ISSN 1472-2739 (on-line) 1472-2747 (printed)

Algebraic & Geometric Topology
Volume 5 (2005) 1719–1732
Published: 17 December 2005

Surgery and involutions on 4-manifolds

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Abstract We prove that the canonical 4-dimensional surgery problems can be solved after passing to a double cover. This contrasts the long-standing conjecture about the validity of the topological surgery theorem for arbitrary fundamental groups (without passing to a cover). As a corollary, the surgery conjecture is reformulated in terms of the existence of free involutions on a certain class of 4-manifolds. We consider this question and analyze its relation to the A, B-slice problem.

AMS Classification 57N13; 57M10, 57M60

Keywords 4-manifolds, surgery, involutions

1 Introduction

The geometric classification techniques — surgery and the s-cobordism theorem — are known to hold in the topological category in dimension 4 for a class of fundamental groups which includes the groups of subexponential growth [2], [7], [10], and are conjectured to fail in general [3]. The unrestricted surgery theorem is known to be equivalent to the existence of a certain family of canonical 4-manifolds with free fundamental group. Recall the precise conjecture concerning these canonical surgery problems [3]:

Conjecture 1.1 *The untwisted Whitehead double of the Borromean Rings,* Wh(Bor)*, is not a freely topologically slice link.*

In this statement the additional "free" requirement is that the complement of the slices in the 4-ball has free fundamental group generated by the meridians to the link components. (The slicing problem is open without this extra condition as well.) Considering the slice complement, the conjecture is seen to be equivalent to the statement that there does not exist a topological 4-manifold M, homotopy equivalent to $\vee^3 S^1$, and whose boundary is homeomorphic to the zero-framed surgery on the Whitehead double of the Borromean rings: $\partial M \cong \mathcal{S}^0(\text{Wh(Bor)})$. In contrast, here we show that there exists a double cover of this hypothetical manifold M:

Theorem 1.2 *There exists a smooth 4-manifold N homotopy equivalent to a double cover of $\vee^3 S^1$ with ∂N homeomorphic to the corresponding double cover of $\mathcal{S}^0(\mathrm{Wh}(\mathrm{Bor}))$. The analagous double covers exist for all generalized Borromean rings.*

Here the generalized Borromean rings are a family of links obtained from the Hopf link by iterated ramified Bing doubling. The Borromean rings are the simplest representative of this class of links, and the slicing problem for the Whitehead doubles of such links provides a set of canonical surgery problems.

It is interesting to note that the double covers of the (hypothetical) canonical 4-manifolds, constructed in theorem 1.2, are *smooth*. The surgery conjecture is known to fail in the smooth category even in the simply-connected case [1], thus there does not exist a smooth free involution on N — corresponding to at least one of the generalized Borromean rings — extending the obvious involution on the boundary ∂N. In the topological category we have the following new reformulation of the surgery conjecture.

Corollary 1.3 *Suppose the topological 4-dimensional surgery and 5-dimensional s-cobordism conjectures hold for free groups. Then there exists a free involution on each manifold N constructed in theorem 1.2, extending the given involution on the boundary. Conversely, suppose such involutions exist. Then the surgery conjecture holds for all fundamental groups.*

To have a statement equivalent just to surgery (independent of the s-cobordism conjecture), one needs to consider involutions on the class of all 4-manifolds with the given boundary and homotopy equivalent to those constructed in theorem 1.2. To prove the corollary, note that if a required involution existed then the quotients provide solutions to the canonical surgery problems. Conversely, the validity of the surgery conjecture implies that a surgery kernel (a direct sum of hyperbolic pairs $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ in $\pi_2 M$) can be represented by embedded spheres is a 4-manifold s-cobordant to M, see [6, 12.3]. The proof of theorem 1.2 (see section 2) involves a construction of a specific manifold M_{Bor} surgery on which would give a canonical manifold as above. Therefore if the surgery conjecture were true there exists a manifold M' s-cobordant to M_{Bor} where the surgery kernel is represented by embedded spheres. Surger them out and consider its double cover N'. If the s-cobordism conjecture holds then N' is homeomorphic to the manifold N constructed in theorem 1.2, so N admits a free involution. □

The proof of theorem 1.2 is given in section 2. Another set of canonical problems is provided by *capped gropes* (they are canonical in the sense that finding an embedded disk in capped gropes is equivalent to both the surgery and the s-cobordism theorems.) Section 2 also contains a proof analogous to theorem 1.2 in this context. In section 3 we start analyzing the approach to the surgery conjecture provided by the corollary 1.3 above. Its relation to the A, B-slice problem is illustrated by showing that an involution cannot have a fundamental domain bounded by certain homologically simple 3-manifolds. The complexity of this problem is in the interplay between the topology of ∂N and the homotopy type of N — we point out how the analogous question is settled when N is closed or has a "simpler" boundary.

