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On the relative slice problem and four dimensional topological surgery

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1. Introduction

A central open problem in the classification theory of topological four-manifolds is to determine the validity of four-dimensional surgery and five-dimensional s-cobordism theorems without fundamental group restrictions. By work of M. Freedman [2], [3] the class of groups for which these theorems hold ("good groups") includes the groups of polynomial growth. Recently Freedman and Teichner [6] showed that, more generally, the groups of subexponential growth are good. It is expected [3] that the theorems fail for free (non-abelian) groups; this conjecture is known as the A-B-*slice problem*. A more precise conjecture states that the Whitehead double of the Borromean Rings $Wh(Bor)$ is not (freely) slice. In this paper we study the "relative-slice" reformulation of this problem, introduced in [4]. Our main theorem may be viewed as a result in link theory, while providing some evidence towards the conjecture. Recall the following definition from [4].

Definition. A pair of disjoint links (L, H) in S^3 is called *relatively slice* if the components of L bound disjoint embedded (topologically flat) disks in the handlebody $B^4 \cup_H$ 2-handles, where the 2-handles are attached to B^4 along the components of H with zero framings.

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This is a generalization of the usual notion of a slice link (which corresponds to the case of an empty link H in the definition above.) The second link H is the "helping" link, and to determine whether (L, H) is relatively slice means to measure, in some sense, the difference between the links L and H. The surgery conjecture for free groups fails if and only if link pairs in a certain infinite family are not relatively slice, see [4] for a precise description. The main result of this paper, based on recent developments ([4], [10], [11]) in *link homotopy theory*, is that a restricted class of link pairs is not relatively slice, see Theorem 1 in Sect. 3. An example of a link pair, shown to be not relatively slice, is given in Fig. 2. The main difference of links considered here from the general case, arising from surgery, is in that L and H are allowed to interact only in a "controlled" way, see definition 3.1. This result may also be thought of as an extension of the Link Composition Lemma of Freedman and Lin (this analogy is made precise in example 3.4.) Some of the techniques developed in the proof may be applied to the general relative-slice problem.

The notion of link homotopy, introduced by J. Milnor [13], is a weaker equivalence relation than the usual isotopy of links (components of a link are allowed to self-intersect during a link homotopy.) Thus links modulo link-homotopy are easier to study; for example, there is a simple algebraic characterization of homotopically trivial links: a link is null-homotopic if and only if its Milnor's $\bar{\mu}$-invariants with non-repeating coefficients vanish. The $\bar{\mu}$-invariants are "higher-order linking numbers", derived from nilpotent qutients of the link group. A motivation behind the relative-slice approach to the surgery conjecture is that while all known obstructions to slicing vanish for $Wh(Bor)$, one may hope that a certain relative version of link homotopy theory will show that all corresponding link pairs are not relatively slice. In fact, Theorem 1 proves that for link pairs in the (restricted) family, the components of L do not even bound *disjoint maps of disks* in $B^4 \cup_H$ 0-framed 2-handles. Similarly to the recent works [4], [7], [10], [11], we use a combination of the classical Milnor's algebraic approach, and of four-dimensional geometric techniques.

The outline of the proof of the main result (Theorem 1) is as follows. Assume a link pair (L, H) is relatively-slice. The disks bounded by the components of L may be assumed to be transverse to the cocores of the 2-handles attached to B^4 along H. Disregarding these 2-handles, L bounds in B^4 disjoint planar surfaces, the other boundary components of which are untwisted parallel copies of the components of H. Given a component h of H, any planar surface may have many boundary components parallel to h. Lemma 3.6, proved in Sect. 6, changes the surfaces, reducing the number of boundary components while preserving their disjointness. After this step is applied the linking of surfaces in the four-ball is reflected, in some

sense, in linking of their boundaries in S^3, which can then be measured using $\bar{\mu}$-invariants. This step cannot, in general, be achieved without introducing gropes. We allow insertion of gropes in the surfaces since in terms of link homotopy disjoint gropes of a sufficiently large class are as good as disjoint disks, compare Grope Lemma 2.8. Now the $\bar{\mu}$-invariants of links in S^3 are used to define homomorphisms between certain free abelian groups. The presumed planar surfaces, connecting these links in B^4, force relations between the $\bar{\mu}$-invariants, giving an overdetermined linear algebraic problem and leading to a contradiction. The Link Composition Lemma, Grope Lemma [4], [11] and additivity of $\bar{\mu}$-invariants [10] play a crucial role in formulating this linear algebraic problem.

In a special case we also present an alternative geometric proof. It is based on a Bing doubling construction for surfaces, due to M. Freedman, which is described in the Appendix. We also give both an algebraic and a geometric argument for the "pull-up procedure" for surfaces in the four-ball, used in formulating linear algebra, and which is the main tool in dealing with the indeterminacy of $\bar{\mu}$-invariants in our proof. We present both of these alternative viewpoints, as it is unclear which of the two approaches may be more beneficial in the search for an obstruction to surgery. The organization of the paper is as follows.

2. Preliminary results in link homotopy.
3. Main theorem: linear algebra and the relative-slice problem.
4. A geometric proof in the Bing double case.
5. Technical lemmas.
6. A pull-up procedure for surfaces in the four-ball.
7. Appendix: Bing doubling a pair of pants (after Michael Freedman).

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2. Preliminary results in link homotopy

In this section we recall background material on Milnor groups, $\bar{\mu}$-invariants and gropes from [13], [14], [7]. We also review results in link-homotopy theory, established in [4], [10], [11]. Of particular importance for applications to

the relative-slice problem are Link Composition Lemma 2.5, Grope Lemma 2.8 and the additivity of $\bar{\mu}$-invariants (Theorem 2.4.)

The free group on generators $g_1, \ldots, g_k$ will be denoted by $F_{g_1,\ldots,g_k}$. Given a group G, its lower central series is defined inductively by $G^1 = G$, $G^2 = [G, G], \ldots, G^q = [G, G^{q-1}]$.

We briefly review the definition of $\bar{\mu}$-invariants from [14]. Let $L = (l_1, \ldots, l_n)$ be an oriented link in S^3. Given a positive integer q, the quotient $\pi_1(S^3 \setminus L)/(\pi_1(S^3 \setminus L))^q$ is generated by meridians $m_1, \ldots, m_n$ to the components of L. Let $w_1, \ldots, w_n$ be some words in $m_1, \ldots, m_n$ which represent the untwisted longitudes in this group, then $\pi_1(S^3 \setminus L)/(\pi_1(S^3 \setminus L))^q$ has the presentation

$$< m_1, \ldots, m_n | [m_1, w_1], \ldots, [m_n, w_n], (F_{m_1,\ldots,m_n})^q > .$$

The Magnus expansion homomorphism $M\colon F_{m_1,\ldots,m_n} \longrightarrow \mathbb{Z}\{x_1, \ldots, x_n\}$ into the ring of formal non-commutative power series in the indeterminates $x_1, \ldots, x_n$ is defined by $M(m_i) = 1 + x_i$, $M(m_i^{-1}) = 1 - x_i + x_i^2 \pm \ldots$ for $i = 1, \ldots, n$. Let

$$M(w_j) = 1 + \Sigma \mu_L(I, j) x_I$$

be the expansion of w_j, where the summation is over all multiindices $I = (i_1, \ldots, i_k)$ with entries between 1 and n, and $x_I = x_{i_1} \cdot \ldots \cdot x_{i_k}$, $k > 0$. This expansion defines for each such multiindex I the integer $\mu_L(I, j)$. Let $\Delta_L(i_1, \ldots, i_k)$ denote the greatest common divisor of $\mu_L(j_1, \ldots, j_s)$ where $j_1, \ldots, j_s$, $2 \leq s \leq k - 1$ is to range over all sequences obtained by cancelling at least one of the indices $i_1, \ldots, i_k$ and permuting the remaining indices cyclicly.

Let $\bar{\mu}_L(I)$ denote the residue class of $\mu_L(I)$ modulo $\Delta_L(I)$. For each multiindex I of length $|I| \leq q$ the residue class $\bar{\mu}_L(I)$ is an *isotopy invariant* of the link L, where $\bar{\mu}_L(I)$ is defined using the quotient $\pi_1(S^3 \setminus L)/(\pi_1(S^3 \setminus L))^q$.

2.1. Link homotopy and Milnor groups.

Two n-component links L and L' in S^3 are said to be *link-homotopic* if they are connected by a 1-parameter family of immersions such that different components stay disjoint at all times. L is said to be *homotopically trivial* if it is link-homotopic to the unlink. L is *almost homotopically trivial* if each proper sublink of L is homotopically trivial.

For a group π normally generated by $g_1, \ldots, g_k$ its *Milnor group* (with respect to $g_1, \ldots, g_k$) $M\pi$ is defined to be the quotient of π by its subgroup $\ll [g_i, g_i^h] : 1 \leq i \leq k, \quad h \in \pi \gg$. $M\pi$ is nilpotent of class $\leq k + 1$,

in particular it is a quotient of $\pi/(\pi)^{k+1}$, and is generated by the quotient images of $g_1, \ldots, g_k$. The Milnor group $M(L)$ of a link L is defined to be $M\pi_1(S^3 \smallsetminus L)$ with respect to its meridians m_i.

Milnor showed in [13] that the Magnus expansion induces a well defined injective homomorphism $MM\colon M(F_{m_1,\ldots,m_k}) \longrightarrow R(x_1, \ldots, x_k)$ into the ring $R(x_1, \ldots, x_k)$ which is the quotient of $\mathbb{Z}\{x_1, \ldots, x_k\}$ by the ideal generated by monomials $x_{i_1} \cdots x_{i_r}$ with some index occuring at least twice. Let $\overline{w}_n \in MF_{m_1,\ldots,m_{n-1}}$ be a word representing l_n in $M\pi_1(S^3 \smallsetminus (l_1 \cup \ldots \cup l_{n-1}))$. Then $\bar{\mu}$-invariants of L with non-repeating coefficients may also be defined by the equation

$$MM(\overline{w}_n) = 1 + \Sigma \mu_L(I, n) x_I$$

where summation is over all multiindices I with non-repeating entries between 1 and $n-1$, and $\bar{\mu}_L(I, n)$ is the residue class of $\mu_L(I, n)$ modulo the indeterminacy $\Delta_L(I, n)$, defined above.

The Milnor group of L is the largest common quotient of the fundamental groups of all links link-homotopic to L, hence one has the following result.

Theorem 2.1 (Invariance under link homotopy [13]) *If L and L' are link homotopic then their Milnor groups are isomorphic. In particular, for any multiindex I with non-repeating entries $\bar{\mu}_L(I) = \bar{\mu}_{L'}(I)$.*

Isotopy of links is a special kind of *concordance*, and it is a result of Stallings that Milnor's invariants are preserved under this more general equivalence relation.

Theorem 2.2 (Concordance invariance [15]) *If L and L' are concordant then all their $\bar{\mu}$-invariants coincide. In fact, if $L \subset S^3 \times \{0\}$ and $L' \subset S^3 \times \{1\}$ are connected in $S^3 \times I$ by disjoint immersed annuli then L and L' are link-homotopic* ([8], [9], [12]).

The next result gives an algebraic criterion for a link to be null-homotopic.

Lemma 2.3 ([13]) *For an n-component link L, the following conditions are equivalent:*

(i) L is homotopically trivial,

(ii) the components of L bound disjoint immersed disks in B^4,

(iii) $M(L) \cong M(F_{m_1,\ldots,m_n})$ with the isomorphism carrying a meridian to l_i to the generator m_i of the free group,

(iv) all $\bar{\mu}$-invariants of L with non-repeating coefficients vanish.

It follows from Lemma 2.3 that L is almost homotopically trivial if and only if all its $\bar{\mu}$-invariants with non-repeating coefficients of length less than n vanish. In particular, if L is almost homotopically trivial then its

$\bar{\mu}$-invariants with non-repeating coefficients of length n are well-defined integers.

