SURFACES IN 4-MANIFOLDS AND THE SURGERY CONJECTURE

VYACHESLAV S. KRUSHKAL

Abstract. We give a survey of geometric approaches to the topological 4-dimensional surgery and 5-dimensional s-cobordism conjectures, with a focus on the study of surfaces in 4-manifolds. The geometric lemma underlying these conjectures is a statement about smooth immersions of disks and of certain 2-complexes, capped gropes, in a 4-manifold. We also mention a reformulation in terms of the $A-B$ slice problem, and the relation of this question to recent developments in the study of the classical knot concordance group.

1. Introduction

The development of the classification theory of topological 4-manifolds in the simply-connected case [4],[8] relied on the surgery program and parallelled the situation in higher dimensions. This analogy with the manifolds of dimension greater than 4 was provided by Freedman's disk embedding theorem [4] allowing one to represent hyperbolic pairs in $\pi_2 M^4$ by embedded spheres and therefore to complete surgery. Surgery and another ingredient of the classification theory – the s-cobordism conjecture – are reduced to closely related statements about immersions of disks. (The embedding problem related to surgery allows certain flexibility in the ambient manifold, so it is conceivable that the surgery conjecture holds while the s-cobordism conjecture has an obstruction.) More recent progress in the subject (cf [9], [13]), extending the class of fundamental groups for which such techniques hold, involved the analysis of maps of certain 2-complexes, *capped gropes*, into 4-manifolds. Such methods so far have fallen short of proving the conjectures in general – the case of free groups is the key question. The study of surfaces in 4-manifolds has been central to the subject, and the purpose of this survey is to mention open questions and available techniques. In particular, it is worth pointing out that the main open question may be formulated in terms of the existence of a *smooth* immersion, satisfying a certain condition on π_1, of a disk into a specific smooth 4-manifold. We also mention another attractive reformulation: the $A-B$ slice problem which concerns smooth decompositions of the 4-ball.

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Capped gropes, certain special 2-complexes embedded in 4-manifolds, turned out to be a useful geometric tool for studying homotopies of surfaces [8], [9], [13]. One can easily find a capped grope in the context of the disk embedding theorem. If one finds a π_1-*null* caped grope then the problem is solved using the foundational theorem [4] that a Casson handle is homeomorphic to the standard 2-handle. We summarize the tools available for manipulating capped gropes in section 3. On the one hand, these techniques allow one to encode the geometric problem in terms of algebra of trees in $\pi_1 M$ (and lead to a solution when the fundamental group has subexponential growth.) On the other hand, this does not immediately lead to an obstruction in the main open case (free fundamental group) since the moves on capped gropes, translating the problem into algebra, capture only a limited class of homotopies of surfaces.

Gropes have recently emerged also as an important tool in a different context: the study of the classical knot concordance group [2]. In both applications gropes provide a filtration from bounding a surface (algebraically: being trivial in homology) to bounding a disk (being trivial in π_1). However they appear in different guises (capped vs uncapped) and therefore the techniques for studying them are quite different. Moreover they serve distinct goals: in knot theory obstructions measure whether a knot bounds an embedded grope of a given height. In the surgery context, one can find a grope of any given height and the problem is to find a disk. Therefore the question relevant for classification theory of 4-manifolds is whether the intersection of the grope filtration on links coincides with the class of slice links.

Outline

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4. Main open problem: canonical examples, free fundamental group.
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7. The $A - B$ slice problem and invariants of decompositions of D^4.

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2. Formulation of the problem

We start by stating the surgery and s-cobordism conjectures. Our main focus will be on the geometry of surfaces underlying these problems.