Acknowledgements I would like to thank Michael Freedman and Frank Quinn for discussions on the subject. I also thank the referee for the comments on the exposition of the paper.

This research was partially supported by the NSF and by the Institute for Advanced Study.

2 Double cover of the canonical problems

In this section we prove Theorem 1.2. The proof consists of an explicit construction of a 4-manifold with the prescribed boundary, and an observation that the surgery kernel in a double cover is represented by embedded spheres. Surgering them out gives a 4-manifold with the required homotopy type. We also present an argument for a different set of canonical disk embedding problems: capped gropes.

Proof of Theorem 1.2 Given a link $L = (l_1, \dots, l_n)$ in S^3, let $L' = (l'_1, \dots, l'_n)$ denote its parallel copy, where the components are pushed off with trivial linking numbers. Consider the 4-manifold M_L obtained by attaching zero framed 2-handles to B^4 along the $(2n)$-component link $L \cup L'$, and introducing a plumbing between the handles attached to l_i and l'_i, for each i. The fundamental group of M_L is the free group F_n, freely generated by n loops, each passing exactly once through a plumbing point. Suppose all linking numbers of L vanish, then the intersection form on $\pi_2(M_L)$ is a direct sum of hyperbolic planes $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

The boundary of M_L is diffeomorphic to the zero-framed surgery on S^3 along the untwisted Whitehead double of L, cf [4], [7]. For convenience of the reader and since the manifold M_L (where $L =$ the Borromean rings) plays an important role in the proof of theorem 1.2, we provide a proof based on Kirby calculus [7].

Lemma 2.1 *$\partial M_L \cong \mathcal{S}^0(\mathrm{Wh}(L))$, with the isomorphism carrying the meridians of the 1-handles to the meridians to $\mathrm{Wh}(L)$.*

There is a $\pm$ ambiguity for the clasp of each component (the sign of the Whitehead doubling corresponds to the sign of the plumbing), and since it is irrevelevant for our discussion, we consider any choice of the sign for each component, so $\mathrm{Wh}(L)$ denotes any of the 2^n resulting links. Recall a well-known fact in Kirby calculus:

Proposition 2.2 *The effect of introducing a $\pm$ plumbing on the underlying Kirby handle diagram of a handlebody is to introduce a new 1-handle and a $\pm$ clasp of the attaching curves of the 2-handles being plumbed over the 1-handle (Figure 1).*

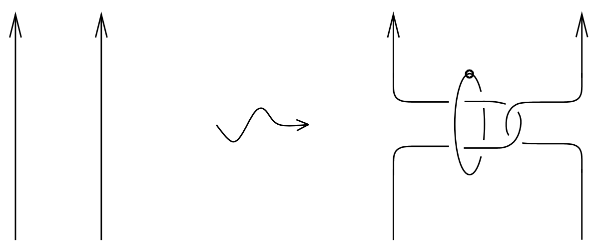

Figure 1: Kirby diagram of a (positive) plumbing

Proof of lemma 2.1 This is a calculation in Kirby calculus in a solid torus, see Figure 2. The solid torus is the complement of the dotted circle in S^3. Figure 2 displays the positive clasp, of course the negative case is treated analogously. The first and third arrows are isotopies of $\mathbb{R}^3$. The second arrow involves replacing a zero-framed 2-handle with a 1-handle (a diffeomorphism of the boundary) and then cancelling a 1- and 2-handle pair. Similarly, the last arrow is a diffeomorphism of the boundary of the 4-manifold. □

The construction of M_L is used in [7] to show that the untwisted Whitehead doubles of a certain subclass of homotopically trivial links are slice. This is done by representing a hyperbolic basis of $\pi_2(M_L)$ by π_1-null transverse pairs of spheres, which by [6, Chapter 6] are s-cobordant to embedded pairs. We show that for the generalized Borromean Rings, one can find *smoothly embedded* transverse pairs in a double cover. (Of course these links are homotopically *essential* and this is the central open case in the surgery conjecture.)