The following two results play a crucial role in formulating the linear-algebraic obstruction in the proof of main theorem in this paper. For two oriented links $L' = (l'_1, \ldots, l'_n)$ and $L'' = (l''_1, \ldots, l''_n)$ in S^3, separated by a 2-sphere, let $L' \sharp L'' = (l_1, \ldots, l_n)$ denote a link the i-th component of which is obtained by taking a connected sum (ambient surgery along an arc) of the components l'_i and l''_i respecting their orientations, $i = 1, \ldots, n$. The sum $L' \sharp L''$ depends in general on the choice of bands in S^3, but in each case the choice will be clear from the context.

Lemma 2.4 (Theorem 1 in [10]) *Let $L' = (l'_1, \ldots, l'_n)$ and $L'' = (l''_1, \ldots, l''_n)$ be two oriented links in S^3, separated by a 2-sphere. Then for any choice of connecting bands (in particular, they may intersect the separating 2-sphere more than once) and for any multiindex I, the indeterminacy $\Delta_{L' \sharp L''}(I)$ is a multiple of $g.c.d.(\Delta_{L'}(I), \Delta_{L''}(I))$, and*

$$\bar{\mu}_{L' \sharp L''}(I) \equiv \bar{\mu}_{L'}(I) + \bar{\mu}_{L''}(I) \; mod \; (g.c.d.(\Delta_{L'}(I), \Delta_{L''}(I))).$$

In particular, if L' and L'' are both almost homotopically trivial, then so is $L' \sharp L''$, and

$$\bar{\mu}_{L' \sharp L''}(1, \ldots, n) = \bar{\mu}_{L'}(1, \ldots, n) + \bar{\mu}_{L''}(1, \ldots, n).$$

We will now recall a version of the Link Composition Lemma, most convenient for our applications. It states that the first non-vanishing $\bar{\mu}$-invariants are multiplicative under composition. Given a link $\widehat{L} = (l_1, \ldots, l_{k+1})$ in S^3 and a link $Q = (q_1, \ldots, q_m)$ in the solid torus $S^1 \times D^2$, their "composition" is obtained by replacing the last component of $\widehat{L}$ with Q. More precisely, it is defined as $C = (c_1, \ldots, c_{k+m}) := (l_1, \ldots, l_k, \phi(q_1), \ldots, \phi(q_m))$, where ϕ: $S^1 \times D^2 \hookrightarrow S^3$ is a 0-framed embedding whose image is a tubular neighborhood of l_{k+1}. The meridian $\{1\} \times \partial D^2$ of the solid torus will be denoted by $\wedge$ and we put $\widehat{Q} := Q \cup \wedge$.

Theorem 2.5 (Link Composition Lemma: Theorem 2.3 in [4], Theorem 3 and remark after its proof in [11]) *If both $\widehat{L}$ and $\widehat{Q}$ are almost homotopically trivial, then so is their composition $C = L \cup \phi(Q)$, and*

$$\bar{\mu}_C(1, \ldots, k+m) = \bar{\mu}_{\widehat{L}}(1, \ldots, k+1) \cdot \bar{\mu}_{\widehat{Q}}(1, \ldots, m, \wedge).$$

In particular, if $\widehat{L}$ and $\widehat{Q}$ are both homotopically essential in S^3 then $L \cup \phi(Q)$ is also homotopically essential.

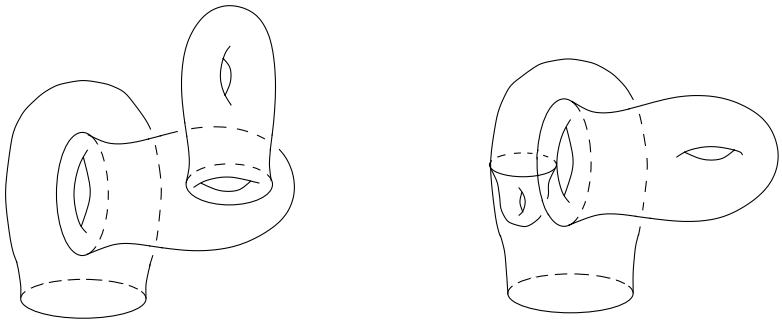


Fig. 1 Two gropes of class 4

2.2. Gropes and the lower central series.

A *grope* is a special pair (2-complex, circle). A grope has a *class* $k = 1, 2, \ldots, \infty$. For $k = 1$ a grope is defined to be the pair (circle, circle). For $k = 2$ a grope is precisely a compact oriented surface Σ with a single boundary component. For k finite a k-*grope* is defined inductively as follow: Let $\{\alpha_i, \beta_i, i = 1, \ldots, \mathrm{genus}\}$ be a standard symplectic basis of circles for Σ. For any positive integers p_i, q_i with $p_i + q_i \geq k$ and $p_{i_0} + q_{i_0} = k$ for at least one index i_0, a k-grope is formed by gluing p_i-gropes to each α_i and q_i-gropes to each β_i.
The proof of the next lemma, and additional properties of gropes may be found in [7], [11].

Lemma 2.6 (Lemma 2.1 in [7]) *For a space X, a loop γ lies in $\pi_1(X)^k, 1 \leq k < \omega$, if and only if γ bounds a map of some k-grope. Moreover, the class of a grope (G, γ) is the maximal k such that $\gamma \in \pi_1(G)^k$.*

Given a surface S, an S-*like grope* of class k is a 2-complex obtained by replacing a 2-cell in S with a k-grope. For example, one has *annulus-like* k-gropes; sphere-like gropes are sometimes also referred to as *closed* gropes. Given a space X, the *Dwyer's subgroup* $\phi_k(X)$ of $H_2(X; \mathbb{Z})$ is the set of all homology classes represented by maps of closed gropes of class k into X.

Theorem 2.7 (Dwyer's Theorem [1]) *Let k be a positive integer and let $f\colon X \longrightarrow Y$ be a map inducing an isomorphism on H_1 and an epimorphism on H_2/ϕ_k. Then f induces an isomorphism on $\pi_1/(\pi_1)^k$.*

If two links are concordant, then by theorem 2.2 they are link-homotopic. Grope Lemma (originally formulated in [4] in the case when one of the links

is trivial) shows that the same conclusion holds if instead of disjoint annuli connecting the links in $S^3 \times [0,1]$ one has disjoint immersed annulus-like gropes of a sufficiently large class.

Theorem 2.8 (Grope Lemma: Theorem 2 in [11]) *Two n-component links in S^3 are link homotopic if and only if they cobound disjointly immersed annulus-like gropes of class n in $S^3 \times I$.*

Corollary 2.9 (Grope-concordance invariance) *Let $L = (l_1, \dots, l_n)$ and $L' = (l'_1, \dots, l'_n)$ be two links in $S^3 \times \{0\}$ and $S^3 \times \{1\}$ respectively. Suppose there are disjoint immersed annulus-like gropes $A_1, \dots, A_n$ of class n in $S^3 \times [0,1]$ with $\partial A_i = l_i \cup l'_i$, $i = 1, \dots, n$. Then for any multiindex I with non-repeating entries $\bar{\mu}_L(I) = \bar{\mu}_{L'}(I)$.*

3. Main theorem: linear algebra and the relative-slice problem

In this section we state the main result, theorem 1, and outline its proof, deferring verification of technical lemmas to sections 5 and 6.

Notation 3.1 Let k and m be positive integers. Consider a chain of solid tori

$$W_{-k}, R_{-(k-1)}, \dots, W_{-2}, R_{-1}, W_{-1}, W_1, R_1, W_2, \dots, R_{m-1}, W_m$$

and links in them, $L_i \subset W_i$, $i = -k, \dots, m$ and $H_j \subset R_j$, $j = -(k-1), \dots, m-1$, as in Fig. 2. Set $L = \cup_i L_i$, $H = \cup_j H_j$. The chain of solid tori will be fixed throughout the proof of theorem 1, they are important only for visualizing the structure of the links. Once these tori are fixed, one may consider various links L_i, H_j in them; theorem 1 applies to an infinite family of such link pairs (L, H).

An essential feature of this definition is that there are two sublinks of L in the center of the chain: L_{-1} and L_1, while to the left and to the right the links L_i and H_j alternate. Note that the chain ends in both directions with sublinks of L.

Given a solid torus $T = S^1 \times D^2$, following the convention of [4], we denote its meridian $\{1\} \times \partial D^2$ by $\wedge_T$, or simply by $\wedge$ when there is no danger of confusion. Given a link K in the interior of T, put $\widehat{K} := K \cup \wedge$. A link K in T is said to be *$\wedge$-homotopically essential*, *$\wedge$-homotopically trivial*, or *$\wedge$-almost homotopically trivial* if $\widehat{K}$ satisfies this property, see Sect. 2.1 for relevant definitions.

Theorem 1 *Let k, m be positive integers, and let (L, H) be a chain of links, as in definition 3.1. Assume that for each $i = -k, \dots, m$,*

(i) the link L_i is $\wedge$-homotopically essential,

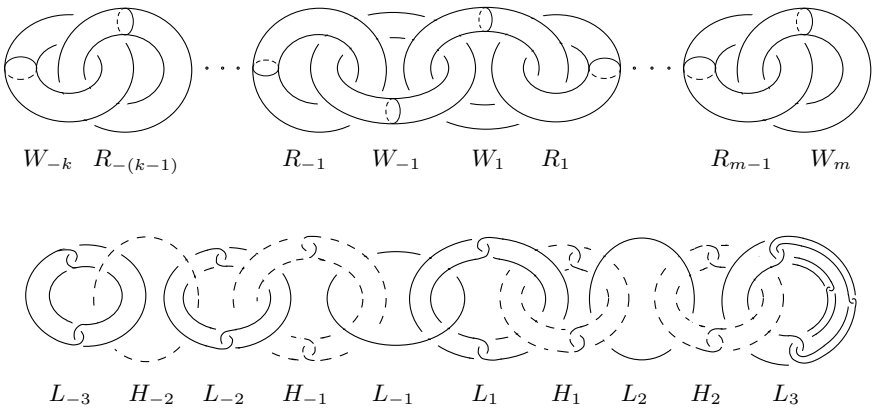


Fig. 2 A chain of solid tori and an example of links in them. The "helping" links H_i are drawn dashed. There are two sublinks of L in the center: L_{-1} and L_1, while to the left and to the right of the center, the links L_i and H_j alternate. The chain ends in both directions with sublinks of L

(ii) the link L_i is almost homotopically trivial in the solid torus W_i, and
(iii) each link L_i and H_j separately is isotopic in S^3 to an unlink.

Then (L, H) is not relatively slice. Moreover, the components of L do not even bound disjoint maps of disks in $B^4 \cup_H 0$-framed 2-handles.

Remarks. Condition (i) is an essential assumption. It may be replaced by the assumption that, for each i, L_i becomes $\wedge$-homotopically essential after adding some number of parallel copies to its components; then Theorem 1 still implies that (L, H) is not relatively slice. (Assume that (L, H) is relatively slice. Add the parallel copies and denote the new link by L_i again. The pair (L, H) is still relatively slice since the new components of L bound parallel copies of the old slices.) Note that there is no such restriction on the links H_j.

Condition (ii) is a technical assumption which is slightly stronger than $\widehat{L}_i$ be almost homotopically trivial in S^3. This latter condition could be assumed without loss of generality by omitting some of the components of L_i if necessary.

We include condition (iii) since it makes arguments technically easier, and it is satisfied by the link pairs, arising in connection with the surgery conjecture [4]. This condition corresponds to the fact that the links describe 1-handles in a Kirby handle diagram of a certain 4-manifold.

It has been emphasized that the chain of links in Theorem 1 ends with sublinks of L in both directions. The conclusion certainly fails in general,

if this is not the case. One can easily modify the following example to see this.

Example 3.2 The simplest example of (L, H) satisfying the assumptions of theorem 1 is the case when each link L_i and H_j consists just of the core circle of the corresponding solid torus W_i (respectively R_j.) One may give in this case an elementary proof, reducing the problem to linear algebra, using linking numbers. The reader may want to keep this example in mind while going through the proof of Theorem 1, since most technical difficulties in the proof correspond to generalizing linking numbers to $\bar{\mu}$-invariants.

Example 3.3 A more general family of examples is obtained from the previous one by (iterated) Bing doubling, as in Fig. 2. (See Sect. 7 for a discussion about Bing doubles.) In Sect. 4 we present a geometric proof of theorem 1 in this special case.