Surgery conjecture. *Let* $f\colon (M,\partial M) \longrightarrow (X,\partial X)$ *be a degree one normal map from a topological* 4*-manifold to a* 4*-dimensional Poincaré duality pair, inducing a homotopy equivalence* $\partial M \longrightarrow \partial X$*. Assume that the surgery obstruction* $\theta(f) \in L_4(\pi_1 X)$ *vanishes. Then* f *is normally bordant to a homotopy equivalence* $h\colon (N,\partial N) \longrightarrow (X,\partial X)$*.*

s-cobordism conjecture. *Let* (W^5, M_0, M_1) *be a* 5*-dimensional topological s-cobordism. Then* W *is homeomorphic to the product* $W \cong M_0 \times I$*.*

The proof of both conjectures proceeds as in the higher-dimensional case, until the following problems remain: in the surgery case one has hyperbolic pairs $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in $\pi_2 M$ which need to be represented by embedded spheres. In the s-cobordism case one can cancel all handles except perhaps for some 2− and 3− handles, which are paired up over $\mathbb{Z}\pi_1$ by the s-cobordism assumption. Considering the surgery kernel in the first case, and the cores/cocores of the handles in the middle level of the cobordism in the second case, the geometric data is: an immersion of a collection of $S^2 \vee S^2$'s into a 4-manifold M, with a "distinguished" intersection point for each pair, and with all extra intersection points between the spheres paired up, with the corresponding Whitney loops contractible in M. In summary, a proof of the following lemma would yield both the surgery and the s-cobordism conjectures:

The disk embedding conjecture. *Let* $(A,\alpha) \longrightarrow M^4$ *be an immersion of a union of disks with algebraically transverse spheres whose algebraic intersections and selfintersection numbers are* 0 *in* $\mathbb{Z}[\pi_1 M]$*. Then* α *bounds in* M *disjoint topologically embedded disks with the same framed boundary as* A*, and with transverse spheres.*

The *algebraically transverse spheres* in the statement above are (framed) spheres $\{S_i\}$ in M with $S_i \cdot A_j = \delta_{i,j}$ (over $\mathbb{Z}[\pi_1 M]$). The conjecture is stated for disks; of course to apply it to immersions of a collection of $S^2 \vee S^2$'s above one punctures one sphere in each pair. Each of the statements above depends on the fundamental group of M. The disk embedding conjecture, proved originally in the simply-connected case in [4], is known to hold for a class of groups including the groups of subexponential growth [9],[13]. The way that the fundamental group enters the proof will be clear from the discussion in the following section. Note that to complete the proof of the surgery conjecture, it would suffice to solve the disk embedding problem *up to s-cobordism*. There are techniques which use this extra freedom [8, Chapter 6] but known applications require additional restrictions on the surgery kernel.

3. Capped gropes and the double point loops.

In this section we review the definitions and terminology for capped gropes and their intersections in 4-manifolds. A more detailed exposition may be found in [8] or [13]. The section ends with a discussion of the proof of the disk embedding conjecture in the subexponential growth case and of some related questions.

Definition 3.1. A *grope* (or *symmetric grope*) (g, γ) is an inductively defined pair (2-complex, base circle). A grope has a height $h \in \mathbb{N}$. For $h = 1$ a grope is a compact oriented surface g with a single boundary component γ. A grope g of height $h+1$ is defined inductively: g is obtained from a grope g_h of height h by attaching surfaces (gropes of height one) to the circles in a symplectic basis for all top stage surfaces of g_h.

A *capped grope* (g^c, γ) of height h is obtained from a grope g of height h by attaching 2-cells (*caps*) to the circles in a symplectic basis for all top stage surfaces of g and introducing finitely many double points among the caps. Capped gropes of height 1 are also called capped surfaces, see figure 1.

Geometric (and of course algebraic) properties of gropes are substantially different from those of capped gropes. For a loop γ in a space X bounding a map of a grope of height n is equivalent to being in the n-th term of the derived series of $\pi_1 X$. On the other hand, specifying a map of a capped grope bounding γ means finding special *null-homotopies* of γ in X. Geometrically, if one considers $X =$ grope of height n then γ does not bound a map of a grope of height $n+1$ in X. However given $X =$ capped grope of height 2, γ bounds in X a capped grope of *any* height ≥ 2. There are several such grope height raising techniques available [8], [9], and the important question about them is the complexity of the group elements their double point represent in $\pi_1 X$ as a function of height. This is discussed in more detail below.

Given a capped grope g^c, its *body* g is the union of all surfaces except for the caps. Note that the caps are not allowed to intersect the body of a capped grope, and the body does not have double points. This definition is based on the fact that such 2-complexes can be found in four-manifolds, bounding Whitney circles, as they arise in the proof of the surgery and s-cobordism theorems [8, Theorems 5.1, 7.1, 11.3].