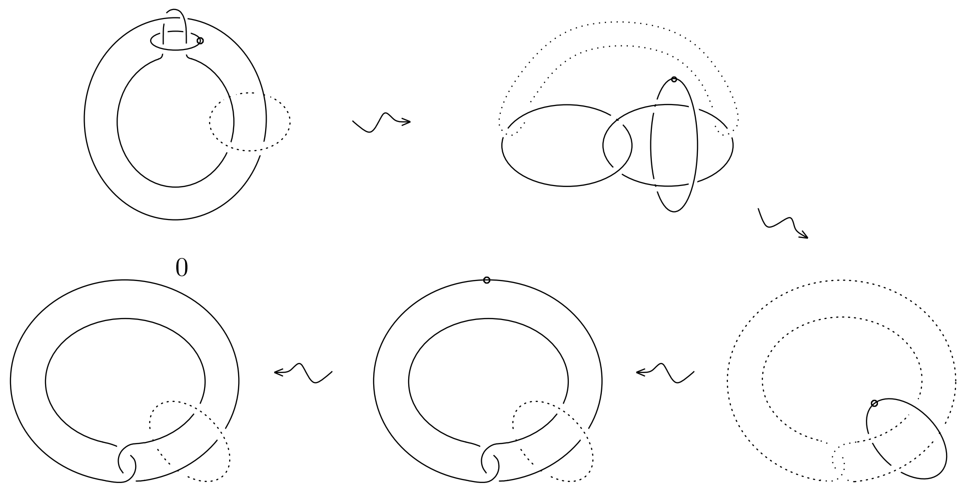

Figure 2: Proof of lemma 2.1

Consider the Borromean Rings $\mathrm{Bor} = (l_1, l_2, l_3)$, and the corresponding manifold M_{Bor}, introduced above. Let $p\colon \widetilde{M}_{\mathrm{Bor}} \longrightarrow M_{\mathrm{Bor}}$ be the double cover induced by the homomorphism $\pi_1(M_{\mathrm{Bor}}) \longrightarrow \mathbb{Z}/2$, which sends each preferred free generator to the non-trivial element. More specifically, the $4-$ball and the $2-$handles of M_{Bor} each have two lifts in the double cover. Denote the two $4-$balls by B^4, $\overline{B}^4$, and consider the lifts Bor, $\overline{\mathrm{Bor}}$ of the Borromean rings and their parallel copies: l_i, l_i' in ∂B^4; $\bar{l}_i, \bar{l}_i'$ in $\overline{B}^4$, $i = 1, 2, 3$, see Figure 3. The cover $\widetilde{M}_{\mathrm{Bor}}$ is obtained from

$$(B^4 \cup_{\mathrm{Bor},\mathrm{Bor}'} 2-\text{handles}) \amalg (\overline{B}^4 \cup_{\overline{\mathrm{Bor}},\overline{\mathrm{Bor}'}} 2-\text{handles})$$

by introducing a plumbing of the handles attached to $l_i, \bar{l}_i'$ and also of the handles attached to $l_i', \bar{l}_i$, for each i. There is an obvious involution τ on $\widetilde{M}_{\mathrm{Bor}}$, with $\widetilde{M}_{\mathrm{Bor}}/\tau \cong M_{\mathrm{Bor}}$, and comparing with Figure 4, one observes that this is the double cover corresponding to the required homomorphism $\pi_1(M_{\mathrm{Bor}}) \longrightarrow \mathbb{Z}/2$.

Observe that $\pi_2(\widetilde{M}_{\mathrm{Bor}})$ consists of six hyperbolic planes, and we represent their bases by 2-spheres as shown in Figure 3. The special intersection point in each hyperbolic pair is a lift in the cover of the plumbing point of the cores of the 2-handles attached to B^4. These cores are capped off with disks (drawn dotted in the figure) bounded by the Borromean rings in the two lifts of the 4-ball, and the extra intersection points between the spheres are the intersections between

