

STICKY CANTOR SETS IN \mathbb{R}^d

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ABSTRACT. A subset of \mathbb{R}^d is called “sticky” if it cannot be isotoped off of itself by a small ambient isotopy. Sticky wild Cantor sets are constructed in \mathbb{R}^d for each $d \geq 4$.

1. INTRODUCTION

Wild Cantor sets in Euclidean spaces have been extensively studied, following the definition of Antoine’s necklace in [1]. Embeddings exhibiting various interesting phenomena have been constructed using decomposition theory, starting with the Bing decomposition of the 3-sphere [2]. The reader may find a general discussion and examples in [5, 8].

A systematic construction of Cantor set embeddings which suitably approximate codimension 2 submanifolds was given in [7]. Accordingly, a notion of “general position” for wild Cantor sets in Euclidean spaces may seem plausible (and has been mentioned in the literature). Indeed, the result of this paper may be thought of as a version of general position for certain wild Cantor sets approximating codimension 2 submanifolds. (It may be interesting to compare this with [3] where wild Cantor sets are constructed in \mathbb{R}^d , $d \geq 3$, which can be “slipped off” every Cantor set.)

This paper concerns the question of whether there exist sticky Cantor sets in \mathbb{R}^d (see [13, Conjecture 1], and also [6, Problem E8]). Here a (wild) Cantor set embedded in \mathbb{R}^d is called *sticky* if it cannot be isotoped off of itself by any sufficiently small ambient isotopy. The question of whether any given Cantor set X can be slipped off *every* Cantor set Y in \mathbb{R}^d has also been asked in [4, Conjecture 1], [3, Conjecture 1.2]. The following theorem states the main result of the paper.

Theorem 1. *For any $d \geq 4$ there exists a sticky Cantor set in \mathbb{R}^d .*

The main ingredients of the proof are the spun Bing decompositions, considered in [9, 10], and the Stallings theorem [12] on the lower central series of groups. The spun Bing decompositions produce wild Cantor sets which approximate codimension 2 submanifolds (more specifically, $S^{d-2} \subset \mathbb{R}^d$), in the sense of [7]. It is an interesting question whether a similar “general position” result holds for the construction in [7].

2. PROOF OF THEOREM 1

Proof. First consider the case $d = 4$. The starting point of the construction is the nested sequence of Bing doubles [2] in the solid torus, figure 1. Spin it to obtain a nested sequence of “Bing doubles of spheres” in $S^2 \times D^2$. (Each stage in this sequence is a collection of $S^2 \times D^2$ ’s.) A well-known theorem [9], [10] (used in particular in the proof of the Double Suspension Theorem in [9]) states that the spun Bing decomposition is shrinkable, so the nested sequence gives rise to a Cantor set in $S^2 \times D^2$. In fact, it is shown in [9] that iterated spun Bing decompositions are shrinkable, yielding a Cantor set in $S^n \times D^2$, for each $n \geq 2$.

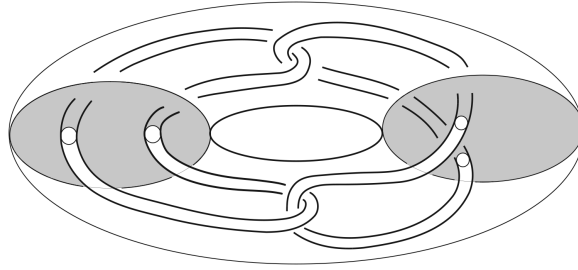


FIGURE 1. Bing double of the core of a solid torus. The spun version is obtained by spinning this figure in \mathbb{R}^4 about the 2-plane which intersects the solid torus in the two indicated shaded disks.

Now consider two standard 2-spheres in \mathbb{R}^4 intersecting in two points, figure 2, and let A, B denote two copies of the Cantor set as above, one in each $S^2 \times D^2$. The proof of theorem 1 relies on the following lemma.

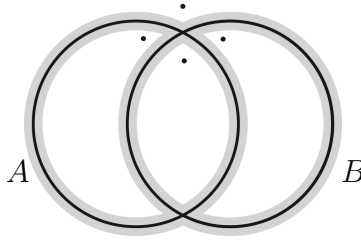


FIGURE 2. Two intersecting spheres in \mathbb{R}^4 and the Clifford torus.