Note that a capped grope (g^c, γ) without double points properly embeds into the upper half space $(\mathbb{R}^3_+,\ \mathbb{R}^2 \times \{0\})$. The "untwisted" 4-dimensional thickening is obtained from a capped grope g^c without double points by first taking its three-dimensional thickening in $\mathbb{R}^3_+$, then crossing with I, and finally introducing finitely many plumbings among the caps. The 2-complex g^c forms a spine of this thickening, and sometimes we will abuse the notation and use g^c to denote both objects. To define such "canonical" thickening for capped gropes with double points, one follows the definition above and then introduces a finite number of plumbings among the thickenings of caps.

Let (S^c, γ) be the untwisted thickening of a capped surface (capped grope of height 1) with the body surface S of genus 1 and with the caps A and B. There are various ways to get a disk on γ in S^c, we mention some of them to fix the terminology for the rest of the paper. Let (α, β) be the symplectic pair of circles in S which serve as the attaching curves for A and B respectively. We say that $S \smallsetminus (\alpha \times I) \cup$ (two parallel copies of A) is the *surgery on S along the cap A* (analogously one has the surgery on S along B.) The intermediate operation – *contraction* (or *symmetric surgery*) – uses

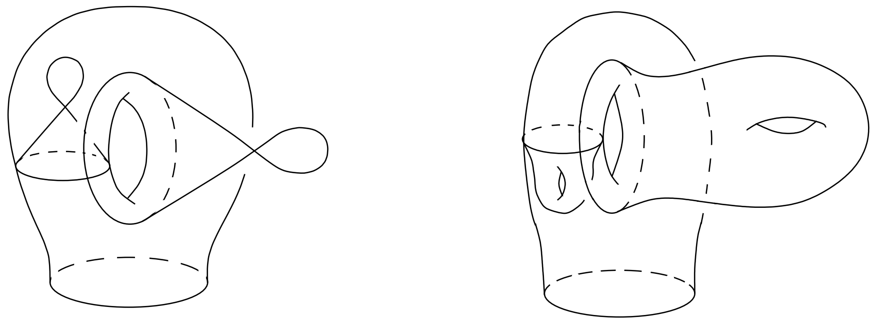

FIGURE 1. A capped surface and a grope of height 2. A *capped* grope of height 2 is obtained from the 2-complex on the right by attaching four caps.

both caps A and B, and is described in detail, together with the associated operation of *pushoff*, in [8, 2.3]. The definitions of surgery and contraction are extended to capped surfaces of higher genus, and more generally to capped gropes with more surface stages.

Let P, Q be surfaces or capped gropes in a 4-manifold M. Choose paths connecting the base point in M to P and Q to assign groups elements to the intersection points of the surfaces. Denote by $\gamma(p) \in \pi_1 M$ the group element associated to an intersection point $p \in P \cap Q$. For self-intersections, one orders the sheets at each intersection point to define the group element. Usually the choice of an ordering is not specified, since it is either not important, or is clear from the context. Given P, an immersed surface or a capped grope in M, to measure the complexity of P in $\pi_1 M$ it is important to fix the generators of $\pi_1 P$. Assume that P is connected. In our applications the source surface (or capped grope) will always be simply connected, and we choose the *double point loops* which pass exactly through one intersection point in the image (one loop for each self-intersection) as the free generators of $\pi_1 P$. Since it is important to know how group elements associated to the intersections change under the basic operations on surfaces, we briefly review these facts.

3.2. Surgery, contraction. Suppose G^c is a capped grope, and a surface P intersects one or both caps in a dual pair of caps A, B of G^c. Contraction of the top stage surface of G to which the caps are attached, or surgery along one of the caps doubles the number of intersections, but the group elements do not change: if p', p'' are the two new intersection points created by surgery or contraction from p, then $\gamma(p') = \gamma(p'') = \gamma(p)$.

3.3. Pushoff. Suppose again that P intersects both caps in a dual pair of caps A, B of G^c, $a \in P \cap A$, $b \in P \cap B$. Pushing P off the contraction of G^c along these two caps eliminates the intersection points a, b and creates two self-intersections p', p'' of P. The new group elements are given by $\gamma(p') = \gamma(p'') = \gamma(a) \cdot \gamma(b)^{-1}$.