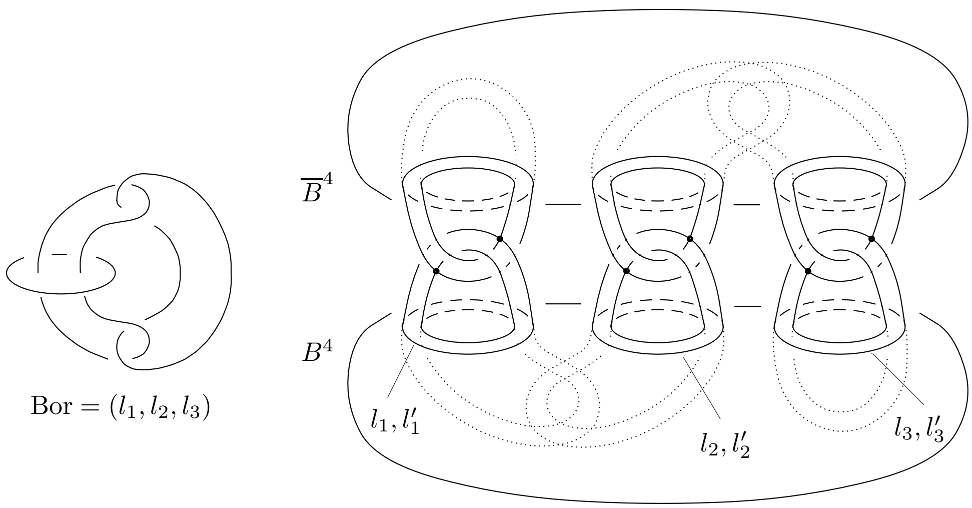

Figure 3: The Borromean rings, and the double cover $\overline{M}_{\text{Bor}}$

the disks. The key point of the argument is the choice of the disks bounded by the Borromean rings. Two different ways of unlinking them are used in the two lifts of the 4−ball: in one of them, B^4, the components l_2, l_2' intersect l_1, l_1'. In the other one, $\overline{B}^4$, $\bar{l}_2, \bar{l}_2'$ intersect $\bar{l}_3, \bar{l}_3'$. To be specific, we introduce a notation for the spheres formed by the cores of the 2−handles capped off with the disks: S_i, S_i', $i = 1, 2, 3$ are the spheres in $B^4 \cup 2-$handles attached to $\text{Bor} \cup \text{Bor}'$. Here the index reflects the component of the link giving rise to the sphere. The spheres in the other lift, $\overline{B}^4 \cup 2-$handles attached along $\overline{\text{Bor}} \cup \overline{\text{Bor}'}$, are denoted by $\overline{S}_i$, $\overline{S}_i'$, $i = 1, 2, 3$. The six hyperbolic planes in $\pi_2(\widetilde{M}_{\text{Bor}})$ are formed by the pairs of 2−spheres $(S_i, \overline{S}_i')$ and $(\overline{S}_i, S_i')$, $i = 1, 2, 3$. The spheres in each pair intersect in precisely one point: the plumbing point of the corresponding 2−handles.

Note that none of the spheres have self-intersections (this is true even without taking a cover, in M_{Bor}). Moreover, $\overline{S}_1, \overline{S}_1', S_3, S_3'$ are embedded disjointly from other spheres, except for the special intersection (plumbing) points, and they provide embedded transverse spheres (cf [6, 1.9]) for $S_1, S_1', \overline{S}_3, \overline{S}_3'$. Due to the choice of the disks bounded by the link components in B^4, $\overline{B}^4$, each of their intersections involves one of the spheres $S_1, S_1', \overline{S}_3, \overline{S}_3'$. Using the embedded duals, one resolves all extra intersection points among the spheres, getting six smoothly embedded hyperbolic pairs. More specifically (see Figure 3) the in-

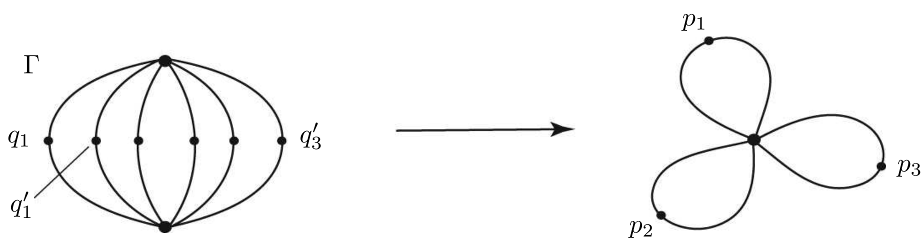

Figure 4: The double cover $\Gamma \longrightarrow S^1 \vee S^1 \vee S^1$, corresponding to the homomorphism $\text{Free}_3 \longrightarrow \mathbb{Z}/2\mathbb{Z}$, sending each "preferred" generator to the non-trivial element.

tersections of S_1, S_1' with S_2, S_2' are resolved by adding parallel copies of $\overline{S}_1, \overline{S}_1'$ to S_2, S_2'. Similarly, the intersections of $\overline{S}_2, \overline{S}_2'$ with $\overline{S}_3, \overline{S}_3'$ are eliminated by adding parallel copies of S_3, S_3' to $\overline{S}_2, \overline{S}_2'$. Surgering out the resulting embedded hyperbolic pairs, one gets a smooth 4-manifold M homotopy equivalent to the double cover Γ of $\vee^3 S^1$.

The double cover in the general case (when the link Bor is further Bing doubled and ramified) is constructed analogously. After additional Bing doubling, the link can still be changed into the unlink by intersecting a single pair of components — there is actually more freedom in choosing the pair of components to intersect. The proof goes through without significant changes also for ramified Bing doubles (when one takes parallel copies of the components before Bing doubling them). Any such link is obtained from the Hopf link $H = (a, b)$ by an iterated application of taking parallel copies and Bing doubling, so that the resulting link has trivial linking numbers. During the first iteration, at least one of the components of H, say a, and all of its parallel copies, are going to be Bing doubled. Label the other component, b, by 1, and the Bing doubles of a and of its parallel copies by 2 and 3. (That is, the two components of each such Bing double are labeled by 2 and 3 respectively — pick any of the two possible labelings. Note that, in particular the resulting link becomes the unlink if one removes all components with the label i, for any given $i \in \{1, 2, 3\}$.) During the following iterations, the parallel copies and Bing doubles inherit the label of the component to which the operation is applied. In this general case, construct the double cover as above. In one lift of the $4-$ball, one only has intersections of the components labeled by 1 and 2. In the other lift, there are only intersections involving the labels 2, 3. The rest of the proof is identical to the above, with individual link components and spheres labeled by i replaced with the collections of all link components and spheres labeled by i, $i = 1, 2, 3$. □