Lemma 2.1. *A cannot be isotoped off of B by any sufficiently small ambient isotopy.*

Proof of lemma 2.1. Consider a Clifford torus for one of the intersection points of the spheres. (The intersection of spheres is locally modeled on $\mathbb{R}^2 \times \{0\} \cup \{0\} \times \mathbb{R}^2 \subset \mathbb{R}^4$. The Clifford torus of radius r is the product $S_r^1 \times S_r^1$ of two circles of radius r in the complement of these two planes in \mathbb{R}^4 . The radius r is chosen so that the Clifford

torus is in the complement of the two $S^2 \times D^2$'s, so it is in the complement of $A \cup B$. The torus is drawn as 4 points ($S^0 \times S^0$) in the 2-dimensional illustration in figure 2.)

Consider the two standard generators of π_1 (Clifford torus): meridians m_A, m_B to the two 2-spheres, in other words the circles $S_r^1 \times *, * \times S_r^1$ linking the 2-spheres. The 2-cell of the torus gives the relation $[m_A, m_B] = 1$.

Suppose an isotopy in the statement of lemma 2 exists. Then some finite stages A_n, B_n of the nested sequences defining A, B are disjoint after the isotopy. The ambient isotopy is assumed to be small enough so that the nested sequences defining A, B stay disjoint from the Clifford torus during the isotopy. To avoid confusion, we will keep the notation A_n, B_n for the n -th stages of the nested sequences *before* the isotopy, and \tilde{A}_n will denote A_n *after* the isotopy. The assumption is $\tilde{A}_n \cap B_n = \emptyset$.

It follows from the previous two paragraphs that

$$(2.1) \quad [m_A, m_B] = 1 \text{ in } \pi_1(\mathbb{R}^4 \setminus (\tilde{A}_n \sqcup B_n)).$$

We will next show that this leads to a contradiction with the Stallings theorem. (The argument below is a generalization of the analogous proof that the two 2-spheres shown in figure 2 (as opposed to their “aproximations” A_n, B_n) cannot be made disjoint by a small isotopy. The reader may consider applying the argument in this case before carrying it out in full generality, discussed below.)

Since $\tilde{A}_n \sqcup B_n$ is a collection of disjointly embedded (thickened) 2-spheres, by Alexander duality $H_1(\mathbb{R}^4 \setminus (\tilde{A}_n \sqcup B_n); \mathbb{Z})$ is generated by meridians (small linking circles) to the 2-spheres, and $H_2(\mathbb{R}^4 \setminus (\tilde{A}_n \sqcup B_n); \mathbb{Z})$ is trivial. Given a group G , its lower central series is defined inductively by $G^1 = G$, $G^2 = [G, G]$, ..., $G^k = [G, G^{k-1}]$.

Theorem 2.2 (Stallings Theorem [12]). *Suppose a map $f: X \rightarrow Y$ induces an isomorphism on $H_1(., \mathbb{Z})$ and a surjection on $H_2(., \mathbb{Z})$. Then for each finite k , f induces an isomorphism $\pi_1(X)/(\pi_1(X))^k \cong \pi_1(Y)/(\pi_1(Y))^k$.*

For brevity of notation denote $Y := \mathbb{R}^4 \setminus (\tilde{A}_n \sqcup B_n)$. Consider the map $f: \vee^N S^1 \rightarrow Y$ from the wedge of circles (one circle for each 2-sphere in the collection $\tilde{A}_n \sqcup B_n$), mapping each circle to a meridian (small linking circle, connected to a basepoint) of the corresponding 2-sphere. This map induces an isomorphism on H_1 and a surjection on $H_2 = 0$. By the Stallings theorem, for any k , $\pi_1(Y)/(\pi_1(Y))^k$ is isomorphic to the corresponding quotient of the free group, $F_N/(F_N)^k$ where $F_N \cong \pi_1(\vee^N S^1)$ is the free group on N generators.

The meridians m_A, m_B may be explicitly written down both in $\pi_1(\mathbb{R}^4 \setminus (A_n \sqcup B_n))$ and in $\pi_1(Y)$. Indeed, each Bing doubling replaces a meridian m to a sphere by a commutator of the meridians m_1, m_2 of the two smaller spheres. This may be read off in the 3-space slice (shown in figure 1), as illustrated in figure 3: m bounds a punctured torus in the complement of the two Bing-doubled curves, with the two

generators of π_1 of the torus corresponding to m_1, m_2 . (This is an important basic example in Milnor's theory of link homotopy [11].)

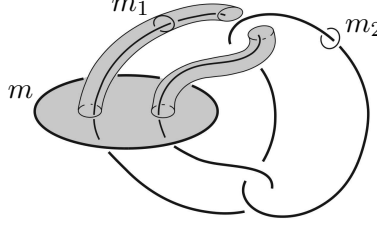


FIGURE 3. $m = [m_1, m_2]$, where m, m_1, m_2 are meridians suitably connected to a basepoint.