3.4. Splitting [13] is an operation on gropes which arranges all surface stages above the first one to have genus 1 and simplifies the pattern of intersections between the caps at the expense of vastly increasing the genus of the bottom stage. This operation increases the number of double points but does not change the group elements represented by them.

3.5. Transverse spheres and gropes. A *transverse sphere* for a surface Σ in M is a framed immersed sphere intersecting Σ in a single point. Similarly a transverse (capped) grope is a (capped) grope whose bottom stage surface intersects Σ in a single point. In a capped grope, the caps and all surfaces, except for the bottom stage, have a standard transverse grope. More specifically, consider a top stage surface S of a grope and a pair of dual caps A, B, attached along the curves α, β as above. Consider the normal circle bundle to S, restricted to a parallel copy of β (denote it by T). This is a transverse torus for A. Note that a parallel copy of B provides a cap for T, and surgering T along this cap gives a transverse sphere for A. This construction can be continued to produce transverse gropes of height greater than 1, cf [13]. If a surface C intersects A, taking the sum of C with the capped grope tranverse to A allows one to translate between the geometry of intersecting surfaces and the algebra of the group elements represented by their double point loops (see below).

The connection to surgery is provided by the following observation. Under the assumptions of the disk embedding conjecture formulated in section 2, the curves α bound in M capped gropes of height 2. The proof involves straightforward manipulations of surfaces [8, 5.1]. Given a capped grope (G, α) of height two, α bounds in it a capped grope of any given height. The subtlety is in the complexity of the double point loops of capped gropes: in the most efficient grope height raising algorithm presently known [9] the length of the double point loops of the grope G_n of height n constructed in the given grope G grows linearly with n (measured in the free group $\pi_1 G$.)

Splitting of a given capped grope (say of height 2) allows one to encode the intersections by trees with edges labelled by group elements. Vertices of these trees correspond to genus one pieces of the first stage of the grope, edges correspond to intersections of caps. For a capped grope of height 2 this is a 4-valent tree. Splitting uniformizes the intersection patterns of caps – in particular all double points of a given cap have the same group element in $\pi_1 M$ associated to them, and this enables one to label the tree edges by well-defined group elements. In fact the result of splitting is stronger, and while locally one sees a tree the global structure of this "intersection graph" is rather complex.

This algebraic structure provides a basis for the proof of the disk embedding theorem for fundamental groups of subexponential growth in [13]: the number of group elements represented by a tree grows exponentially with height, while the subexponential growth of $\pi_1 M$ allows one to find trivial group elements. (Recall that a group G generated by $g_1, \ldots, g_k$ has *subexponential growth* if, given any $\alpha > 0$ its growth function gr satisfies $gr(n) < e^{\alpha n}$ for large n. Here the growth function $gr\colon\ \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$ assigns to each positive integer n the number of elements in G that can be represented as products of at most n generators $g_{i_1} \cdots g_{i_n}$.) The translation from algebra back to geometry of gropes is provided by the moves described above, which yield a capped grope such that all double point loops are trivial in $\pi_1 M$. The proof is concluded by appealing to the theorem [4] that a Casson handle is homeomorhic to the standard 2-handle.

Such methods have so far fallen short of giving a proof in the general case. In particular, taking $M^4 =$ the untwisted thickening of a capped grope provides a "canonical" problem with free fundamental group:

3.6. π_1-null disk conjecture. *Let (G^c, γ) denote the untwisted thickening of a 2-stage capped Grope. Then γ bounds a π_1-null disk in G^c.*

Recall that an immersion $f\colon\ D^2 \longrightarrow M^4$ is π_1-*null* if the inclusion $f(D^2) \hookrightarrow M$ induces the trivial map on π_1. This conjecture implies both the surgery and the s-cobordism theorems, since in both contexts one can find a (thickening of a) capped grope. Note that to complete the proof of the surgery conjecture, one needs to find a π_1-null disk only up to an s-cobordism of the ambient 4-manifold.

