

SURGERY ON CLOSED 4-MANIFOLDS WITH FREE FUNDAMENTAL GROUP

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The 4-dimensional topological surgery conjecture has been established for a class of groups, including the groups of subexponential growth (see [6], [13] for recent developments), however the general case remains open. The full surgery conjecture is known to be equivalent to the question for a class of canonical problems with free fundamental group [5, Chapter 12]. The proof of the conjecture for “good” groups relies on the disk embedding theorem (see [5]), which is not presently known to hold for arbitrary groups. However, in certain cases it may be shown that surgery works even when the disk embedding theorem is not available for a given fundamental group (such results still use the disk-embedding theorem in the simply-connected setting, proved in [3].) For example, this may be done when the surgery kernel is represented by π_1 -null spheres [4], or more generally by a π_1 -null submanifold satisfying a certain condition on Dwyer’s filtration on second homology [7]. Here we state another instance when the surgery conjecture holds for free groups. The following results are stated in the topological category.

Theorem 1. *Let X be a 4-dimensional Poincaré complex with free fundamental group, and assume the intersection form on X is extended from the integers. Let $f: M \rightarrow X$ be a degree 1 normal map, where M is a closed 4-manifold. Then the vanishing of the Wall obstruction implies that f is normally bordant to a homotopy equivalence $f': M' \rightarrow X$.*

In the canonical surgery problems, X has free fundamental group and trivial π_2 , however what makes them harder to analyze is the interplay between the homotopy type of X , and the topology of the boundary. Our result sidesteps this by considering closed manifolds. We also prove a related splitting result:

Theorem 2. *Let M be a closed orientable 4-manifold with free fundamental group, and suppose the intersection form on M is extended from the integers. Then M is s -cobordant to a connected sum of $\sharp^n S^1 \times S^3$ with a simply-connected 4-manifold.*

Note that if the surgery conjecture fails for free groups, then for both theorems above there is, in general, no extension to 4-manifolds with boundary. The assumption on the intersection pairing in theorem 2 is necessary, since there are forms

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not extended from the integers, for example for $\pi_1 \cong \mathbb{Z}$ [8]. It follows from the classification of 4-manifolds with infinite cyclic fundamental group that under the assumptions of theorem 2, if $\pi_1(M) \cong \mathbb{Z}$ then M is *homeomorphic* to a connected sum of $S^1 \times S^3$ with a simply-connected 4-manifold [5]. The s -cobordism conjecture for free non-abelian groups remains open.

A brief outline of the proof of theorem 1 is as follows. It may be assumed that $\pi_1(f)$ is an isomorphism, and $\ker(\pi_2(f))$ is a direct sum of standard planes. Using the assumption on the intersection form, we construct a complex $K = \vee^n S^1 \vee K_0$, where K_0 is simply-connected, and a map $X \rightarrow K$, inducing isomorphisms on π_1 and π_2 . The inverse preimage of a collection of points $\{p_i\}$, one in each circle summand of K , under the composition $h: M \rightarrow X \rightarrow K$, is arranged to be a disjoint union of 3-spheres in M , at the expense of further stabilizing M . Now consider 2-spheres in M , representing a hyperbolic basis of $\ker(\pi_2(f))$; they are surgered along disks lying in the 3-spheres, and we show that the resulting elements of π_2 also form a hyperbolic basis of the surgery kernel. However, the new 2-spheres lie in a simply-connected 4-manifold, thus the disk embedding theorem yields embedded transverse pairs of spheres, concluding the argument. In the proof of theorem 2 one also has to keep track of the Lagrangians in the surgery kernel, so that the constructed cobordism is a $\mathbb{Z}\pi_1$ -homology product.

Theorem 2 also follows from the results of [1], [2]. However, there the authors additionally assume that M is smooth, while the conclusion is still topological. Under the assumption $\pi_2(X) = 0$, the splitting theorem 2 is also stated in [9]. Our result is entirely in the topological category, and the line of argument is different from the above papers. In particular, instead of using 5-dimensional surgery theory, our proof gives a more explicit geometric construction of the s -cobordism.

Remark. The idea of our proof extends in certain cases to 4-manifolds with boundary. Recall from the outline above that the proof considers point inverses $h^{-1}(p_i)$, changes them into 3-spheres, up to a cobordism of M , and reduces the problem to the simply-connected setting. For M with non-empty boundary, if there is a map $h: M \rightarrow K$ with $h^{-1}(p_i) \cap \partial M = S^2$, for each i , then the same proof yields 3-disks, and the argument goes through. Compare this with the general case: for example if $h^{-1}(p_i) \cap \partial M$ are tori, they cannot necessarily be arranged to bound disjoint solid tori, even up to an s -cobordism of M , see [12]. This illustrates the difference between the closed case, considered here, and the canonical surgery problems.