The finite stages A_n, B_n of the nested sequences defining A, B correspond to a certain number of iterations of Bing doubling. This exhibits the meridians m_A, m_B as commutators of fixed length l in the meridians to the 2-spheres forming A_n, B_n , where l is the number of 2-spheres in the each of the collections A_n, B_n . (To relate this to the notation following the statement of the Stallings theorem, $N = 2l$.)

This calculation is unchanged by a small isotopy which moves A_n to \tilde{A}_n . Indeed, the expression for m_B , read off in a 3-dimensional slice, is unchanged (see figure 4). The ambient isotopy applied to the meridian m_A moves it to a curve \tilde{m}_A which is contained in a small tubular neighborhood of m_A . The expression for \tilde{m}_A in the complement of \tilde{A}_n is identical to the expression for m_A in the complement of A_n , since they are related by an ambient isotopy. Moreover, since \tilde{m}_A is contained in a small tubular neighborhood of m_A , these two curves are isotopic within this tubular neighborhood. In particular, m_A is isotopic to \tilde{m}_A in the complement of \tilde{A}_n and of B_n . It follows that the expression for m_A as an l -fold commutator of the meridians of A_n is unchanged by the isotopy.

Then $[m_A, m_B]$ is a commutator of length $2l$ in $\pi_1(\mathbb{R}^4 \setminus (\tilde{A}_n \sqcup B_n))$. For the remainder of the argument fix an integer $k > 2l = N$. $[m_A, m_B]$ is seen to be non-trivial in $\pi_1(Y)/(\pi_1(Y))^k \cong F_N/(F_N)^k$. (The underlying reason for this is that the only relations in $F_N/(F_N)^k$ are: {all commutators of length k are trivial}. A shorter commutator $[m_A, m_B]$, of length $2l < k$ is not a consequence of these relations. A rigorous proof uses the Magnus expansion [11].)

This contradiction with (2.1) completes the proof of Lemma 2.1. \square

Returning to the proof of theorem 1, define the Cantor set C to be the union of A, B . Suppose there is an arbitrarily small ambient isotopy, pushing C off of itself. In particular, then A may be isotoped off of B by a small ambient isotopy. This contradiction with Lemma 2 concludes the proof of Theorem 1 in the case $d = 4$.

The construction in higher dimensions d starts from a $(d - 3)$ -spun Bing decomposition. Analogously to the case $d = 4$, take two standard $(d - 2)$ -spheres in general

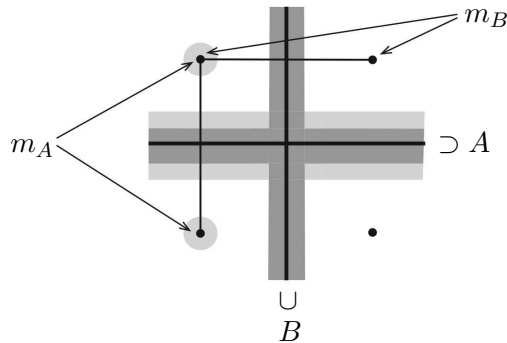


FIGURE 4. A more detailed illustration of the spheres in figure 2 near one of the intersection points. As above, the Clifford torus is shown as 4 points. A_n, B_n are contained in neighborhoods of the 2-spheres which are indicated with darker shading. Lighter shading shows the range of a small ambient isotopy.

position (intersecting in a $(d - 4)$ -sphere) in \mathbb{R}^d . All steps in the proof above go through: the Clifford torus in figure 2 (corresponding to a codimension = $(d - 4)$ -slice) gives a relation $[m_A, m_B] = 1$ in the fundamental group of the complement. The contradiction with the Stallings theorem relies only on homological information which holds due to Alexander duality. \square

Remark. Michael Freedman suggested a refinement of the construction in theorem 1, giving a *locally* sticky Cantor set. That is, the intersection of such a Cantor set with any open set is itself sticky with respect to ϵ -isotopies, for some ϵ depending on the open set. The idea is to start with the Hopf link and then replace each component with four: a Bing pair, and in addition the two meridional circles, then iterate - always replacing each component with four. (In the usual Bing decomposition each component is replaced with a Bing double, shown in figure 1. In addition, now the two meridional curves are also included.) This gives a shrinkable decomposition. The idea is then to use a spun version of it, with “minimal” intersections of the 2-spheres $\times D^2$ in 4-space to define the desired Cantor set. (The meridional curves create linking numbers, resulting in intersections between the spheres in 4-space at all scales.)

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