We conclude this section with a discussion of the class of "good" groups for which the disk embedding conjecture is known to hold. As mentioned above, it is known to hold for fundamental groups of subexponential growth. Moreover, it is not difficult to show that the class of groups for which it holds is closed under extensions and direct limits. It is an interesting question whether the disk embedding conjecture holds for amenable groups. The class of groups containing all groups of subexponential growth and closed under direct limits and extensions is contained in the class of amenable groups. The question of whether these two classes actually coincide has been open until recently: [1] announced a proof that the inclusion is proper and therefore gave an example of an amenable group for which the disk embedding conjecture is open.

4. Canonical surgery problems, $Wh(Bor)$, and the COT filtration

Let $Wh(Bor)$ denote the untwisted Whitehead double of the Borromean rings. This is a 3-component link – there is a choice of a $\pm$ clasp for each component, but this choice is not important for the problem. The slicing question for $Wh(Bor)$ and a related family of links, with an additional assumption on π_1 of the slice complement (see below), is a set of canonical surgery problems. (An alternative set of "canonical" problems is given by the disk embedding conjecture in 4-dimensional thickenings of capped gropes discussed above.) Therefore the following question is central to the surgery program:

Question [5] *Does there exist a* 4*-manifold* M *homotopy equivalent to* $S^1 \vee S^1 \vee S^1$*, with the boundary homeomorphic to* $\mathcal{S}^0(Wh(Bor))$*?*

The existence of M is equivalent to $Wh(Bor)$ being "freely slice" (slice with the additional requirement that the fundamenetal group of the slice complement in D^4 is freely generated by meridians to the link components). The equivalence is shown by considering $M =$ slice complement.

Cochran, Orr and Teichner [2] defined a filtration of the classical knot concordance group and introduced an obstruction theory to show that the quotients of the consecutive terms of the filtration are non-trivial. A similar result for links has been established by Harvey [10]. There are two closely related (and perhaps equivalent) definitions of a filtration, one of them is in terms of gropes: a knot K is in $\mathcal{F}_n$ if it bounds in D^4 an embedded symmetric grope of height n.

The following proposition shows that $Wh(Bor)$ lies in the intersection $\cap\mathcal{F}_n$ of the filtration, and so is not detected by the aforementioned obstruction theory. Therefore the question, interesting from the point of view of the surgery conjecture, is whether the intersection of the filtration coincides with the class of slice links.

Proposition 4.1. For any n, $Wh(Bor)$ bounds an embedded symmetric grope of height n in D^4.

Proof. This proposition holds for a larger class of links (good boundary links). To be specific we first consider $Wh(Bor)$ and then we give a more general proof for good boundary links.

Each component of $Wh(Bor)$ bounds the obvious genus one Seifert surface $\Sigma_i, i = 1, 2, 3$ – these are disjoint and lie in S^3. A symplectic basis of curves for these surfaces is given by the components l_i of the undoubled link, and a "short" curve which bounds an embedded disk, disjoint from everything else but which is not framed. (The linking number of this curve with a parallel push-off on the surface in S^3 is ± 1.) To correct the framing, replace the latter curve by the $(1, \pm 1)$ curve l'_i on the punctured torus. The components of Bor bound disjoint embedded surfaces in D^4. More specifically, one of the components l_1 bounds a genus one surface S; the other two components l_2, l_3 bound disks D_2, D_3. The genus one surface is easily seen in S^3; we push it in D^4 slightly. The two caps of S intersect the disks D_2, D_3 respectively. Note that l_i, l'_i bound disjoint parallel copies of these respective surfaces (S, S' for $i = 1$, D_i, D'_i for $i = 2, 3$.)