Example 3.4 Consider the trivial case $k = m = 1$, that is, the link H is empty, and $L = L_{-1} \cup L_1$. Since L_{-1} and L_1 are $\wedge$-homotopically essential, Link Composition Lemma 2.5 implies that L is homotopically essential, hence by lemma 2.3 its components do not bound disjoint maps of disks in B^4. Thus theorem 1 may be thought of as a generalization of the Link Composition Lemma.

Briefly the idea of the proof of theorem 1 is as follows. We will show that the components of L do not even bound disjoint maps of disks in $B^4 \cup_H$ 0-framed 2-handles. Assume on the contrary that L bounds disjoint maps of disks. Since a small perturbation will leave these maps disjoint, one may assume that the disks bounded by the components of L are smoothly immersed. The disks may also be assumed to be transverse to the cocores of the 2-handles attached to B^4 along H. Disregarding these 2-handles, L bounds in B^4 disjoint planar surfaces, the other boundary components of which are untwisted parallel copies of the components of H. Given a component h of H, any planar surface may have many boundary components parallel to h. In particular, each surface may have many boundary components in every solid torus R_j. Lemma 3.6, stated below, changes the surfaces, preserving their disjointness, so that the surface bounded by each component of L has precisely one boundary component in each solid torus R_j. After this step is applied the linking of surfaces in the four-ball is reflected, in some sense, in linking of their boundaries in S^3, which can then be measured using $\bar{\mu}$-invariants. This step cannot, in general, be achieved without introducing gropes. We allow insertion of gropes in the surfaces since in terms of link homotopy disjoint gropes of a sufficiently large class are as good as disjoint disks, compare Grope Lemma 2.8. Now we can formulate a linear-algebraic obstruction. The $\bar{\mu}$-invariants of links in the solid tori $\{W_i\}$

and $\{R_j\}$ are used to define a homomorphism between certain free abelian groups. (One can show that the $\bar{\mu}$-invariants in question are well-defined integers.) The presumed planar surfaces, connecting these links in B^4, force relations between the $\bar{\mu}$-invariants, giving an overdetermined linear algebraic problem and leading to a contradiction. Link Composition Lemma 2.5, Grope Lemma 2.8 and additivity of $\bar{\mu}$-invariants (Lemma 2.4) play a crucial role in formulating this linear algebraic problem.

Notation 3.5 For each component l of L, let P_l denote the (immersed) planar surface it bounds in B^4. Also for each sublink K of L, denote $\cup_{l\in K} P_l$ by P_K. Fix an orientation for each surface P_l, and let the components of L be oriented as their boundaries. Let $\partial = \cup_{l\in L}\partial P_l$ denote the union of boundaries of all surfaces.

The proof of the next result is given in Sect. 6; also see the remarks after lemma 6.3.

Lemma 3.6 *Let (L, H) be a pair of links as in Theorem 1, and let n be a positive integer. Then the associated planar surfaces $\{P_l\}_{l\in L}$ can be modified, possibly changing their boundary components other than L, introducing self-intersections and inserting gropes of class n so that*

i) *For each component l of L and for each $j = -(k-1), \ldots, m-1$, P_l has exactly one boundary component in the solid torus R_j,*
ii) *$P_l \cap P_{l'} = \emptyset$ if $l \neq l'$,*
iii) *$(\partial \setminus L)$ is a ribbon link contained in $\cup_j R_j$.*

Proof of Theorem 1. Suppose, as above, that the components of L bound disjoint maps of disks in $B^4 \cup_H$ 0-framed 2-handles or, equivalently, that they bound in B^4 disjoint immersed planar surfaces, the other boundary components of which are untwisted parallel copies of the components of H. Apply lemma 3.6 with $n = |L|$, the number of components of L, so from now on we will assume conditions i)-iii) in 3.6. Let n_i denote the number of components of L_i. By assumptions of Theorem 1, $\widehat{L}_i$ is homotopically essential and almost homotopically trivial, hence its $\bar{\mu}$-invariants with non-repeating coefficients of length $n_i + 1$ are well-defined integers and by lemma 2.3 at least one of them is non-zero. Order the components so that $\bar{\mu}_{\widehat{L}_i}(1, \ldots, n_i, \wedge) \neq 0$. (By cyclic symmetry of $\bar{\mu}$-invariants [14] one may assume without loss of generality that there is a non-trivial $\bar{\mu}$-invariant with last index $\wedge$.) We will fix this order on the components of each link L_i for the rest of the proof.

Notation 3.7 Let W (respectively R) denote the free abelian group with a free generator for each solid torus W_i (respectively R_j):

$$W = \mathbb{Z} < W_{-k}, \ldots, W_{-1}, W_1, \ldots, W_m >,$$

$$R = \mathbb{Z} < R_{-(k-1)}, \ldots, R_{-1}, R_1, \ldots, R_{m-1} > .$$

These groups have ranks $(k+m)$ and $(k+m-2)$ respectively.

The location of the links $\{L_i\}$ and $\{H_j\}$ in S^3, and the presumed planar surfaces, connecting them in B^4, define homomorphisms

$$A\colon W \longrightarrow R \quad \text{and} \quad B\colon R \longrightarrow W$$

as follows. Fix $-k \leq i \leq m$ and $-(k-1) \leq j \leq m-1$. Recall that by lemma 3.6 the (planar surface)-like grope bounded by each component of L_i has exactly one boundary component in the solid torus R_j. Hence $L_{i,j} := \partial P_{L_i} \cap R_j$ is an n_i-component link, where n_i is the number of components of L_i. The link $L_{i,j}$ and the gropes P_{L_i} are ordered, according to the order on L_i. Let each link $L_{i,j}$ be oriented as the boundary of P_{L_i}.

Lemma 3.8 *The link $\widehat{L}_{i,j}$ is almost homotopically trivial for each $(i,j) \in \{-k, \ldots, -1\} \times \{1, \ldots, m-1\} \cup \{1, \ldots, m\} \times \{-(k-1), \ldots, -1\}$. In particular, for any such pair (i,j), $\bar{\mu}_{\widehat{L}_{i,j}}(1, \ldots, n_i, \wedge_{R_j})$ is a well-defined integer.*

The proof of lemma 3.8 is given in Sect. 5. Let R^* and W^* denote the dual abelian groups, and define $A'\colon W \longrightarrow R^*$, $B'\colon R \longrightarrow W^*$ by

$A'(W_i)(R_j) = \bar{\mu}_{\widehat{L}_{i,j}}(1, \ldots, n_i, \wedge_{R_j})$

if $(i,j) \in \{-k, \ldots, -1\} \times \{1, \ldots, m-1\} \cup \{1, \ldots, m\}$
$\times \{-(k-1), \ldots, -1\}$,

$A'(W_i)(R_j) = 0$ otherwise;

$B'(R_j)(W_i) = \bar{\mu}_{\widehat{L}_i}(1, \ldots, n_i, \wedge_{W_i})$ if the tori R_j and W_i link,

$B'(R_j)(W_i) = 0$ otherwise.

In this definition the links L_i and $L_{i,j}$ are labelled by $1, \ldots, n_i$, respecting the fixed order. The homomorphisms $A\colon W \longrightarrow R$ and $B\colon R \longrightarrow W$ are obtained from A' and B' via the isomorphisms $W^* \cong W$, $R^* \cong R$ defined by the chosen bases. Let C denote the composition

$$C = B \circ A\colon W \longrightarrow W.$$

Here B is a fixed homomorphism determined by the links $\{L_i\}$ in Theorem 1. The map A is "variable", and is given by the presumed slices for L. The

goal is to find an obstruction for any A. Note that the (non-trivial) entries of the matrices A and B with respect to the fixed bases are given by

$$A_{j,i} = \bar{\mu}_{\widehat{L}_{i,j}}(1, \ldots, n_i, \wedge_{R_j}), \;\; B_{i,j} = \bar{\mu}_{\widehat{L}_i}(1, \ldots, n_i, \wedge_{W_i}).$$

Here i ranges from $-k$ to m and j ranges from $-(k-1)$ to $m-1$.

Remarks. The $\bar{\mu}$-invariants in the definition of B are well-defined integers and are non-trivial, by the assumptions of Theorem 1. By Lemma 3.8, the $\bar{\mu}$-invariants defining A are also well-defined. It was necessary to fix an order on the components of the links L_i and $L_{i,j}$ for further arguments since the $\bar{\mu}$-invariants of a given link depend, in general, on the order of indices. Note that all constructions do not depend on the links $\{H_j\}$.

The proof of the following result essentially reduces to Link Composition Lemma 2.5 and grope-concordance invariance 2.9, and is given in Sect. 5.

Lemma 3.9 *Let (L, H) be a link pair, satisfying the assumptions of theorem 1. Then the associated matrix C is skew-symmetric.*

With respect to the chosen bases C may be written as a block matrix

$$C = \begin{pmatrix} 0 & C'' \\ C' & 0 \end{pmatrix}$$

where C' is an $m \times k$ matrix, C'' is a $k \times m$ matrix, and Lemma 3.9 states that $C' = -(C'')^t$. Consider the entry $C_{-1,1}$ of the matrix C:

$$C_{-1,1} = \sum_{i=-(k-1)}^{m-1} B_{-1,i} \cdot A_{i,1}.$$

This sum is equal to $B_{-1,-1} \cdot A_{-1,1}$, since $B_{-1,i} = 0$ unless $i = -1$. Recall that

$$B_{-1,-1} = \bar{\mu}_{\widehat{L}_{-1}}(1, \ldots, n_{(-1)}, \wedge_{W_{-1}}) \text{ and}$$
$$A_{-1,1} = \bar{\mu}_{\widehat{L}_{1,-1}}(1, \ldots, n_1, \wedge_{R_{-1}}).$$

By Link Composition Lemma 2.5,

$$C_{-1,1} = B_{-1,-1} \cdot A_{-1,1} = \bar{\mu}_{L_{-1} \cup L_{1,-1}}(1, \ldots, n_{(-1)} + n_1).$$

Similarly,

$$C_{1,-1} = \bar{\mu}_{L_{-1,1} \cup L_1}(1, \ldots, n_{(-1)} + n_1).$$

By lemma 3.9, $C_{-1,1} = -C_{1,-1}$. The proof of 3.9 only uses the fact that the solid tori in definition 3.1 with *negative* indices are linked in a chain, and also

that the tori with *positive* indices are linked. It would also hold if the tori W_{-1} and W_1 in the center were not linked. We will now use this remaining piece of information to find a contradiction. Consider the four central solid tori R_{-1}, W_{-1}, W_1, R_1, the links L_{-1}, L_1 and the (planar surfaces)-like gropes $P_{L_{-1}}, P_{L_1}$ they bound in B^4. These gropes have boundary components in each solid torus R_j, $j = -(k-1), \ldots, m-1$. Disregarding all other gropes, we will now modify $P_{L_{-1}}$ and P_{L_1} so that they satisfy

(i) $\partial(P_{L_{-1}} \cup P_{L_1}) \subset R_{-1} \cup W_{-1} \cup W_1 \cup R_1$, and

(ii) $\partial P_{L_1} \cap R_1 = \emptyset$, $\partial P_{L_{-1}} \cap R_{-1} = \emptyset$,

and are still disjoint from each other. In other words, the pairs of the corresponding components of L_{-1} and $L_{-1,1}$, and of L_1 and $L_{1,-1}$ will cobound in B^4 disjointly immersed annulus-like gropes of class $n = |L|$.

The boundary components of the gropes in question, lying in the solid tori $\cup_{j \neq -1,1} R_j$, form a slice link by condition (iii) of Lemma 3.6. Attach a collar $B^3 \times I$ to B^4 near each R_j for $j \neq -1, 1$ and let these links bound disjoint disks in the attached collars. This takes care of condition (i).