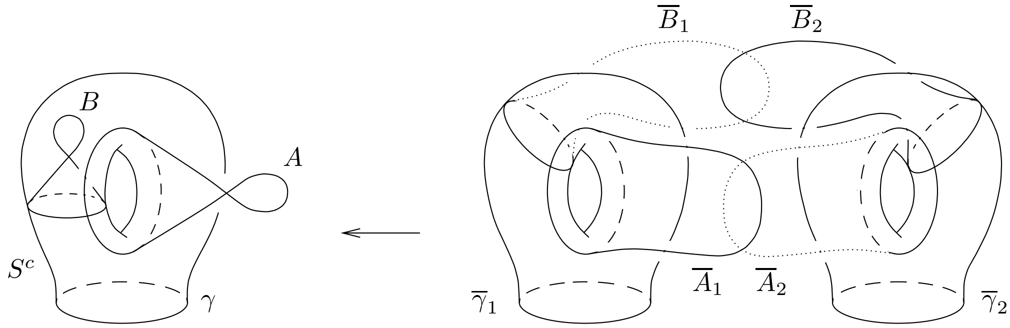

Figure 5: The double cover, corresponding to the homomorhism $\pi_1 S^c \to \mathbb{Z}/2$ which sends both double point loops to the non-trivial element. The lifts $\overline{\gamma}_1$, $\overline{\gamma}_2$ bound (non-equivariant) disjoint embedded disks — surgeries along the caps $\overline{A}_1$ and $\overline{B}_2$ respectively. (The caps which are not used are drawn dotted.)

A different class of canonical problems — for the disk embedding conjecture — is provided by capped gropes, see [6, 2.1]. Recall that the disk embedding conjecture is the lemma underlying the proofs of the surgery and of the s-cobordism theorems for good groups (in fact it is equivalent to both of these theorems for any fundamental group). Capped gropes are thickenings of certain special 2-complexes. They provide a class of canonical problems in the sense that they may be found in the setting of the disk embedding conjecture [6, 5.1] and therefore finding a flat embedded disk in them, with a given boundary, is equivalent to the embedding problem in the general case. Here we present the analogue of Theorem 1.2 in this context.

Consider the simplest case (which however captures the point of the argument): a capped surface of genus one and with just one self-intersection point for each cap. Recall the definition of a capped surface: start with a surface S with one boundary component γ. A capped surface S^c is obtained by attaching disks to a symplectic basis of curves in S. Finally, intersections are introduced among the caps (only self-intersections of the caps, in the current example). The interiors of the caps are disjoint from the base surface S. Abusing the notation, we use S^c also to denote a special "untwisted" 4-dimensional thickening of this 2-complex, see [6]. The capped surface S^c is homotopy equivalent to the wedge of two circles, in particular $\pi_1(S^c)$ is the free group on two generators (a "preferred" set of generators is given by the double point loops). Consider the homomorphism $\pi_1(S^c) \longrightarrow \mathbb{Z}/2$, sending each preferred generator to the non-trivial element, and consider the corresponding double cover $\widetilde{S}^c$. It is an exercise in elementary topology to check that $\widetilde{S}^c$ is given by two copies of the capped surface of genus one, say with caps $\overline{A}_1, \overline{B}_1$ and $\overline{A}_2, \overline{B}_2$ respectively, such that $\overline{A}_1$ intersects $\overline{A}_2$ in two points, and similarly $\overline{B}_1$ intersects $\overline{B}_2$, Figure 5. Note that no cap in the cover has self-intersections, and Figure 5 shows that the two lifts $\overline{\gamma}_1$, $\overline{\gamma}_2$ of the boundary curve γ bound disjoint embedded disks in (the thickening of) $\widetilde{S}^c$.

Consider the general genus one case: there may be self-intersections of each cap, and also intersections between the two caps. A slight variation is necessary in this case: consider the homomorphism $\pi_1 S^c \to \mathbb{Z}/2$ which sends all double point loops, corresponding to self-intersections, to 1, and the double point loops corresponding to the intersections of dual caps, to 0. The same choice of caps as in the case above works here: the caps $\overline{A}_1$, $\overline{B}_2$ are still embedded and disjoint (compare with the notation in Figure 5), since the intersections of the dual caps lift to intersections between $\overline{A}_1$ and $\overline{B}_1$, and also to intersections between $\overline{A}_2$, $\overline{B}_2$.