Now assemble the surfaces: $Wh(l_1)$ bounds a 2-stage capped grope G_1 with the base surface Σ_1 (pushed slighly into D^4), second stage surfaces S, S' and four caps two of which intersect D_1 and D'_1, the other two intersect D_2, D'_2. $Wh(l_2)$, $Wh(l_3)$ bound capped surfaces G_2, G_3 with bases Σ_1, Σ_2 (pushed into D^4), D_i, D'_i provide *embedded* caps for them. This collection of capped gropes/surfaces doesn't satisfy one requirement for being a collection of capped gropes *of height* 2: the caps of G_1 intersect D_i, D'_i which should be considered as second stage surfaces for G_2, G_3. To resolve these interestions, note that each second stage disk of Σ_2, Σ_3 has a geometrically transverse embedded sphere. To be specific, consider D_2. To see the transverse sphere, consider the transverse capped torus (described in 3.5) and surger it along a cap provided by a parallel copy of D_2'. Now use these transverse spheres to resolve the intersections between the caps of G_1 and the second stage surfaces of G_2, G_3. The result of this construction consists of: embedded disks bounded by $Wh(l_2)$ and $Wh(l_3)$ and a 2-stage capped grope bounded by $Wh(l_1)$. As mentioned in section 3, any of the grope height raising techniques can be used at this point to find a capped grope of any height in a neighborhood of G_1. Note that the capped gropes we constructed for $Wh(Bor)$ are of course π_1-null in D^4 but this is not sufficient to apply the available embedding techniques (compare with 3.6) to conclude that the link is slice. To be useful, the nullhomotopies for the double point loops of the caps have to be disjoint from the body of the gropes – and finding capped gropes satisfying this requirement is an open problem.

We will now sketch a proof of the statement for all good boundary links. By definition here we consider boundary links $L = \partial\Sigma$, $\Sigma \subset S^3$, such that there is a basis for $H_1(\Sigma, \mathbb{Z})$ in which the Seifert pairing has the form $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Examples of good boundary links, important from the perspective of the surgey conjecture, are given by the untwisted Whitehead doubles of links with trivial linking numbers.

Consider a Seifert surface Σ for L satisfying the good boundary condition, and let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be curves representing a basis as above. Consider immersed disks A_i, B_i bounded in D^4 by the curves α_i, β_i respectively. Pushing the surfaces Σ into D^4 gives a collection of capped surfaces bounded by L in the 4-ball. As in 3.5 each disk A_i has a transverse torus T_i. Consider the sphere S_i given by the surgery on T_i along its cap B_i. Using the assumptions on the Seifert pairing observe that $\{A_i, S_i\}$ form a collection of disks with algebraically transverse spheres, satisfying the assumptions of the disk embedding conjecture. Here the 4-manifold M is the complement of Σ in D^4. Now (as explained in section 3) one finds capped gropes of height 2, and therefore of any given height, bounded by L in D^4.

□

The Borromean rings is the simplest example of a link with trivial linking numbers but which is homotopically *essential* (its components do not bound disjoint maps of disks in D^4.) While the slicing question for $Wh(Bor)$ is a key open problem, it is known [9] that the untwisted Whitehead doubles of (a slightly smaller subclass of) homotopically *trivial* links are slice.

It is worth pointing out that while the existence of a 4-manifold M above ($M \simeq S^1 \vee S^1 \vee S^1$, $\partial M \cong \mathcal{S}^0(Wh(Bor))$) is a long-standing open problem, one can construct [11] a double cover N of this hypothetical manifold M (and of all other canonical manifolds) – in fact N is a smooth manifold. Therefore the surgery conjecture is equivalent to the existence of a free topological involution on a certain class of 4-manifolds. Such involutions don't exist *smoothly* on at least some of these manifolds, since by a result of Donaldson [3] the 4-dimensional surgery conjecture fails in the smooth category.

The existence of the manifold M described above and of a closely related family of 4-manifolds (equivalently: the free-slice problem for a class of links) is equivalent to the surgery conjecture. There are no similar canonical problems presently known specifically for the s-cobordism conjecture. A recent paper [14] develops new techniques in this direction.

5. Decompositions of D^4 and the $A-B$ slice problem

The $A-B$ slice problem, introduced in [6], is a reformulation of the surgery conjecture. Assume the canonical surgery problem, described in section 4, has a solution: that is, there exists a 4-manifold M homotopy equivalent to $S^1 \vee S^1 \vee S^1$, with the boundary homeomorphic to $\mathcal{S}^0(Wh(Bor))$. It is shown in [6] that the compactification of the universal cover $\widetilde{M}$ is the 4-ball. The group of covering transformations (the free group on three generators) acts on D^4 with a prescribed action on the boundary, and roughly speaking the $A-B$ slice problem asks whether such action exists – see [6] for a precise formulation. Here we will state an attractive, less technical formulation, implicitly contained in [7]. I would like to thank Michael Freedman for explaining this approach and for allowing me to include it in this paper.