To get condition (ii) notice that by Lemma 3.6 (iii), $(\partial P_{L_1} \cup \partial P_{L_{-1}}) \cap R_1$ is a ribbon link, so it is concordant in a collar $S^3 \times [0,1]$ on B^4 to the unlink. By theorem 2.2 this concordance may be changed into a link homotopy $A \subset S^3 \times [0,1]$. Let the solid torus W_1 move between times 0 and 1 by an isotopy in the complement of A. W_{-1} is a small torus linking W_1 at each time. In $S^3 \times \{1\}$, the solid torus W_1 lies in the complement of the unlink $(\partial P_{L_1} \cup \partial P_{L_{-1}}) \cap R_1$. Fix a component l of L_1. By the almost triviality of L_1 in the solid torus W_1 (assumption (ii) of Theorem 1), there is a further link homotopy in $S^3 \times [1,2]$ supported in W_1 so that all components of L_1 except l become small unlinked circles, and l is a long curve in $W_1 \subset S^3 \times \{2\}$. Now the corresponding boundary component $\partial P_l \cap R_1$ may be taken off the rest of the link and capped off with a disk, after possibly introducing self-intersections of P_l. Applying this argument to each component l of L_1 between times 1 and 2, and then running the link-homotopy A backwards in $S^3 \times [2,3]$ gives the first part of (ii). Its second part is achieved analogously.

The result of this argument is that the links L_{-1}, $L_{-1,1}$ and L_1, $L_{1,-1}$ cobound in B^4 disjoint immersed annulus-like gropes of class n. An argument, similar to the proof of grope-concordance invariance 2.9 shows that under these conditions

$$\begin{aligned}&\bar{\mu}_{L_1 \cup L_{-1,1}}(1, \ldots, n_{(-1)} + n_1) + \bar{\mu}_{L_1 \cup L_{-1}}(1, \ldots, n_{(-1)} + n_1)\\ &\quad + \bar{\mu}_{L_{1,-1} \cup L_{-1}}(1, \ldots, n_{(-1)} + n_1) = 0.\end{aligned}$$

This fact is stated and proved rigorously as lemma 5.2 in Sect. 5. Since $\widehat{L}_{-1}$ and $\widehat{L}_1$ are homotopically essential, and due to the choice of the labeling of their components, Link Composition Lemma 2.5 implies

$$\bar{\mu}_{L_{-1}\cup L_1}(1,\ldots,n_{(-1)}+n_1)\neq 0, \text{ so}$$

$$\begin{aligned} C_{1,-1} &= \bar{\mu}_{L_{-1,1}\cup L_1}(1,\ldots,n_{(-1)}+n_1) \\ &\neq -\bar{\mu}_{L_{-1}\cup L_{1,-1}}(1,\ldots,n_{(-1)}+n_1) \\ &= C_{-1,1}. \end{aligned}$$

However, Lemma 3.9 implies $C_{-1,1}=-C_{1,-1}$, and this contradiction concludes the proof of Theorem 1. □

4. A geometric proof of Theorem 1 in the Bing double case

In this section we use a geometric construction described in the appendix (lemma 7.1) to prove Theorem 1 in the special case when each link L_i is an iterated Bing double of the core circle of the corresponding solid torus W_i, see Fig. 2 and the Appendix. We state this result in the following lemma.

Lemma 4.1 *Let (L,H) be a chain of links, as in definition 3.1, where L_i is an iterated Bing double of the core circle of the corresponding solid torus W_i, for each i. Assume that the link H is isotopic in S^3 to the unlink. Then the pair (L,H) is not relatively slice.*

Proof. Let T denote a solid torus obtained from W_1 by enlarging it to include also the links $H_1, L_2, \ldots, H_{m-1}, L_m$, compare Figs. 2 and 3. Consider this solid torus as $T=T\times\{1\}\subset T\times[0,1]$. Consider the links in it as a Kirby handle diagram, where $\{H_i\}$ describe 1-handles, and $\{L_i\}$ are the attaching curves of 0-framed 2-handles, Fig. 3. Let M denote the 4-manifold with the attaching region $T\times\{0\}$, defined by this Kirby diagram:

$$\begin{aligned} M := T\times[0,1] &\smallsetminus (\text{standard slices for } H_1\cup\ldots\cup H_{m-1}) \\ &\cup_{L_1\cup\ldots\cup L_m\subset T\times\{1\}} 0\text{-framed } 2\text{-handles.} \end{aligned}$$

Define T' and M' analogously, using the links $L_{-1}, H_{-1}, \ldots, H_{-(k-1)}$, L_{-k}, Fig. 4.

Proposition 4.2 *There exists a $\wedge$-homotopically essential link K (a symmetric iterated Bing double of the core) in the attaching region $T\times\{0\}$ of M, such that the components of K bound disjoint disks in M. Similarly, there is a link K' with the analogous properties in the attaching region $T'\times\{0\}$ of M'.*

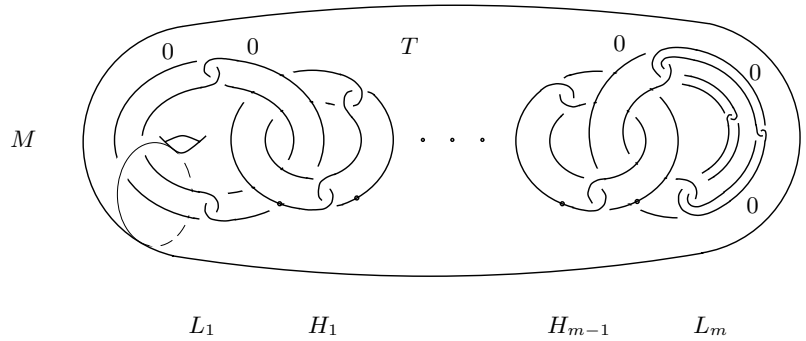


Fig. 3

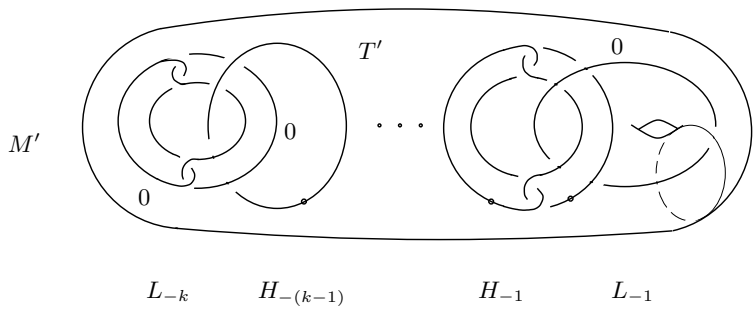


Fig. 4

Remark. The 2-handles of M, attached along $L_1 \subset T \times \{1\}$, do not provide the required disks: their attaching regions cannot, in general, be pushed down to $T \times \{0\}$ since the slices for H_1 are missing from the collar.

Proof of Proposition 4.2. Recall that the links L_i, H_j are contained in the solid tori W_i, $R_j \subset T = T \times \{1\} \subset M$ respectively, $i = 1, \ldots, m$, $j = 1, \ldots, m-1$. For each i, let c_i denote the core circle of W_i. Note that there is a planar surface P in $T \times [0,1]$, cobounded by the core c of $T \times \{0\}$ and by the curves $c_1, \ldots, c_m$, which is disjoint from the links $\{H_j\}$ and from the slices they bound, Fig. 5. Choose a large n, so that the n-iterated symmetric Bing double is a refinement of each iterated Bing double $L_1, \ldots, L_m$ (see the Appendix for definitions.) An application of Corollary

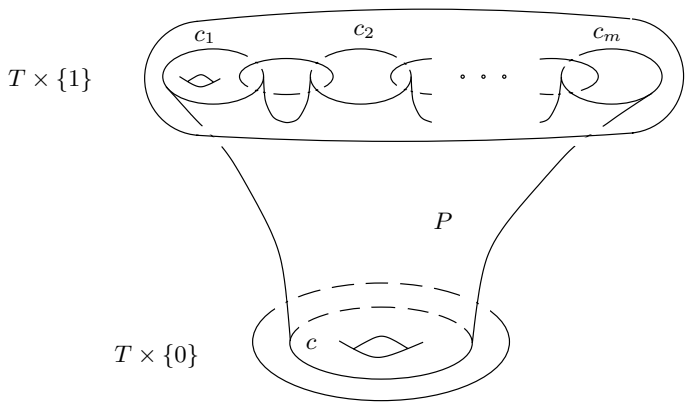


Fig. 5

7.2 gives planar surfaces $P_1, \ldots, P_{2^n}$, all boundary components of which in $T \times \{1\}$ bound disjoint disks in the attached 2-handles. Their union gives the disks required by lemma. $\square$

Let $f\colon T \hookrightarrow S^3$ and $f'\colon T' \hookrightarrow S^3$ be 0-framed embeddings such that $f(T)$, $f'(T')$ is a standard pair of Hopf-linked solid tori.

Proposition 4.3 *Let (L, H) be a pair of links as in Lemma 4.1, and suppose that (L, H) is relatively slice. Then there exist disjoint embeddings of the handlebodies M, M' into the four-ball D^4, extending the embeddings f, f', fixed above, of their attaching regions into $S^3 = \partial D^4$.*

Proof. Consider the links L, H in $f(T) \cup f'(T') \subset S^3 = \partial B^4$, and let D^4 denote the 4-ball obtained from B^4 by attaching a collar $S^3 \times [0,1]$, identifying $S^3 \times \{1\}$ with ∂B^4, so that $\partial D^4 = S^3 \times \{0\}$. Since H is an unlink, the 2-handles attached to B^4 with $0-$framings along H in the relative slicing of (L, H) may be disjointly embedded in $S^3 \times [0,1]$ in a standard way.

For M, a collar on the attaching region, union with the 1-handles, is mapped diffeomorphically onto $(f(T) \times [0,1] \setminus$ 2-handles attached to B^4 along $H_1 \cup \ldots \cup H_{m-1}$). Consider the analogous embedding of the 1-handles of M'. An embedding of the 2-handles of M and M' is provided by the relative-slice assumption on (L, H). $\square$

Consider the $\wedge$-essential links $f(K)$ and $f(K')$, given by Proposition 4.2, in the Hopf-linked solid tori $f(T)$, $f(T') \subset S^3 = \partial D^4$. By Link Composition Lemma 2.5 the link $f(K) \cup f'(K')$ is homotopically essential.

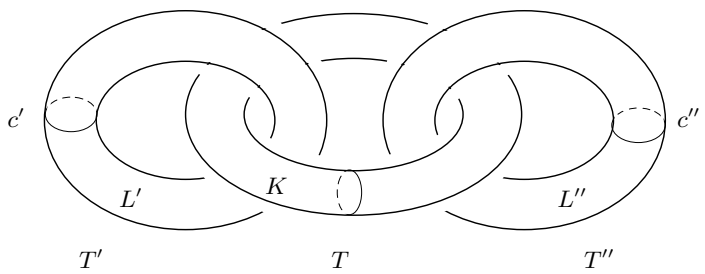


Fig. 6

However, Propositions 4.2 and 4.3 imply that the components of $f(K) \cup f'(K')$ bound disjoint disks in D^4, hence by lemma 2.3 it is homotopically trivial. This contradiction concludes the proof of lemma 4.1. $\square$

5. Technical lemmas

First we will prove two lemmas which establish additivity of $\bar{\mu}$-invariants in the presence of planar surfaces in B^4 bounded by links. These results are used in the proof of theorem 1.