Now consider the general case, given a capped grope (G^c, γ) of height $n \geq 1$, whose bottom stage surface S has genus g. G^c is obtained from S by attaching capped gropes of height $n-1$ along a symplectic basis of curves $\{\alpha_i, \beta_i\}$, $i = 1 \ldots, g$ in S. Divide these capped gropes of height $n-1$ into two collections, A and B, according to whether they are attached to one of the curves α_i, or one of the β_i, respectively. In particular, all caps of G^c are labeled A or B. $\pi_1 G^c$ is a free groups generated by the double point loops, and generalizing the construction above, consider the homomorphism $\pi_1 G^c \to \mathbb{Z}/2$ which sends all double point loops, corresponding to $A-A$ or $B-B$ intersections, to 1, and the double point loops corresponding to $A-B$ intersections, to 0. The double cover $\widetilde{G}^c$ consists of two capped gropes, and neither of them has any $A-A$ or $B-B$ intersections, since none of the double point loops of this type in G^c lift to the cover. (Here the two lifts of each cap of G^c in the cover inherit the label, A or B, of the cap.) Moreover, no $A-$caps of one of the gropes in the cover intersect $B-$caps of the other one, since each double point loop of type $A-B$ in G^c lifts to a closed loop in the cover. Now the two lifts γ_1, γ_2 of γ bound disjoint embedded disks: surgery along the $A-$caps in one grope, and surgery along the $B-$caps in the other one. Thus we have proved:

Lemma 2.3 *Let (G^c, γ) be a capped grope of height ≥ 1, with the attaching curve γ. Then there exists a double cover $\overline{G}^c \longrightarrow G^c$ such that both lifts $\overline{\gamma}_1$, $\overline{\gamma}_2$ bound disjoint smooth disks in $\overline{G}^c$.*

Instead of using the lifts of different caps to surger the gropes in the cover, one could also use all caps, together with the operation of contraction/pushoff ([6], Chapter 2.3). For example, in the cover shown in Figure 5 the capped surfaces can be contracted, and then the $\overline{A}_1$ — $\overline{A}_2$ intersections are pushed off one contracted surface, while the $\overline{B}_1$ — $\overline{B}_2$ intersections are pushed off the other one, to get disjoint embedded disks.

We note that the idea used here is different from the usual strategy for the proof of the disk embedding theorem. Rather than trying to improve the intersections between all caps, we pick certain "good" caps, sufficient for surgering the surface into a disk, and discard the rest of the caps.

3 Involutions and fundamental domains.

In this section we start the analysis of the existence of free involutions on the family of 4-manifolds constructed in Theorem 1.2. This provides an approach to solving the canonical surgery problems. Conversely, an obstruction to the existence of such involutions would be an obstruction to surgery or to the s-cobordism theorem for free groups. We will point out, in particular, that the analogous problem in the closed case has a simple solution.

The argument used in our analysis is familiar to the experts in the A, B-slice problem. Stronger results are available in a related context (cf [5], [8]) although a translation to this setting is not immediate. We include this discussion to show a connection of our new reformulation of the surgery conjecture with the previous developments in the field, and to illustrate the flavor of the problem.

Suppose there exists a required free involution on N (we use the notations of theorem 1.2). Then the quotient is a 4-manifold M homotopy equivalent to $\vee^3 S^1$ and with $\partial M \cong \mathcal{S}^0(\text{Wh(Bor)})$. Consider a homotopy equivalence $M \longrightarrow \vee^3 S^1$ and lift it to a homotopy equivalence f of double covers, $f\colon\ N \longrightarrow \Gamma$. Choose $p_1, p_2, p_3 \in \vee^3 S^1$, one point in the interior of each circle, and let $X_i = f^{-1}(p_i)$ (the transversality is provided by [11]). Consider the preimages $q_i, q_i' \in \Gamma$ of p_i, $i = 1, 2, 3$, Figures 4, 6. Let $Y_i = f^{-1}(q_i)$, $Y_i' = f^{-1}(q_i')$. Denote $X = \cup X_i$, $Y = \cup_i Y_i$, $Y' = \cup Y_i'$. Note that due to the $\mathbb{Z}/2$-equivariance of f, Y_i is diffeomorphic to Y_i', for each i. Using the standard surgery arguments, one may assume that each X_i (and Y_i, Y_i') is connected. Denote the two connected components of $N \smallsetminus (Y \cup Y')$ by N_0 and N_1.

