G).$$

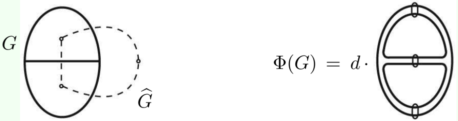
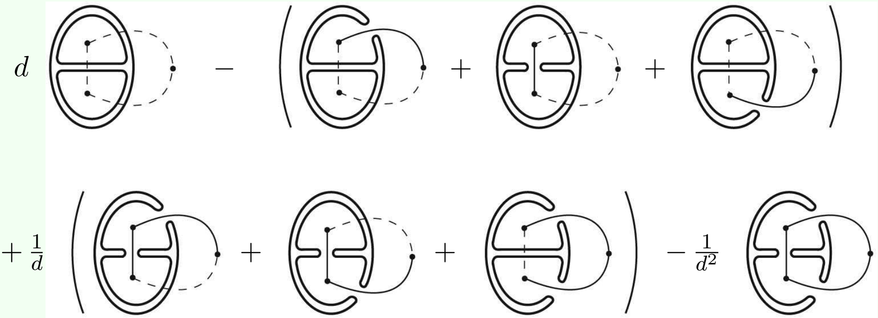


Figure:



The expansions of $Q^{-1}\,\chi_Q(\widehat{G})$, $\Phi(G)$ where G is the theta graph.

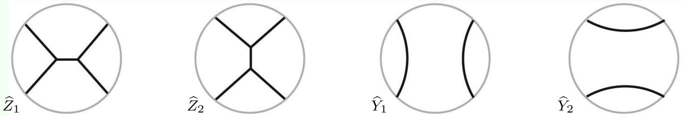
The trace radical of an algebra A consists of elements a such that $\langle a, b \rangle = 0$ for all b in A.

Corollary: The pullback of the trace radical in TL^{d}_{2n} to $\mathcal{C}^{d^2}_{n}$ is in the trace radical of the chromatic algebra.

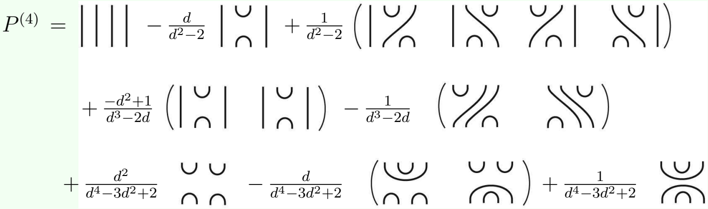
The trace radical of the Temperley-Lieb algebra is well-understood (Jones, Wenzl, Goodman):

It is non-trivial precisely for $d = 2\, cos(\pi j/n)$, and for these values it is generated by the Jones-Wenzl projector.

$$\widehat{Z}_1 + \widehat{Z}_2 = \phi^{-3}\,[\,\widehat{Y}_1 + \widehat{Y}_2\,]$$



Φ maps the dual of Tutte's relation to the 4-th Jones-Wenzl relation (at $d = \phi$):



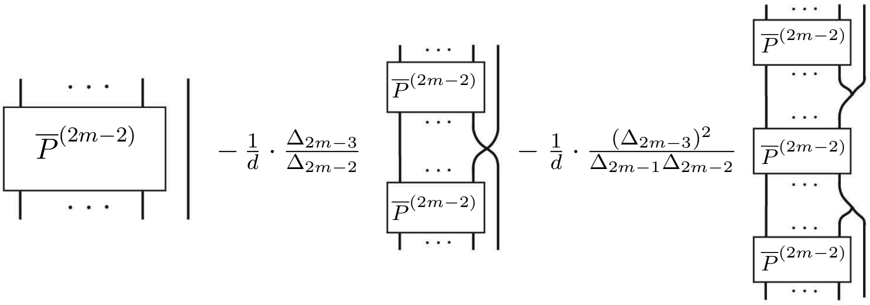


Figure: A recursive formula for the pull-back $\overline{P}^{(2m)}$ of the Jones-Wenzl projector $P^{(2m)}$ in the chromatic algebra.

Theorem

For a planar triangulation $\widehat{G}$*,*

$$\chi_{\widehat{G}}(\phi+2) = (\phi+2)\,\phi^{3\,V(\widehat{G})-10}\,(\chi_{\widehat{G}}(\phi+1))^2 \qquad (1)$$

where $V(\widehat{G})$ *is the number of vertices of* $\widehat{G}$*.*

Theorem

For a planar triangulation $\widehat{G}$,

$$\chi_{\widehat{G}}(\phi+2) = (\phi+2)\,\phi^{3\,V(\widehat{G})-10}\,(\chi_{\widehat{G}}(\phi+1))^2 \qquad (2)$$

where $V(\widehat{G})$ is the number of vertices of $\widehat{G}$.