Lemma 5.1 *Let $L' = (l'_1, \ldots, l'_k)$, $K = (l_{k+1}, \ldots, l_n)$ and $L'' = (l''_1, \ldots, l''_k)$ be $\wedge$-almost homotopically trivial oriented links in three linked solid tori T', T and T'' as in Fig. 6. Let $L' \sharp L'' = (l_1, \ldots, l_k)$ denote a connected sum of L' and L'', such that the connecting bands lie in the complement of T. Then $(L' \sharp L'') \cup K$ is almost homotopically trivial, and*

$$\bar{\mu}_{(L' \sharp L'') \cup K}(1, \ldots, n) = \bar{\mu}_{L' \cup K}(1, \ldots, n) + \bar{\mu}_{L'' \cup K}(1, \ldots, n).$$

Proof. Let c denote the core circle of the middle solid torus T, and let c', c'' be the meridians of T' and T'' respectively. The link $(L' \sharp L'') \cup c$ is a connected sum of $L' \cup c'$ and $L'' \cup c''$, hence Lemma 2.4 implies

$$\bar{\mu}_{(L' \sharp L'') \cup c}(1, \ldots, k, c) = \bar{\mu}_{L' \cup c'}(1, \ldots, k, c') + \bar{\mu}_{L'' \cup c''}(1, \ldots, k, c'').$$

The link $(L' \sharp L'') \cup K$ may be viewed as a composition of $(L' \sharp L'') \cup c$ and of K. Link Composition Lemma 2.5 and the equality above give

$$\bar{\mu}_{(L' \sharp L'') \cup K}(1, \ldots, n) = \bar{\mu}_{(L' \sharp L'') \cup c}(1, \ldots, k, c) \cdot \bar{\mu}_{\widehat{K}}(k+1, \ldots, n, \wedge_T) =$$

$$(\bar{\mu}_{L' \cup c'}(1, \ldots, k, c') + \bar{\mu}_{L'' \cup c''}(1, \ldots, k, c'')) \cdot \bar{\mu}_{\widehat{K}}(k+1, \ldots, n, \wedge_T) =$$

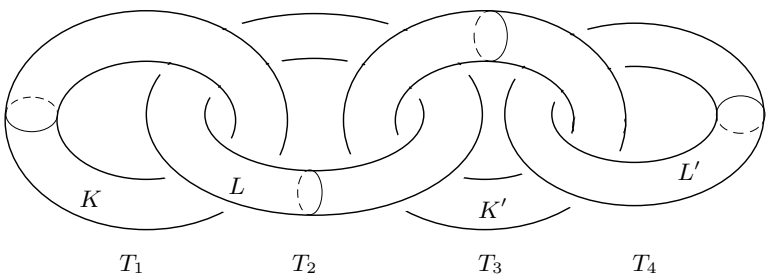


Fig. 7

$$\bar{\mu}_{L'\cup K}(1,\ldots,n)+\bar{\mu}_{L''\cup K}(1,\ldots,n).$$

□

Lemma 5.2 *Let $T_1,\ldots,T_4$ be a chain of solid tori in S^3, containing links $K=(l_1,\ldots,l_p)$ $L=(l_{p+1},\ldots,l_r)$, $K'=(l'_1,\ldots,l'_p)$, $L'=(l'_{p+1},\ldots,l'_r)$ respectively, as in Fig. 7. Assume there are immersed annulus-like gropes A_i of class r in B^4 with $\partial A_i=l_i\cup l'_i$, $i=1,\ldots,r$ and $A_i\cap A_j=\emptyset$ for $i\neq j$, and let the links be oriented as boundaries of the gropes. Assume K' and L are almost homotopically trivial in the solid torus and homotopically trivial in S^3 and that K and L' are $\wedge$-almost homotopically trivial. Then*

$$\bar{\mu}_{K\cup L}(1,\ldots,r)+\bar{\mu}_{K'\cup L}(1,\ldots,r)+\bar{\mu}_{K'\cup L'}(1,\ldots,r)=0.$$

Remarks. By Link Composition Lemma 2.5 the links $K\cup L$, $K'\cup L$ and $K'\cup L'$ are almost homotopically trivial, and the $\bar{\mu}$-invariants above are well-defined integers. Note that the statement of lemma 5.2 is well-defined with respect to the choice of orientations of the gropes. If orientation of one of the gropes is reversed, each term in the equality above changes its sign.

Proof of Lemma 5.2. Let M^4 denote $B^4\smallsetminus(A_2\cup\ldots\cup A_r)$. For each i, fix a meridian m_i to l_i and m'_i to l'_i. Clearly, $H_1(M^4;\mathbb{Z})$ is freely generated by $m_1,\ldots,m_r$. As in the proof of Theorem 1 in [11], one can show that $H_2(M^4;\mathbb{Z})/\phi_{r+1}$ is freely generated by the tori – circle normal bundles over $l_1,\ldots,l_r$ and by the Clifford tori in the neighborhoods of self-intersection points of the A_i. Here ϕ_{r+1} denotes a term in the Dwyer's filtration, defined in Sect. 2.2. The relations given by the Clifford tori are among the defining relations of the Milnor group on meridians to first stages of the gropes. By Dwyer's theorem 2.7,

$$M\pi_1(M^4)\cong<m_2,\ldots,m_r|[m_2,l_2],\ldots,[m_r,l_r],MF_{m_2,\ldots,m_r}>,$$

where F denotes the free group generated by $m_2, \ldots, m_r$. Since the components l_1 and l_1' are missing, it follows from assumptions on the links and by Link Composition Lemma 2.5 that the link $(l_2, \ldots, l_r, l_2', \ldots, l_r')$ is homotopically trivial, so its Milnor group is the free Milnor group generated by $m_2, \ldots, m_r, m_2', \ldots, m_r'$:

$$M(l_2 \cup \ldots \cup l_r \cup l_2' \cup \ldots \cup l_r') \cong M(F_{m_2,\ldots,m_r,m_2',\ldots,m_r'}).$$

In particular, the relations $[m_i, l_i]$ are consequences of the relations in this free Milnor group, so $M(\pi_1 M^4) \cong M(F_{m_2,\ldots,m_r})$. Consider the commutative diagram

$$\begin{array}{ccc} M(l_2 \cup \ldots \cup l_r \cup l_2' \cup \ldots \cup l_r') & \xrightarrow{\ i\ } & M\pi_1(M^4) \\ {\scriptstyle M_1}\big\downarrow & & {\scriptstyle M_2}\big\downarrow \\ R(x_2, \ldots, x_r, x_2', \ldots, x_r') & \xrightarrow{\ \phi\ } & R(x_2, \ldots, x_r) \end{array}$$

where i is the map induced by inclusion, $i(m_i) = m_i$, $i(m_i') = m_i^{g_i}$ for some $g_i \in M\pi_1(M^4)$; $\phi(x_i) = x_i$, $\phi(x_i') = (1+\gamma_i)(1+x_i)(1+\bar{\gamma}_i) - 1$, where $M_2(g_i) = \gamma_i$, $M_2(g_i^{-1}) = \bar{\gamma}_i$. The homomorphisms M_1 and M_2 are the Magnus expansions. Notice that i is a well-defined homomorphism of Milnor groups, since $i(m_i)$ and $i(m_i')$ commute with their conjugates in $M\pi_1(M^4)$.

Let μ_1 and μ_2 be the coefficients of $x_2 x_3 \cdots x_r$ in the expansions $M_2(l_1)$ and $M_2(l_1')$ respectively. The components l_1 and $\bar{l}_1'$ are conjugate in $M\pi_1(M^4)$ since they cobound a grope of class r in M^4, hence $\mu_1 = -\mu_2$. Here $\bar{l}_1'$ denotes l_1' with the opposite orientation. The coefficient μ_1 is equal to the sum of coefficients of all terms of the form $x_2^{(')} x_3^{(')} \ldots x_r^{(')}$ in $M_1(l_1)$, where the notation indicates that each multiple is either x_i or x_i', and the analogous statement holds for μ_2.

Since $(l_2', \ldots, l_p')$ is homotopically trivial in the solid torus T_3, $\mu_1 = \bar{\mu}_{K\cup L}(1, \ldots, r)$. To compute μ_2 notice that by the almost triviality of K' in the solid torus T_3 a non-trivial term of the form above has to contain $x_2' \cdots x_p'$. Similarly by the almost triviality of L in the solid torus T_2 it has to contain either $x_{p+1} \cdots x_r$ or $x_{p+1}' \cdots x_r'$. The only two possibilities are $x_2' \cdots x_p' x_{p+1} \cdots x_r$ and $x_2' \cdots x_p' x_{p+1}' \cdots x_r'$, so $\mu_2 = \bar{\mu}_{K\cup L}(1, \ldots, r) + \bar{\mu}_{K'\cup L}(1, \ldots, r)$. This concludes the proof of Lemma 5.2. $\square$

The remaining part of this section contains the proofs of Lemmas 3.8 and 3.9.

Proof of Lemma 3.8. Fix $-k \le i \le -1$ and $1 \le j \le m$, the case $(i, j) \in \{1, \ldots, m\} \times \{-(k-1), \ldots, -1\}$ is treated analogously. Suppose $\widehat{L}_{i,j}$ is

not almost homotopically trivial. By Lemma 3.6 (iii), $L_{i,j}$ is homotopically trivial in S^3, hence it contains a $\wedge_{R_j}$-essential proper sublink. Let F be the family of all $\wedge$-essential sublinks of the links $L_{i,1}, \ldots, L_{i,m-1}$ and let n be the minimal number of components among the links in F. By assumption, $n < n_i$, where $n_i = |L_i|$. Consider a link M in F which has n components, and which is rightmost among all such links. In other words, if $M \subset H_p$ and $M' \subset H_{p'}$ is another link in F with $p' > p$, then $|M'| > n$. Denote the components of L_i, corresponding to M, by $k_1, \ldots, k_n$.

Let R_p, $1 \leq p \leq m-1$, be the solid torus containing M, so M is a sublink of $L_{i,p}$. The solid torus W_{p+1} links R_p on the right, and by Link Composition Lemma 2.5 the link $M \cup L_{p+1}$ is homotopically essential and, by the minimality property of M, $M \cup L_{p+1}$ is almost homotopically trivial. Also by the choice of M, the link $M' := (\partial P_{k_1} \cup \ldots \cup \partial P_{k_n}) \cap R_{p+1}$ is $\wedge$-homotopically trivial (in the case $p = m-1$, this is a vacuous link).

Now consider the link $(k_1 \cup \ldots \cup k_n) \cup L_{p+1}$ and the (planar surfaces)-like gropes it bounds. Consider the solid tori R_i, W_i, R_{i+1} and R_p, W_{p+1}, R_{p+1}. As in the part of the proof of Theorem 1 contained in Sect. 3 (independent of this lemma) the gropes in question may be modified so that their boundary components are contained in these six solid tori, and so that $\partial P_{L_{p+1}} \cap R_{p+1} = \emptyset$, $\partial P_{L_{p+1}} \cap R_p = \emptyset$. The link $(k_1, \ldots, k_n)$ in W_i is a proper sublink of L_i, so by assumption (ii) in Theorem 1 it is homotopically trivial in the solid torus W_i. This means that the link under consideration in $R_i \cup W_i \cup R_{i+1}$ is homotopically trivial. Attach a collar to B^4 and let this link bound disjoint immersed disks in it.

The link left in the boundary of the four-ball is $(M \cup L_{p+1} \cup M') \subset R_p \cup W_{p+1} \cup R_{p+1}$. In the case $p = m-1$ this already gives a contradiction with the Grope Lemma 2.8: $M \cup L_{p+1}$ is a homotopically essential link bounding in B^4 disjoint gropes of a large class. The contradiction finishes the proof of lemma 3.8 in this case.

Suppose $p < m-1$. The link M' is $\wedge$-homotopically trivial, however it might be not homotopically trivial in the solid torus R_{p+1}. We will use lemma 5.1 to find a contradiction with the Grope Lemma. Connect the corresponding components of M and M' by arcs in the annulus-like gropes they bound in B^4. By Lemma 6.4 these arcs may be pulled up to $(S^3 \setminus W_{p+1})$ without introducing intersections between the gropes bounded by different components of M, but they might now intersect $P_{L_{p+1}}$. These intersections are resolved by first pushing them down to the first stages of the gropes (see 2.5 in [5]) and then performing finger moves on $P_{L_{p+1}}$ that create new boundary components for $P_{L_{p+1}}$ – small circles linking the arcs in S^3. Each intersection point between the k-th and the l-th stages of two gropes creates 2^{k+l} small circles.

The following argument shows how to modify the gropes in order to eliminate these new boundary components, without changing the rest of the link. Fix one of these new circles, c, and recall that c and some component l of L_{p+1} coubound a punctured (planar surface)-like grope, and are allowed to intersect. The link $(L_{p+1} \smallsetminus l)$ is homotopically trivial in the solid torus W_{p+1}. Let A be a link-homotopy in a collar $S^3 \times [0,1]$ on B^4 which restricts to null-homotopy of $(L_{p+1} \smallsetminus l)$ in W_{p+1}. The link $M \cup M'$ is ribbon, so the whole link $K := (M \cup (L_{p+1} \smallsetminus l) \cup M' \cup$ small circles) is concordant, hence by Theorem 2.2 link-homotopic in $S^3 \times [1,2]$ to the unlink. (The component l moves between times 0 and 2 by an isotopy in the complement of K). In $S^3 \times [2,3]$, c can be just taken off the rest of the link $K \cup l$, possibly introducing intersections between c and l, and capped off with a disk. Now run the homotopy A backwards (except for c) in $S^3 \times [3,4]$. This procedure is repeated to eliminate all small circles.