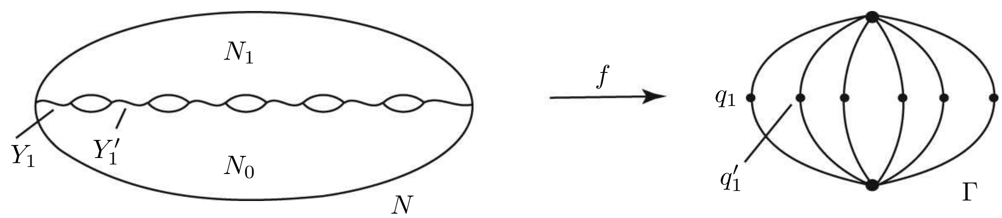

Figure 6

We will consider homotopy equivalences f such that $\partial Y_i = Y_i \cap \partial N \cong \partial Y_i' = Y_i' \cap \partial N$ is a torus, for each i. We need a precise description of these tori in ∂N. The manifold $\mathcal{S}^0(\mathrm{Wh}(\mathrm{Bor}))$ has the following convenient description. Let Bor' denote an untwisted parallel copy of the Borromean rings Bor. Then $\mathcal{S}^0(\mathrm{Wh}(\mathrm{Bor}))$ is obtained from S^3 by cutting out tubular neighborhoods of $\mathrm{Bor} \cup \mathrm{Bor}'$ and identifying the corresponding boundary tori T_i, T_i', exchanging the meridian and the longitude, $i = 1, 2, 3$. (The proof follows from lemma 2.1.) Abusing the notation, we denote the resulting tori in $\mathcal{S}^0(\mathrm{Wh}(\mathrm{Bor}))$ by T_i again. Start with a map $\mathcal{S}^0(\mathrm{Wh}(\mathrm{Bor})) \longrightarrow \vee^3 S^1$ with these given point inverses T_i, and consider its lift $f_\partial \colon \partial N \longrightarrow \Gamma$. We will consider homotopy equivalences f which are extensions of f_∂.

It follows from the above, in particular, that ∂N is obtained from two copies of $S^3 \smallsetminus (\mathrm{Bor} \cup \mathrm{Bor}')$ by identifying the boundary torus of a regular neighborhood of a component l_i of Bor in one copy of S^3 with the boundary torus for the component l_i' in the other copy of S^3, via the diffeomorphism exchanging the meridian and the longitude, for each i.

Definition 3.1 In the general context (not assuming the existence of an involution on N) we say that a homotopy equivalence $f\colon N \longrightarrow \Gamma$ is *weakly equivariant* if the restriction of f to ∂N is equivariant with respect to the obvious $\mathbb{Z}/2$ action, and there is a diffeomorphism from $Y_i = f^{-1}(q_i)$ to $Y_i' = f^{-1}(q_i')$ for $i = 1, 2, 3$, extending the diffeomorhism of their boundary tori $\partial Y_i \longrightarrow \partial Y_i'$, given by the involution on ∂N.

The existence of a weakly equivariant homotopy equivalence is a necessary condition for the existence of an involution on N. (Given an involution, both Y, Y' are diffeomorphic to the point inverses X in the quotient.) Note that this definition does not require that N_0 is homeomorphic to N_1, and it does not impose any equivariance conditions on the inclusions of Y_i, Y_i' into N_0, N_1. However one has that ∂N_0, ∂N_1 are diffeomorphic, and are obtained from S^3 by cutting out a neighborhood of $\mathrm{Bor} \cup \mathrm{Bor}'$ and gluing in Y, Y' where the attaching maps of Y_i, Y_i' differ by the diffeomorphism of the torus exchanging the meridian and the longitude.

Note that in the analogous context: given a homotopy equivalence f from a *closed* 4-manifold N to a graph, the point inverses may be arranged to be 3-spheres [9], up to an s-cobordism of N. (Changing N by an s-cobordism is fine for the applications to the surgery conjecture.) Similarly, if N has boundary but $\partial Y_i, \partial Y_i'$ are $2-$*spheres* then there exists a homotopy equivalence f such that the point inverses are $3-$balls. In the problem we consider here $\partial Y_i, \partial Y_i'$ are *tori*, and we will now show that a similar, naive, guess that there is a homotopy equivalence with $Y_i \cong Y_i' =$ solid torus is not realized. (The complexity of this problem is precisely in the interrelation between the homotopy type and the boundary of the manifolds M, N.)

Lemma 3.1 *There does not exist a weakly equivariant homotopy equivalence $f\colon N \longrightarrow \Gamma$ such that $Y_i = f^{-1}(q_i)$ is an integer homology $S^1 \times D^2$ for each i.*

Note the similarity with the *A-B slice problem* introduced in [4], see also [5], [8]. Assuming the existence of M as above, it is shown in [4] that the compactification of the universal cover $\widetilde{M}$ is the 4-ball. The group of covering transformations (the free group on three generators) acts on D^4 with a prescribed action on the boundary. Roughly speaking, this approach to finding an obstruction to surgery is in eliminating the possibilities for the fundamental domains for such actions. Our present approach is in terms of the fundamental domains in the double cover of M, and in terms of the closely related analysis of the point inverses. Lemma 3.1 eliminates just the most basic possibility for the point inverses Y, Y'. Note that this is not solely a program for finding an *obstruction* to surgery: conversely, solving the problem in the affirmative would construct an involution on N.