Idea of the proof: Construct a map

$$\mathcal{C}^{\phi+2} \longrightarrow \mathcal{C}^{\phi+1} \times \mathcal{C}^{\phi+1}$$

and apply the trace:

$$\begin{array}{ccc} \mathcal{C}^{\phi+2} & \longrightarrow & (\mathcal{C}^{\phi+1}/R) \times (\mathcal{C}^{\phi+1}/R) \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{=} & \mathbb{C} \end{array}$$

More conceptually, Tutte's golden identity is a consequence of level-rank duality for $SO(N)$ topological quantum field theories.

In particular, the level-rank duality implies that the $SO(3)_4$ and $SO(4)_3$ theories are isomorphic, and the latter splits into a product of two copies of $SO(3)_{3/2}$,

$$SO(3)_4 \longrightarrow SO(3)_{3/2} \otimes SO(3)_{3/2}$$

The partition function of an $SO(3)$ theory is given in terms of the chromatic polynomial, specifically $\chi(\phi+2)$ for $SO(3)_4$ and $\chi(\phi+1)$ for $SO(3)_{3/2}$.

The Tutte polynomial of a graph G:

$$T_G(X,Y) = \sum_{H \subset G} X^{c(H)-c(G)}\, Y^{n(H)}.$$

The summation is taken over all spanning subgraphs H of G.

$c(H)$ denotes the number of connected components of the graph H, and $n(H)$ is the *nullity* of H, defined as the rank of the first homology group $H_1(H)$.

($n(H)$ may also be computed as $c(H) + e(H) - v(H)$, where e and v denote the number of edges and vertices of H, respectively.)

Basic properties of the Tutte polynomial: the contraction-deletion rule, and the duality

$$T_G(X,Y) = T_{G^*}(Y,X)$$

where G is a planar graph, and G^* is its dual.

(The vertices of G^* correspond to the connected regions in the complement of G in the plane, and two vertices are connected by an edge in G^* whenever the two corresponding regions are adjacent.)

Now suppose G is a ribbon graph (a graph embedded in a surface Σ). Consider the polynomial

$$P_{G,\Sigma}(X,Y,A,B) \; = \; \sum_{H\subset G} X^{c(H)-c(G)} \; Y^{k(H)} \; A^{s(H)/2} \; B^{s^{\perp}(H)/2}$$

Here $s(H)$ dentes the genus of the surface obtained as a regular neighborhood of the graph H in Σ, and $s^{\perp}(H)$ is the genus of the surface obtained by removing a regular neighborhood of H from Σ. Denote by i the embedding $G \longrightarrow \Sigma$, and define

$$k(H) := \dim\left(\ker\left(i_*\colon\; H_1(H;\mathbb{R}) \longrightarrow H_1(\Sigma;\mathbb{R})\right)\right).$$

The polynomial P satisfies the contraction-deletion rule,

$$P_G = P_{G \smallsetminus e} + P_{G/e},$$

and it satisfies a duality relation, analogous to the duality of the Tutte polynomial os planar graphs:

$$P_G\,(X, Y, A, B) \;=\; P_{G^*}\,(Y, X, B, A).$$

Reference: V. Krushkal, *Graphs, links, and duality on surfaces*, arXiv:0903.5312

The well-known Bollobás-Riordan polynomial of ribbon graphs is defined by

$$BR_{G,S}(X,Y,Z) \;=\; \sum_{H\subset G} (X-1)^{r(G)-r(H)}\, y^{n(H)}\, Z^{c(H)-bc(H)+n(H)}.$$

Let $v(H), e(H)$ denote the number of vertices, respectively edges, of H, and let $c(H)$ be the number of connected components. Then $r(H) = v(G) - c(H)$, $n(H) = e(H) - r(H)$, and $bc(H)$ is the number of boundary components of the surface S.

The polynomial $BR_{G,S}$ is a universal polynomial of ribbon graphs, satisfying the contraction-deletion rule.

The Bollobás-Riordan polynomial of a ribbon graph may be obtained as a specialization of the polynomial P_G:

$$BR_{G,S}(X,Y,Z) \;=\; Y^g\, P_{G,\Sigma}(X-1, Y, YZ^2, Y^{-1}).$$