Now take the band sum of M and M' along the arcs we have in $S^3 \smallsetminus W_{p+1}$ and apply Lemma 5.1 to conclude that the link $(M \sharp M') \cup L_{p+1}$ is homotopically essential. By Grope Lemma 2.8 this contradicts the existence of gropes of large class it bounds in B^4. This means that the link $\widehat{L}_{i,j}$ is in fact almost homotopically trivial and concludes the proof of Lemma 3.8. □

Proof of Lemma 3.9. Fix $-k \leq i \leq -1$ and $1 \leq j \leq m$, and consider the solid tori R_{j-1}, W_j and R_j. Let $L_{i,j-1} \sharp L_{i,j} = (l_1, \ldots, l_{n_i})$ denote a connected sum of $L_{i,j-1}$ and $L_{i,j}$ such that the connecting bands lie in the complement of the solid torus W_j. Label the components of L_j by $n_i + 1, \ldots, n_i + n_j$. (The labelings of both links should obey the order defined in the proof of Theorem 1.) The following proposition provides a geometric interpretation of the map C.

Proposition 5.3

$$C_{i,j} = \bar{\mu}_{L_i \cup (L_{j,i-1} \sharp L_{j,i})}(1, \ldots, n_i + n_j)$$

for $-k \leq i \leq -1$ *and* $2 \leq j \leq m-1$*;*

$$C_{i,1} = \bar{\mu}_{L_{i,1} \cup L_1}(1, \ldots, n_i + n_1),$$

$$C_{i,m} = \bar{\mu}_{L_{i,m-1} \cup L_m}(1, \ldots, n_i + n_m).$$

The analogous equalities also hold for $1 \leq i \leq m$ *and* $-(k-1) \leq j \leq -1$.

Proof. For $j \neq 1, m$,

$$C_{i,j} = \sum_{p=-(k-1)}^{m-1} B_{i,p} \cdot A_{p,j} = B_{i,i} \cdot A_{i,j} + B_{i,i+1} \cdot A_{i+1,j},$$

since $B_{i,p} = 0$ unless $p = i-1, i$. Recall that the (non-trivial) entries of the matrices A, B are given by

$$B_{i,p} = \bar{\mu}_{\widehat{L}_i}(1, \ldots, n_i, \wedge_{W_i}), \;\; A_{p,i} = \bar{\mu}_{\widehat{L}_{i,k}}(1, \ldots, n_i, \wedge_{R_k}).$$

Hence

$$\begin{aligned} C_{i,j} = {} & \bar{\mu}_{\widehat{L}_i}(1, \ldots, n_i, \wedge) \cdot \bar{\mu}_{\widehat{L}_{j,i-1}}(1, \ldots, n_j, \wedge) \\ & + \bar{\mu}_{\widehat{L}_i}(1, \ldots, n_i, \wedge) \cdot \bar{\mu}_{\widehat{L}_{j,i}}(1, \ldots, n_j, \wedge). \end{aligned}$$

By Composition Lemma 2.5 (i) and Lemma 5.1 this is equal to

$$\begin{aligned} & \bar{\mu}_{L_i \cup L_{j,i-1}}(1, \ldots, n_i + n_j) + \bar{\mu}_{L_i \cup L_{j,i+1}}(1, \ldots, n_i + n_j) \\ & = \bar{\mu}_{L_i \cup (L_{j,i} \sharp L_{j,i})}(1, \ldots, n_i + n_j). \square \end{aligned}$$

The proof of Lemma 3.9 is divided into two steps. We will first prove that $C_{i,j} = -C_{j,i}$ for $(i,j) \neq (-1,1), (1,-1)$. Consider two groups of three solid tori each: R_{i-1}, W_i, R_i and R_{j-1}, W_j, R_j, the links L_i, L_j and the (planar surfaces)-like gropes P_{L_i}, P_{L_j} they bound in B^4. As in the proof of Theorem 1 in Sect. 3, one may assume that $\partial P_{L_i} \cap (R_{i-1} \cup R_i) = \emptyset$ and $\partial P_{L_j} \cap (R_{j-1} \cup R_j) = \emptyset$. One may also assume that ∂P_{L_i} and ∂P_{L_j} are disjoint from all other solid tori except the six ones under consideration.

Connect the corresponding components of $L_{i,j-1}$ and of $L_{i,j}$ by arcs in the gropes they bound. They can be pulled up to arcs in $S^3 \setminus W_j$, but possibly intersections between P_{L_i} and P_{L_j} are introduced. These intersections can be resolved by finger moves which produce new boundary components for P_{L_j} – small circles linking the connecting arcs in S^3. Just as in the proof of lemma 3.8, these circles can be disregarded. Apply the same arguments to the triple of tori R_{i-1}, W_i, R_i.

Now there are two $(n_i + n_j)$-component links in S^3: $L_i \cup (L_{j,i-1} \sharp L_{j,i})$ and $L_j \cup (L_{i,j-1} \sharp L_{i,j})$, separated by a 2-sphere and disjoint singular annulus-like gropes in B^4 connecting them. An application of Proposition 5.3 and the grope-concordance invariance (Corollary 2.9) conclude the proof of step 1.

It remains to be shown that $C_{-1,1} = -C_{1,-1}$. This is a formal linear-algebraic argument that uses the result of step 1 and the fact that C factors through the near-diagonal matrix B. It follows from the definition that A, B and C are block matrices

$$A = \begin{pmatrix} 0 & A'' \\ A' & 0 \end{pmatrix}, \;\; B = \begin{pmatrix} B' & 0 \\ 0 & B'' \end{pmatrix}, \;\; C = \begin{pmatrix} 0 & -C'' \\ C' & 0 \end{pmatrix}$$

We have proved in step 1 that C' and C'' are related as follows:

$$C' = \begin{pmatrix} v & c' \\ D & w \end{pmatrix}, \;\; C'' = \begin{pmatrix} v^t & D^t \\ c'' & w^t \end{pmatrix}.$$

Here $c' = C_{1,-1}$ and $c'' = C_{-1,1}$ are integers, v is a row of length $(k-1)$, w is a column of height $(m-1)$ and D is an $(m-1)\times(k-1)$-matrix. We need to show that $c' = c''$. The blocks B', B'' of B are of the form

$$B' = \begin{pmatrix} b_{-k} & 0 & 0 & 0 & 0 \\ b_{-(k-1)} & b_{-(k-1)} & 0 & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & b_{-3} & b_{-3} & 0 \\ 0 & 0 & 0 & b_{-2} & b_{-2} \\ 0 & 0 & 0 & 0 & b_{-1} \end{pmatrix},$$

$$B'' = \begin{pmatrix} b_1 & 0 & 0 & 0 & 0 \\ b_2 & b_2 & 0 & 0 & 0 \\ 0 & b_3 & b_3 & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & b_{m-1} & b_{m-1} \\ 0 & 0 & 0 & 0 & b_m \end{pmatrix}$$

where $b_i = \bar{\mu}_{\widehat{L}_i}(1,\ldots,n_i,\wedge_{W_i}) \neq 0,\ \ i = -k,\ldots,m$. Let B_1, B_2 denote the rows with the entries

$$B_1 = (b_{-1}/b_{-2}, -b_{-1}/b_{-3}, \ldots, (-1)^k b_{-1}/b_{-k}),$$
$$B_2 = (b_1/b_2, -b_1/b_3, \ldots, (-1)^m b_1/b_m).$$

Note that

$$c' = B_2 \cdot w, \quad v = B_2 \cdot D, \quad c'' = B_1 \cdot v^t, \quad w^t = B_1 \cdot D^t.$$

Hence

$$c' = B_2 \cdot w = B_2 \cdot (B_1 \cdot D^t)^t = B_2 \cdot D \cdot B_1^t,$$
$$c'' = B_1 \cdot v^t = B_1 \cdot (B_2 \cdot D)^t = B_1 \cdot D^t \cdot B_2^t,$$

so $c' = (c'')^t = c''$. This concludes the proof of Lemma 3.9. $\square$

6. A pull-up procedure for surfaces in the four-ball

The purpose of this section is to describe a "pull-up" procedure for arcs in surfaces, properly immersed in (B^4, S^3). In some sense linking of surfaces in B^4 is reflected in linking of their boundaries in S^3 after the procedure is applied. Consider two examples as an elementary illustration of this idea.

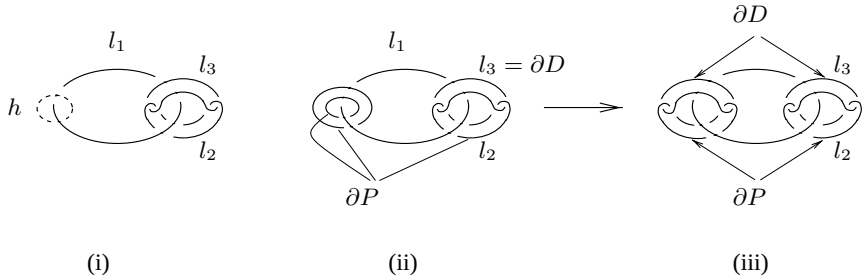


Fig. 8 Example 6.1

Example 6.1 Let $L = (l_1, l_2, l_3)$ be the Borromean rings and let $H = (h)$ be a small circle linking l_1, Fig. 8 (i). As before, H is drawn dashed. L may be unlinked by intersecting l_1 and l_2 two times, with the opposite signs. These intersections may be resolved by letting l_2 go twice (algebraically trivially) over the 2-handle attached to h. Forgetting the 2-handle, this exhibits l_1 and l_3 as boundaries of disks, and l_2 bounds in B^4 a disjoint from them pair of pants P, the other two boundary components of which are parallel copies of h, Fig. 8 (ii). Just looking at ∂P in S^3, it is unclear whether these two boundaries of P are "essential", or whether they can be cancelled to replace the pair of pants P by a disk bounded by l_2.

Connect this algebraically cancelling pair of components of ∂P by an arc α in P, and pull this arc by an ambient isotopy to $S^3 = \partial B^4$. The arc α links in B^4 the disk D bounded by l_3, and to preserve disjointness of P and D, α will pull to S^3 a little patch of D. Puncturing D will introduce a new boundary component of D, linking α in S^3. Taking connected sum of two components of ∂P along α gives Fig. 8 (iii), where P and D are annuli. Now the link seen in the tubular neighborhood N of h is clearly essential (considered with the meridian of N.)

Example 6.2 Consider the unlink $L = (l_1, l_2)$, and let $H = (h)$ be a meridian to l_1, as in the previous example, Fig. 9 (i). Consider disjoint disks bounded by l_1 and l_2, pushed from S^3 into B^4. Move the disk P, bounded by l_2, by an ambient isotopy of $B^4 \cup_h$ 2-handle, so that it goes geometrically twice over the 2-handle. Figure 9 (ii) shows the link in $S^3 = \partial B^4$ (disregarding the 2-handle.) Note that the link in the neighborhood of h is identical to that in the previous example. Applying to P the same procedure as above, however, takes ∂P off l_1, Fig. 9 (iii).

To state the general result, let (L, H) be a relatively slice pair of links, so the components of L bound disjoint disks in $B^4 \cup_{H,0-framings} 2-$handles.

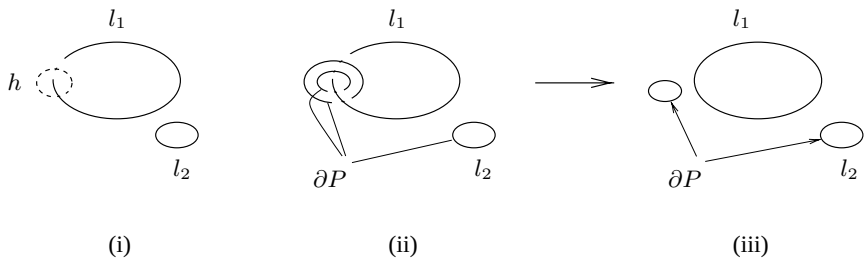


Fig. 9 Example 6.2

For each component h of H, let N_h denote its tubular neighborhood – the attaching region of the corresponding 2-handle. One may assume that all solid tori N_h are disjoint from each other and from L. Disregarding the 2-handles, L bounds in B^4 disjoint planar surfaces, the other boundary components of which are untwisted parallel copies of the components of H. For each component l of L, let P_l denote the surface it bounds in B^4, and let $\partial = \cup_{l\in L}\partial P_l$ denote the boundary of all surfaces.