Proof First assume that Y_i, Y_i' are diffeomorphic to $S^1 \times D^2$ for each i, and consider the Mayer-Vietoris sequence for the decomposition $N = N_0 \cup_{Y,Y'} N_1$. The sequence splits, and we have

$$0 \longrightarrow H_2(Y \cup Y') \longrightarrow H_2(N_0) \oplus H_2(N_1) \longrightarrow 0,$$

$$0 \longrightarrow H_1(Y \cup Y') \longrightarrow H_1(N_0) \oplus H_1(N_1) \longrightarrow 0$$

(all homology groups are considered with the integer coefficients). It follows that $H_2(N_0) = H_2(N_1) = 0$. Since $H_1(Y \cup Y') \cong \mathbb{Z}^6$, the rank of one of the groups $H_1(N_0)$, $H_1(N_1)$ is ≥ 3 — suppose this condition holds for N_0. It also follows from the sequence above that the homomorphism $H_1(\partial N_0) \longrightarrow H_1(N_0)$ is onto.

On the other hand, ∂N_0 is obtained from $S^3 \smallsetminus (\mathrm{Bor} \cup \mathrm{Bor}')$ by gluing in $Y \cup Y'$, where the attaching map of ∂Y_i differs from the attaching map of $\partial Y_i'$ by the diffeomorphism of the torus exchanging the meridian and the longitude. Let (p_i, q_i) be the slope of the curve in T_i which bounds in the solid torus Y_i. The coordinates are given by (meridian, longitude) in T_i which is considered as the boundary of a tubular neighborhood of a component of Bor in S^3. Then (q_i, p_i) is the slope in T_i' which bounds in Y_i'. That is, ∂N_0 is the Dehn surgery on S^3 along $\mathrm{Bor} \cup \mathrm{Bor}'$, with the surgery coefficients (p_i, q_i) for the component l_i of Bor and (q_i, p_i) for the component l_i' of Bor', $i = 1, 2, 3$.

It follows from this description that, considering various possibilities for the pairs (p_i, q_i), the maximal possible rank of $H_1(\partial N_0)$ is 3. Combining this with the facts that $rk(H_1(N_0)) \geq 3$ and the map $H_1(\partial N_0) \longrightarrow H_1(N_0)$ is onto, we conclude that $rk(H_1(\partial N_0)) = rk(H_1(N_0)) = 3$. Moreover, if any of the pairs (p_i, q_i) is not equal to $(1, 0)$ or $(0, 1)$ then the rank of $H_1(\partial N_0)$ is less than 3. Therefore assume that each pair (p_i, q_i) is equal to either $(1, 0)$ or $(0, 1)$. In each of these cases $\partial N_0 = \mathcal{S}^0\mathrm{Bor}$, the zero-framed surgery on the Borromean rings, and $\pi_1(\partial N_0)$ is abelian.

Since $H_1(\partial N_0) \longrightarrow H_1(N_0)$ is an isomorphism and $H_2(\partial N_0) \longrightarrow H_2(N_0)$ is onto, by Stallings theorem [12] the inclusion $\partial N_0 \hookrightarrow N_0$ induces an isomorphism on π_1/π_1^k for all k. Here π^k denotes the kth term of the lower central series of a group π. Another application of Stallings theorem, to an inclusion $\vee^3 S^1 \hookrightarrow N_0$, implies that $\pi_1(N_0)/\pi_1(N_0)^k$ is isomorphic to F_3/F_3^k for all k, where F_3 is the free group on three generators. This is a contradiction since $\pi_1(\partial N_0)$ is abelian.

In the the general case, Y_i is an integer homology $S^1 \times D^2$, for each i. Consider the degree one maps $Y_i \longrightarrow S^1 \times D^2, Y_i' \longrightarrow S^1 \times D^2$, which are diffeomorphisms of the boundaries. Gluing these maps with the identity on $S^3 \smallsetminus (\mathrm{Bor} \cup \mathrm{Bor}')$, one has a map $\partial N_0 \longrightarrow \mathcal{S}^0\mathrm{Bor}$ inducing an isomorphism on homology. By Stallings theorem, this map induces an isomorphism of nilpotent quotients of the fundamental groups, in particular $\pi_1(\partial N_0)$ is abelian. The Mayer-Vietoris calculation for the decomposition $N \smallsetminus (Y \cup Y') = N_0 \cup N_1$ is still valid, showing that $\pi_1(N_0)/\pi_1(N_0)^k \cong F_3/F_3^k$ for all k and giving a contradiction as above. □

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Received: 17 May 2005