Before proceeding with the proof of the theorems, we state a preliminary result. Here we introduce a 2-complex K which will serve as a reference for the homotopy data while the cobordisms are being constructed.

Lemma 3. *Let X be a 4-dimensional Poincaré complex with free fundamental group, and suppose the intersection form on X is extended from the integers. Then*

there is a simply-connected complex K_0 and a map $g: X \longrightarrow K = \vee^n S^1 \vee K_0$, inducing isomorphisms on π_1 and π_2 .

Proof. By [17, §0.2] we may replace X by a homotopy equivalent 4-complex with a single top cell. Following [14], consider the 3-type (π_1, π_2, k) of its 3-skeleton $X^{(3)}$. Here $\pi_1 = \pi_1(X)$, $\pi_2 = \pi_2(X)$ is a module over π_1 , and the invariant k vanishes, since k is an element of $H^3(\pi_1; \pi_2) = 0$. The assumption on the intersection form implies that $\pi_2(X) \cong A \otimes_{\mathbb{Z}} \mathbb{Z}\pi_1$ where A is an abelian group, and the intersection form on $\pi_2(X)$ is induced from a form on A . Let $K_0 = K(A, 2)$, and set

$$K = K(\pi_1, 1) \vee K(A, 2) = \vee^n S^1 \vee K_0,$$

then $K^{(3)}$ has the same algebraic 3-type as $X^{(3)}$. It is proved in [14] that any homomorphism of 3-types is induced by a map between the 3-complexes. Hence there is a map $g: X^{(3)} \longrightarrow K$, inducing isomorphisms on π_1 and π_2 .

We claim that the obstruction o to extending g over the top cell of X , $o \in H^4(X; \pi_3(K))$, vanishes. Since $\pi_3(K_0) = 0$, by Hilton-Milnor's theorem $\pi_3(K)$ is generated by the Whitehead products $[\alpha \cdot a, \beta \cdot b]$, where $a, b \in \pi_2(K_0)$, and $\alpha \neq \beta \in \pi_1$. Suppose o does not vanish, then the value of o on the fundamental cycle in X is non-trivial in $\pi_3(K)$. Since g induces isomorphisms on π_1 and π_2 , the attaching map of the 4-cell has a non-trivial component $[\alpha \cdot a, \beta \cdot b] \in \pi_3(X^{(3)})$, where $\alpha \neq \beta \in \pi_1$, and $a, b \in A$. However, the intersection number of two classes in $\pi_2(X)$ is determined by the value of the attaching map of the 4-cell on their Whitehead product in $X^{(3)}$. In particular, in the situation above the intersection $(\alpha a) \cdot (\beta b)$ is in \mathbb{Z} , contradicting the assumption on the intersection pairing on X , and the assumption $\alpha \neq \beta$. \square

Proof of theorem 1. Following the proof on the higher-dimensional surgery theorem [17] (see also [5, Chapter 11]), we may assume that f induces an isomorphism on π_1 , and the kernel of $\pi_2(f)$ is a direct sum of standard planes. Since the intersection pairing on X is induced from the integers, the same is true for the intersection form on M . Consider a map $g: X \longrightarrow K$ given by lemma 3, and arrange the composition $h = gf: M \longrightarrow K$ to be transverse to a collection of points p_1, \dots, p_n , different from the basepoint, one in each circle summand of K . (See [16] or [5, §9.6] for the statement of transversality in the topological category.) Denote the 3-manifold $h^{-1}(p_i)$ by P_i , and set $P = \amalg P_i$. Changing the map h by a homotopy if necessary, we may assume that P_i is connected, for each i .

For $i = 1, \dots, n$, consider a framed link $L_i \subset P_i$ such that the surgery on P_i along L_i gives the 3-sphere (cf [10]), and let $L = \amalg L_i$. If the components of L bounded disjoint embedded disks with interiors in $M \setminus P$ and with appropriate framings, then P could be ambiently surgered to get a collection of disjoint 3-spheres, geometrically dual to the generators of $\pi_1(M)$. Since this cannot be expected in general, we perform surgery along the link L on the 4-manifold M , and denote the result by N . Here

for the surgeries on M we use the framing of L , determined by the framing of L in P . The components of L bound disjoint embedded disks with the required framings in N , thus the map h is bordant to $h': N \rightarrow K$, with $(h')^{-1}(p_i) = S^3$.