Lemma 6.3 *Let (L, H) be a pair of links with H an unlink, and let n be a positive integer. Assume that the components of L bound disjoint disks in $B^4 \cup_H$ 0-framed 2-handles. Then the associated planar surfaces $\{P_l\}_{l\in L}$ can be modified, possibly changing their boundary components other than L, introducing self-intersections and inserting gropes of class n, so that*

i) for each $l \in L$ and for each $h \in H$, P_l has exactly one boundary component in N_h,
ii) $P_l \cap P_{l'} = \emptyset$ if $l \neq l'$,
iii) $(\partial \smallsetminus L)$ is a ribbon link contained in $\cup_{h\in H} N_h$.

Remarks. Property (i) is the main result of the lemma, while (iii) says that it can be achieved without making the link in S^3 much worse – to start with, $(\partial \smallsetminus L) \subset \cup_{h\in H} N_h$ was an unlink. The conclusion that the link is ribbon implies, in particular, that all its $\bar{\mu}$-invariants vanish. In the applications of lemma 6.3, n will be the number of components of L. We allow insertion of gropes of class n in the surfaces because in terms of link homotopy, disjoint gropes of a sufficiently large class are as good as disjoint disks, compare Grope Lemma 2.8.

Lemmas 3.6 and 6.3 slightly differ in that the solid torus R_j in the statement of 3.6 is replaced by the tubular neighborhood N_h here (so that lemma 6.3 could potentially be applicable to more general link pairs than those considered in Theorem 1.) However, the proofs of these lemmas are identical.

We give two proofs of lemma 6.3. The first proof is more elementary, and is reduced to an algebraic lemma about nilpotent groups. The second proof is geometric and is more explicit. We present two arguments, since both may be useful in the study of the general relative-slice problem. For the first proof we need the following lemma.

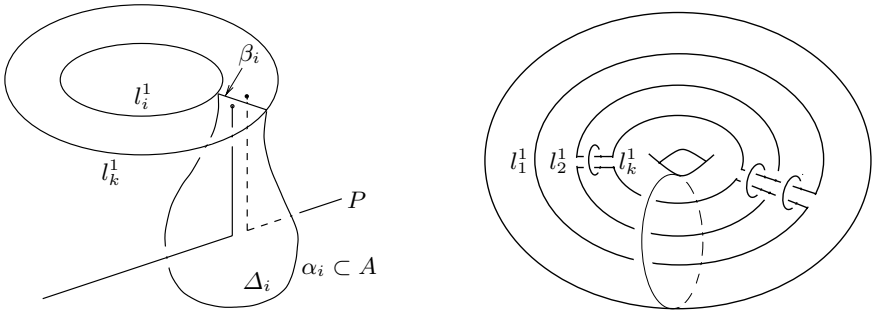
Lemma 6.4 (Lemma 14 in [10]) *Let $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_k$ be a collection of properly immersed disjoint compact connected surfaces in B^4 with $\partial\Sigma_i \neq \emptyset$ for each $i = 1, \ldots, k$. Let $(\alpha, \partial\alpha)$ be an arc in $(B^4 \smallsetminus \Sigma, S^3 \smallsetminus \partial\Sigma)$, and let n be a positive integer. Then there exists an arc $\beta \subset S^3 \smallsetminus \partial\Sigma$ with $\partial\beta = \partial\alpha$ such that $\alpha \cup \beta$ bounds an immersed grope G of class n in $B^4 \smallsetminus \Sigma$.*

Remarks. One can easily construct an example of surfaces Σ and an arc α such that $\alpha \cup \beta$ does not bound in $B^4 \smallsetminus \Sigma$ an immersed *disk* for any choice of β. Lemma 6.4 also holds if the surfaces Σ are replaced by a collection of properly immersed disjoint *gropes*.

Proof of Lemma 6.3. For each component h of H and for each planar surface P with $\partial P \cap N_h = \emptyset$, introduce a thin finger leading from P to $N_h \subset S^3$, disjoint from all other surfaces, so that a new boundary component of P – a small circle in N_h – is introduced. Now if some planar surface Q has more than one boundary component in N_h, say, $\partial Q \cap N_h = q_1 \cup \ldots \cup q_s$, $s > 1$, connect each q_i, $i = 1, \ldots, s-1$ by an arc α_i in Q with q_s. Since each surface has a boundary component in N_h, Lemma 6.4 provides an arc $\beta_i \subset N_h$ such that $\alpha_i \cup \beta_i$ bounds an immersed grope G_i in the complement of other surfaces. Singular surgeries on Q along G_i, $i = 1, \ldots, s-1$ improve Q to satisfy condition (i) above without violating (ii) and (iii). Applying this procedure to each torus N_h and surface Q, we get a collection of (planar surfaces)-like gropes $\{P_l\}$ of class n bounded by the components of L satisfying conditions (i)-(iii). $\square$

Alternative proof of Lemma 6.3. Fix a component h of H. For each surface P with $\partial P \cap N_h = \emptyset$ introduce a thin finger leading to $N_h \subset S^3$, disjoint from all other surfaces, so that a new boundary component of P – a small circle in N_h – is introduced. The proof consists of $(n+1)$ steps which gradually improve the planar surfaces.

Step 1. Suppose a surface A has more than one boundary component in N_h. Denote $\partial A \cap N_h$ by $l_1^1, \ldots, l_k^1$ (the superscript indicates the number of the step). Connect the components l_i^1, $i = 1, \ldots, k-1$ with l_k^1 by embedded arcs $\alpha_i \subset A$ and $\beta_i \subset (N_h \smallsetminus \partial)$. Let Δ_i denote an immersed disk bounded by $\alpha_i \cup \beta_i$ in B^4. Suppose $P \neq A$ is a surface intersecting Δ_i. Perform finger moves on P along arcs in Δ_i connecting the intersection points $P \cap \Delta_i$ with β_i. This makes P disjoint from Δ_i but introduces new boundary components



(i) (ii)

Fig. 10

of P – small circles linking β_i in S^3. Figure 10 (i) illustrates this move in the lower dimension (B^3, S^2).

Apply this procedure to every surface $P \neq A$ intersecting Δ_i, making all finger moves disjoint from each other. Now perform singular surgery on A along Δ_i, so that the two boundary components l_i^1 and l_k^1 are connected by a band along β_i, Fig. 10 (ii). The disk Δ_i was made disjoint from all planar surfaces except A, so the only singularities possibly created by the surgery are self-intersections of A. An application of this construction to each $1 \leq i \leq k-1$ implies property (i) for A and N_h, but possibly creates in N_h many new boundary components of other surfaces. Apply this to each planar surface A that has more than one boundary component in N_h, not taking into consideration the small circles created by the previous A's during this step. Notice that these small circles are better than the original components l_i^1's in the sense that they bound gropes of class 2 (genus one surfaces) in S^3 disjoint from each other and from all other curves (punctured circle bundles over l_k^1.) This is the progress achieved by step 1.

If step 1 did not introduce any small linking circles to the bands, this would be the end of the process (for the component h): each surface would have exactly one boundary component in N_h.

Step k, $2 \leq i \leq n$. Suppose after step $(k-1)$ a surface B has more than one boundary component in N_h. Denote $\partial B \cap N_h$ by $l^{k-1}, l_1^k, \ldots, l_m^k$ where $l_1^k, \ldots, l_m^k$ are small circles and l^{k-1} is the "long curve" created by step $k-1$. Connect $l_1^k, \ldots, l_m^k$ to l^{k-1} by arcs in B and in N_h and proceed as in Step 1. Many new little circles l_j^{k+1}'s are introduced, however they bound in S^3 disjoint gropes of class k. Their stages 2 through k are parallel

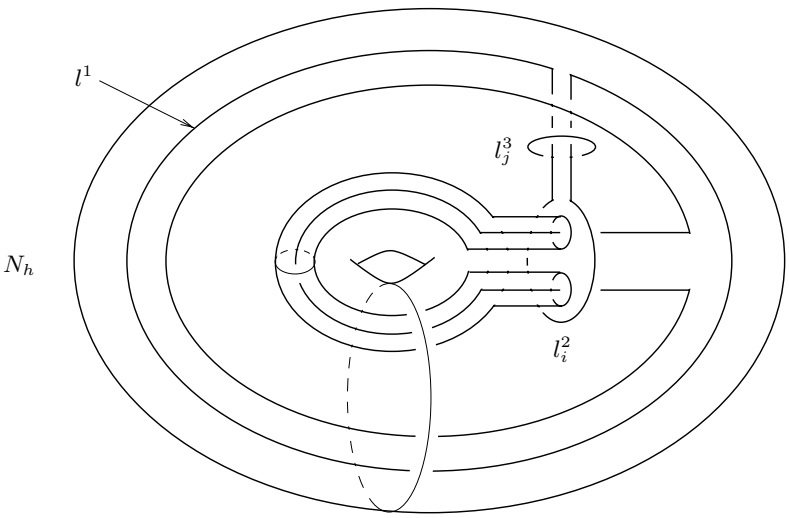


Fig. 11 l_j^3 bounds a grope of class 3

copies of the gropes of class $(k-1)$, bounded by l_i^k's. Figure 11 shows the situation after step 2.

Step $(n+1)$. The small circles introduced during step n bound disjoint from each other and from all curves in S^3 gropes of class n. Push these gropes into B^4 and consider them as parts of "planar surfaces". Now each (planar surface)-like grope has exactly one boundary component in N_h. Condition (iii) holds since to start with $(\partial \setminus L)$ was an unlink, and the singularities introduced at each step are ribbon. $\square$

7. Appendix. Bing doubling a pair of pants (after Michael Freedman)

This section describes a geometric construction, lemma 7.1, which is used in Sect. 4 to give an alternative proof of theorem 1 in the special case when each link L_i is a Bing double. We also present a related construction, lemma 7.3, which may be applied directly to the A-B-slice problem (see [4]) to show that homotopically essential links are not A-B-slice for certain decompositions (A, B). Given a link $L = (l_1, \ldots, l_n)$ which is assumed to be (A, B)-slice, the idea is to find for each i a $\wedge$-essential link in the attaching region of one of the handlebodies A_i, B_i the components of which bound disjoint immersed disks in the handlebody. If L is homotopically essential, this gives a contradiction with the Link Composition Lemma.

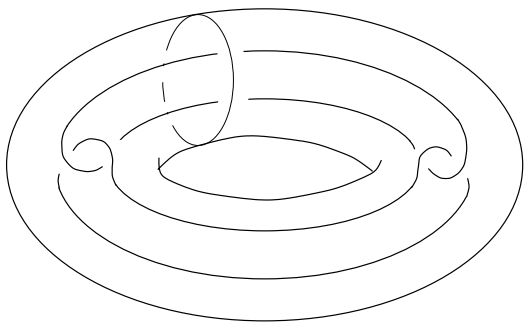


Fig. 12 Bing double

An *untwisted Bing double* $B(L)$ of a link L is obtained by replacing each component of L within its neighborhood by two components of the form
An untwisted *iterated* Bing double of L is defined inductively, starting with L and at each step Bing doubling some of the components of the link given by the previous step. A special case – an untwisted *symmetric* k-iterated Bing double of L is $B^k(L)$. Note that L is trivially an iterated Bing double of itself.

Lemma 7.1 (Bing doubling a pair of pants) *Let P be a pair of pants (disk with three punctures) with $\partial P = \alpha \cup \beta \cup \gamma$ and set $M = P \times D^2$. Let α' and α'' denote the components of the untwisted Bing double of the core circle $\alpha \times \{0\}$ in $\alpha \times D^2$; define analogously $\beta', \beta'' \subset \beta \times D^2$ and $\gamma', \gamma'' \subset \gamma \times D^2$.*

(i) Then there exist disjoint embedded pairs of pants $P', P'' \subset M$ with $\partial P' = \alpha' \cup \beta' \cup \gamma'$ and $\partial P'' = \alpha'' \cup \beta'' \cup \gamma''$.