Let $\mathcal{M}$ denote the set of (smooth) codimension 0 submanifolds M of D^4 with $M \cap \partial D^4 =$ standard $S^1 \times D^2 \subset \partial D^4$. We choose a "distinguished" curve $\gamma \subset \partial M$ whose neighborhood is the solid torus specified above.

Consider invariants of such submanifolds: $I : \mathcal{M} \longrightarrow \{0, 1\}$ satisfying axioms 1-3:

Axiom 1. I is a topological invariant: if (M, γ) is diffeomorphic to (M', γ') then $I(M, \gamma) = I(M', \gamma')$.

Axiom 2. If $(M, \gamma) \subset (M', \gamma')$ and $I(M, \gamma) = 1$ then $I(M', \gamma') = 1$.

For a codimension 0 submanifold A of D^4 let $\partial A = \partial^- A \cup \partial^+ A$ where $\partial^+ A = \partial A \cap \partial D^4$. We say that $D^4 = A \cup B$ is a *decomposition* of D^4 if A, B are codimension zero submanifolds, and $\partial D^4 = \partial^+ A \cup \partial^+ B$ is the standard genus one Heegaard decomposition of S^3 (of course $\partial^- A = \partial^- B$.)

Axiom 3. If (A, B) is a decomposition of D^4 then $I(A) + I(B) = 0$.

Note that I is an invariant of a submanifold and does not depend on an embedding $M \hookrightarrow D^4$. One can require an invariant to be defined on the class of all pairs (4-manifold M, distinguished circle in ∂M) but only submanifolds of D^4 are relevant for the surgery conjecture.

There is an elementary example of such an invariant (in any dimension) given by homology: define $I_h(M, \gamma) = 0$ or 1 depending on whether $\gamma \neq 0$ or $\gamma = 0$ respectively in $H_1(M)$ (with any fixed coefficients). Axioms 1 and 2 are satisfied automatically and axiom 3 follows from Alexander duality.

Are there any other invariants? In particular, the property connecting this question to 4-dimensional topology and which is relevant for the (A, B)-slice problem is summarized in the additional axiom stated below. Given $(M', \gamma'), (M'', \gamma'') \in \mathcal{M}$, define the "double" $D(M', M'') = (S^1 \times D^2 \times I) \cup (M' \cup M'')$ where M', M'' are attached to $S^1 \times D^2 \times \{1\}$ along the distinguished solid tori in their boundaries, so that γ', γ'' form the Bing double of the core of the solid torus. (Note that if both M', M'' embed in D^4 then so does $D(M', M'')$, and therefore $D(M', M'')$ is an element of $\mathcal{M}$.) Let γ denote the core of $S^1 \times D^2 \times \{0\}$.

Axiom 4. Let $(M', \gamma'), (M'', \gamma'') \in \mathcal{M}$ be such that $I(M', \gamma') = I(M'', \gamma'') = 1$. Then $I(D(M', M''), \gamma) = 1$.

The homology candidate above clearly doesn't satisfy this last axiom: consider $(M', \gamma') = (M'', \gamma'') = (D^2 \times D^2, \{0\} \times \partial D^2)$. Then $I_h = 1$ for both M', M''. However $D(M', M'')$ is obtained from the collar $S^1 \times D^2 \times I$ by attaching two 2-handles along the Bing double of the core of the solid torus, so this core is not trivial in homology of $D(M', M'')$ and $I_h(D(M', M'')) = 0$.

An invariant of decompositions satisfying axioms 1-4 would be a very good candidate for an obstruction to surgery: the connection is made by considering fundamental domains of a hypothetical action of the free group on D^4, discussed above. A new approach to constructing an invariant of decompositions is outlined in [12]. That paper defines an invariant of 4-manifolds using the notion of link-homotopy; in a certain sense it is designed to satisfy axiom 4. It would provide an obstruction to surgery if it satisfied "Alexander duality" (axiom 3).

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Department of Mathematics, University of Virginia, Charlottesville, VA 22904

E-mail address: krushkal@virginia.edu