Since L is null-homotopic in M , its components bound disjoint embedded disks in M . This implies that N is homeomorphic to M , connected summed with several copies of $S^2 \times S^2$, and also with copies of the twisted bundle $S^2 \tilde{\times} S^2$. The framed link L may be chosen so that $P \times I \cup_L 2$ -handles is spin [10], so if $\omega_2(M) = 0$ then $\omega_2(N)$ is also trivial. If $\omega_2(M) \neq 0$, note that $H_2(M; \mathbb{Z})$ consists of spherical classes, thus $M \sharp S^2 \tilde{\times} S^2 \cong M \sharp S^2 \times S^2$. In either case, we may assume $N \cong M \sharp^k (S^2 \times S^2)$, and we have a normal bordism from the map $f: M \rightarrow X$ to a map $N \rightarrow X$.

At this point we caution that without a restriction on the intersection pairing on M , one could still consider a map $M \rightarrow \vee^n S^1$, classifying π_1 , and construct N as above. (A similar construction is used in [9] in a proof of the stable 4-dimensional Kneser's conjecture.) Now the intersection form on N (the form on M , stabilized by adding several hyperbolic pairs), is extended from the integers, since N contains 3-spheres, geometrically dual to the generators of π_1 . Certainly, in this general case one cannot hope to de-stabilize N while preserving the 3-spheres.

Returning to the proof, denote $N_0 = \text{closure}(N \setminus (\mathbb{I}S^3 \times I))$. Then $\pi_2(N) \cong \pi_2(N_0) \otimes_{\mathbb{Z}} \mathbb{Z}\pi_1$, and moreover

$$\ker[\pi_2(N) \rightarrow \pi_2(K)] \cong \ker[\pi_2(N_0) \rightarrow \pi_2(K_0)] \otimes_{\mathbb{Z}} \mathbb{Z}\pi_1.$$

Consider a standard hyperbolic basis for $\ker[\pi_2(N) \rightarrow \pi_2(K)]$, say $\{a_i, b_i\}$. Arrange these 2-spheres to be transverse to $\mathbb{I}S^3$, and let the circles of intersection bound maps of disks in the 3-spheres. Consider two copies of each disk, lying in $S^3 \times \{-\epsilon\}$ and $S^3 \times \{\epsilon\}$ respectively, and use them to surger the spheres a_i, b_i . In other words, we cut out an annulus out of each 2-sphere, and glue in the disks described above. The resulting 2-spheres lie in the complement of $\mathbb{I}S^3$, and we connect them to the basepoint by arcs in N_0 . The constructed classes $\alpha_i, \beta_i \in \pi_2(N_0)$ are homologous, but not necessarily homotopic to a_i, b_i . For example, suppose b_i intersects S^3 in a circle. Cutting b_i as above, we get two spheres b'_i and b''_i with $b_i = b'_i + b''_i \in \pi_2(N)$, while $\beta_i = b'_i + g b''_i$. Here g is the generator of $\pi_1(N)$ dual to the given S^3 .

Since $\{\alpha_i, \beta_i\}$ are homologous to the original hyperbolic basis, these classes freely generate

$$\ker[H_2(N_0) \rightarrow H_2(K_0)] \cong \ker[\pi_2(N_0) \rightarrow \pi_2(K_0)],$$

hence they also freely generate

$$\ker[\pi_2(N) \rightarrow \pi_2(K)] = \ker[\pi_2(N) \rightarrow \pi_2(X)]$$

as a module over π_1 . Moreover, $\{\alpha_i, \beta_j\}$ is a collection of (algebraically) transverse pairs of spheres in N_0 , and the disk embedding theorem in the simply-connected

setting [3], [5, §5.1] gives a collection of embedded transverse pairs, homotopic to $\{\alpha_i, \beta_i\}$. Surgering them out yields a homotopy equivalence $f': M' \rightarrow X$. \square

Proof of theorem 2. Since M is homotopy equivalent to a Poincaré complex (cf [11, Chapter 3]), there is a map $f: M \rightarrow K$, satisfying the conclusions of lemma 3. As in the proof of theorem 1, arrange it to be transverse to a collection of points p_1, \dots, p_n , one in each circle summand of K , and denote $P_i = f^{-1}(p_i)$, $P = \amalg^n P_i$. We may assume that P_i is connected, for each i .