(ii) Let N denote $M/$self-plumbings, a thickening of $P/$self-intersections. Then there exist disjoint immersed pairs of pants $P', P'' \subset N$ with $\partial P' = \alpha' \cup \beta' \cup \gamma'$ and $\partial P'' = \alpha'' \cup \beta'' \cup \gamma''$.

Proof. M will be thought of as a subset of $S^3 \times [0,1]$ with $\alpha \times D^2 \subset S^3 \times \{0\}$ and $\beta \times D^2, \gamma \times D^2 \subset S^3 \times \{1\}$ and such that the obvious Morse function on $S^3 \times I$ has just one critical point on P. The pairs of pants P', P'' in case (i) are described in the time slices $S^3 \times \{t\}$, $0 \leq t \leq 1$, in Fig. 13. The Morse function has two critical points on $P \times D^2$, they correspond to the first arrow and the last arrow in the figure. (More precisely, the critical points lie on $\partial(P \times D^2)$.) The middle arrow, between times $1/2$ and $3/4$, corresponds to the critical points of index one on the surfaces P' and P'', one critical point for each surface.

To prove (ii) notice that after an isotopy, the singular points of the quotient map $\pi\colon P \longrightarrow P/$*self-intersections* may be assumed to lie in a collar

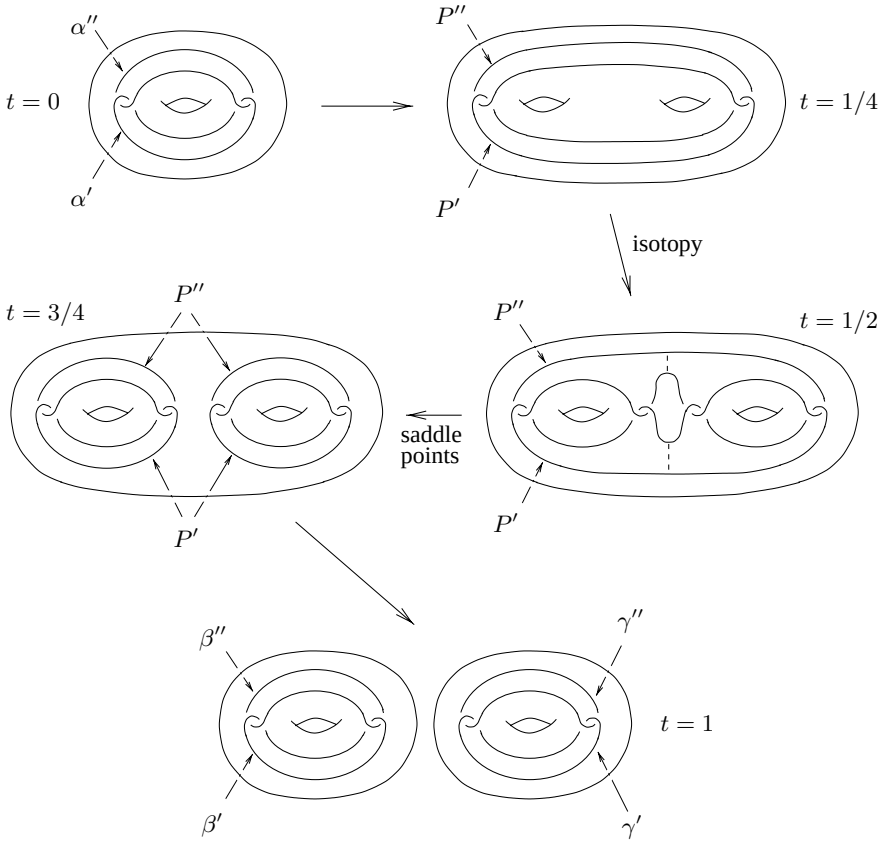


Fig. 13 Bing doubling a pair of pants

$\alpha \times [0, \epsilon] \subset P$, below all critical points of the Morse function. Now consider the two curves of the Bing double on Fig. 12. One of them can be shrunk by an ambient isotopy of the solid torus to be very short, at the expense of making the other curve long. In other words, the image of one curve under the projection $S^1 \times D^2 \longrightarrow S^1$ lies in a small neighborhood of a point, while the image of the other curve covers most of the circle. Thus it may be assumed that only one of the pairs of pants P', P'' described in (i) intersects the singular set of the quotient map $M \longrightarrow N$. This concludes the proof of Lemma 7.1. $\square$

Corollary 7.2 *Let P be a planar surface with the boundary components $\alpha, \beta_1, \ldots, \beta_n$ and set $N = P \times D^2/$self-plumbings. Fix an integer $k \geq 1$ and let $\beta_i^1, \ldots, \beta_i^{(2^k)}$ denote the components of the untwisted symmetric k-*

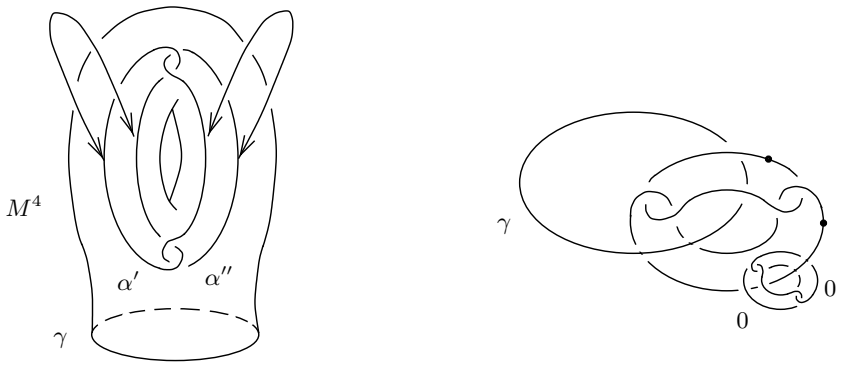


Fig. 14 A schematic picture, and a Kirby handle diagram for M^4

iterated Bing double of β_i in $\beta_i \times D^2$, $i = 1, \ldots, n$; define analogously $\alpha^1, \ldots, \alpha^{(2^k)} \subset \alpha \times D^2$. Then there exist disjoint immersed (embedded if $N = P \times D^2$) planar surfaces $P^1, \ldots, P^{(2^k)} \subset N$ with $\partial P^j = \alpha^j \cup \beta_1^j \cup \ldots \cup \beta_n^j$, $j = 1, \ldots, 2^k$.

Proof. Assume $N = P \times D^2$, the general case with self-plumbings follows as in the proof of Lemma 7.1. The proof is by induction on the number of boundary components of P. The case $n = 2$ is proved by induction on k. If $k = 1$, this is Lemma 7.1, and the case $k > 1$ follows by an application of Lemma 7.1 to the pairs of pants for $(k-1)$-iterated Bing doubles, provided by the induction hypothesis on k. Assume the statement holds for $i < n$. Given P with $\partial P = \alpha \cup \beta_1 \cup \ldots \cup \beta_n$, choose a circle γ in the interior of P cutting P into a pair of pants P_1 with $\partial P_1 = \gamma \cup \beta_1 \cup \beta_2$ and a planar surface P_2 with $\partial P_2 = \alpha \cup \gamma \cup \beta_3 \cup \ldots \cup \beta_n$. By Lemma 7.1 applied to P_1 and the induction hypothesis applied to P_2 the statement also holds for $i = n$. $\square$

We will now describe a related construction which may be applied to the A-B-slice problem to show that homotopically essential links are not A-B-slice for certain decompositions (A, B).

Lemma 7.3 *Let (S, γ) be a once-punctured torus, $N = S \times D^2$, and let $\alpha \subset \partial N$ be an embedded curve representing a standard generator of $H_1(N; \mathbb{Z})$. Denote by α', α'' the components of the untwisted Bing double of α in a tubular neighborhood $\alpha \times D^2_\epsilon \subset \partial N$ and set $M = N \cup_{\alpha', \alpha''} 2-$handles. The precise description of M is given in terms of the Kirby handle diagram, Fig. 14.*

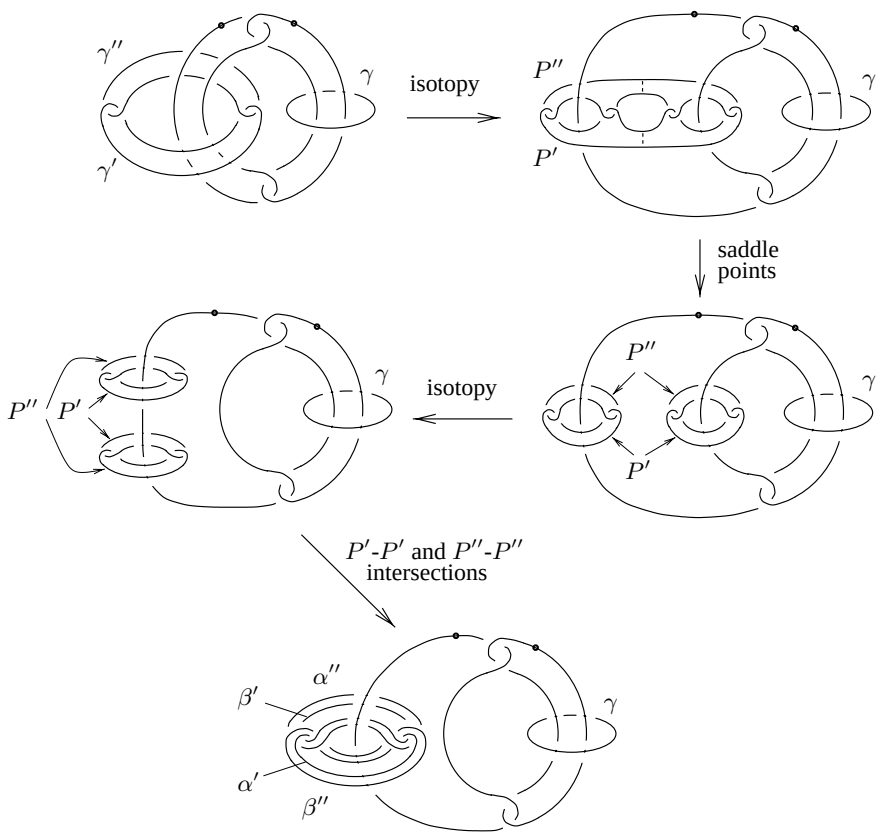


Fig. 15

Then the components γ' and γ'' of the untwisted Bing double of γ in $\gamma \times D^2$ bound in M disjoint immersed disks D', D''.

Proof. Figure 15 describes disjoint immersed pairs of pants P', $P'' \subset N$ with $\partial P' = \alpha' \cup \beta' \cup \gamma'$, $\partial P'' = \alpha'' \cup \beta'' \cup \gamma''$. Here β' is a parallel copy of α'' and β'' is a parallel copy of α'. Now D', D'' are obtained from P', P'' by attaching the cores of the corresponding 2-handles and their parallel copies to α', α'', β', β''. Note that each disk D', D'' goes over both 2-handles. $\square$

Remark. More generally, in the notations of lemma 7.3, *let $\alpha_1, \ldots, \alpha_n$ denote the components of an untwisted iterated Bing double of α in $\alpha \times D^2_\epsilon$, and let $\overline{M}$ denote $N \cup_{\alpha_1,\ldots,\alpha_n} 2-$handles. Then there exists an integer $k \geq 1$*

such that the components $\gamma_1, \ldots, \gamma_{(2^k)}$ *of the untwisted symmetric* k*-iterated Bing double of* γ *bound in* $\overline{M}$ *disjoint immersed disks* $D_1, \ldots, D_{(2^k)}$

Proof. Let k be an integer large enough so that the symmetric k-iterated Bing double of α is a refinement of the given Bing double. (Notice that its components bound disjoint embedded disks in the attached $2-$handles.) Let P', P'' be the pairs of pants as in the proof of lemma 7.3. Now an iterated application of Lemma 7.1 to P', P'' concludes the proof of this remark. □

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