For $i = 1, \dots, n$, consider a framed link $L_i \subset P_i$ such that the surgery on P_i along L_i gives the 3-sphere, and let $L = \amalg L_i$. We denote the surgery on M^4 along the link L with the corresponding framings by N . Define

$$W_1 = M \times [0, 1] \cup 2\text{-handles}; \quad \partial W_1 = M \amalg N.$$

As in the proof of theorem 1, the link L may be chosen so that $N \cong M \sharp^k S^2 \times S^2$. The intersection form over $\mathbb{Z}\pi_1$ on N is the form on M , plus k standard planes. We fix notation, $\{a_i, b_i\}$, for a hyperbolic basis of $\ker[\pi_2(N) \rightarrow \pi_2(K)]$, where $\{b_i\}$ correspond to the belt spheres of the 2-handles of W_1 . Let A, B be the $\mathbb{Z}\pi_1$ -submodules of $\pi_2(N)$, (freely) generated by the $\{a_i\}$ and $\{b_i\}$ respectively, then

$$\ker[\pi_2(N) \rightarrow \pi_2(K)] \cong A \oplus B.$$

Note that the homomorphism $\phi: A \rightarrow B$, induced by the intersection pairing: $\phi(a) = \sum_j (a \cdot b_j) b_j$, is an isomorphism: $\phi(a_i) = b_i$ for each i .

The components of L bound in N disjoint embedded framed disks, provided by the 4-dimensional surgeries. Use these disks to ambiently surger each $P_i = f^{-1}(p_i)$ in N into the 3-sphere. Denote $N_0 = \text{closure}(N \setminus (\amalg S^3 \times I))$. As in the proof of theorem 1, consider 2-spheres $\alpha_i, \beta_i \subset N_0$, homologous (but not necessarily homotopic) to a_i, b_i . These are obtained by arranging the intersections $a_i, b_i \cap \amalg S^3$ to be transverse; each circle of intersection bounds a map of a disk in S^3 , and we surger a_i and b_i along these disks. Finally, connect the resulting spheres to the basepoint by arcs in N_0 to get α_i, β_i .

Proposition 4. *The classes $\{\alpha_i\}$ freely generate the $\mathbb{Z}\pi_1$ -module A .*

Proof. The map $f: M \rightarrow K$ extends to a map $g: N \rightarrow K \vee^k (S^2 \times S^2)$, inducing isomorphisms on π_1 and π_2 , and so that $\pi_2(g)$ maps a_i, b_i to the generators \bar{a}_i, \bar{b}_i of the corresponding $\pi_2(S^2 \times S^2)$. Here k is the number of components of the link L . Set

$$K'_0 = K_0 \vee^k (S^2 \times S^2) \cup 3\text{-cells}, \quad K' = \vee^n S^1 \vee K'_0$$

where the 3-cells are attached to $\{\bar{a}_i\}$, and note that $A = \ker[\pi_2(N) \rightarrow \pi_2(K')]$. Since $\{\alpha_i\}$ are homologous to $\{a_i\}$, the collection $\{\alpha_i\}$ freely generates

$$\ker[H_2(N_0) \longrightarrow H_2(K'_0)] \cong \ker[\pi_2(N_0) \longrightarrow \pi_2(K'_0)];$$

thus they are also free generators over $\mathbb{Z}\pi_1$ of

$$\ker[\pi_2(N_0) \longrightarrow \pi_2(K'_0)] \otimes_{\mathbb{Z}} \mathbb{Z}\pi_1 \cong \ker[\pi_2(N) \longrightarrow \pi_2(K')] \cong A. \quad \square$$

The spheres $\{\alpha_i, \beta_j\}$ form a collection of algebraically transverse pairs in N_0 , and the disk embedding theorem in the simply-connected setting [3], [5, §5.1] implies that they are homotopic to embedded transverse pairs α'_i, β'_i . Set

$$W_2 = N \times [0, 1] \cup 3\text{-handles}$$

where the 3-handles are attached to the spheres $\alpha'_i \subset N \times \{1\}$, and let M' be the corresponding surgery on N . Consider the cobordism $W = W_1 \cup_N W_2$ between M and M' . Since there are only 2- and 3-handles, the chain complex for the relative $\mathbb{Z}\pi_1$ homology groups is $0 \longrightarrow C_3 \longrightarrow C_2 \longrightarrow 0$, where C_i is the $\mathbb{Z}\pi_1$ -module, freely generated by the i -handles, cf [15]. The boundary homomorphism is given by the intersection numbers of the attaching spheres of the 3-handles with the belt spheres of the 2-handles. Using proposition 4, observe that the homomorphism $C_3 \longrightarrow C_2$ is identified with the isomorphism $\phi: A \longrightarrow B$ considered earlier in the proof. Thus W is an h -cobordism, and since $\pi_1(M)$ is free, its Whitehead group is trivial, and so W is an s -cobordism as asserted in the theorem. \square

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