q-series invariants and lattice cohomology of plumbed 3-manifolds

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Joint work with R. Akhmechet and P. Johnson

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Outline:

- Quantum invariants and the $\widehat{Z}$ invariant of 3-manifolds
- Lattice cohomology
- A new invariant unifying lattice cohomology and $\widehat{Z}$
- Details and properties
- Questions

Quantum invariants and the $\widehat{Z}$ invariant of 3-manifolds

Denote by $Z_K(M)$ the $\mathrm{SU}(2)$ Witten-Reshetikhin-Turaev invariant of a compact connected orientable 3-manifold M at $\zeta_K = e^{2\pi i/K}$.

Consider the Poincaré homology sphere $\Sigma(2,3,5)$, and let $W(\zeta_K)$ denote its renormalized WRT invariant

$$W(\zeta_K) = \zeta_K\,(\zeta_K - 1)\,Z_K(\Sigma(2,3,5)).$$

R. Lawrence and D. Zagier, 1999: For $|q| < 1$ consider

$$A(q) = \sum_{n=1}^{\infty} \chi_+(n) q^{(n^2-1)/120} = 1 + q + q^3 + q^7 - q^8 - q^{14} - q^{20} - \dots$$

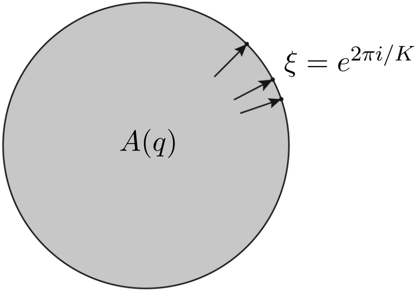
where $\chi_+ \colon \mathbb{Z} \longrightarrow \{-1, 0, 1\}$ is given by:

$n \pmod{60}$	1	11	19	29	31	41	49	59	(other)
$\chi_+(n)$	1	1	1	1	-1	-1	-1	-1	0

$A(q)$ is a holomorphic function in the unit disk.

$$A(q) = \sum_{n=1}^{\infty} \chi_+(n) q^{(n^2-1)/120} = 1 + q + q^3 + q^7 - q^8 - q^{14} - q^{20} - \dots$$

Theorem (Lawrence-Zagier, 1999) Let ξ be a root of unity. Then the radial limit of $1 - \frac{1}{2}A(q)$ as q tends to ξ equals $W(\xi)$, the renormalized WRT-invariant of the Poincaré homology sphere.



(Building on calculations of the WRT invariants by R. Lawrence and L. Rozansky)

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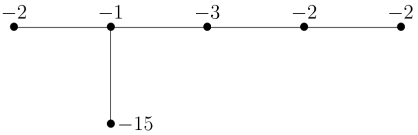
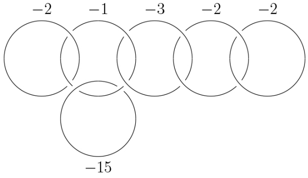
- More generally, Lawrence-Zagier proved the analogous result for three-fibred Seifert integer homology spheres.
- This result led to Zagier's notion of a *quantum modular form*.
- Gukov-Pei-Putrov-Vafa (2020): The $\widehat{Z}$-invariant for a more general class class of plumbed 3-manifolds (discussed next), based on the theory of BPS states.

Plumbed 3-manifolds

A negative definite plumbing Γ and its associated framed link $\mathcal{L}(\Gamma)$. The 3-manifold $Y(\Gamma)$ is the Brieskorn sphere $\Sigma(2,7,15)$:



(A) A plumbing tree Γ.



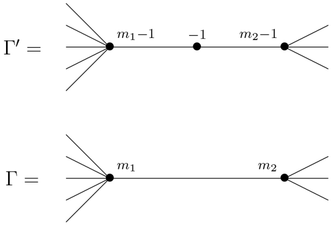
(B) The framed link $\mathcal{L}(\Gamma)$.

Weights (framings) $m\colon \mathcal{V}(\Gamma) \longrightarrow \mathbb{Z}$.

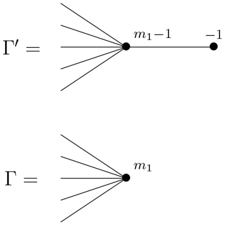
The plumbing tree is negative definite if the associated symmetric matrix $M = M(\Gamma)$ is negative definite:

$$M_{i,j} = \begin{cases} m_i & \text{if } i = j, \\ 1 & \text{if } i \neq j, \text{ and } v_i \text{ and } v_j \text{ are connected by an edge}, \\ 0 & \text{otherwise.} \end{cases}$$

Two negative definite plumbing trees represent diffeomorphic 3-manifolds if and only if they are related by a finite sequence of type (a) and (b) Neumann moves:



(A) Type (a) move.



(B) Type (b) move.

The $\widehat{Z}$ invariant of 3-manifolds

Gukov-Pei-Putrov-Vafa (2020): Given a negative definite plumbed 3-manifold Y with a spin^c structure a, consider

$$\widehat{Z}_{Y,a}(q) = q^{-\frac{3s+\sum_v m_v}{4}} \cdot v.p. \oint_{|z_v|=1} \prod_{v \in \mathcal{V}(\Gamma)} \frac{dz_v}{2\pi i z_v} \left(z_v - \frac{1}{z_v} \right)^{2-\delta_v} \cdot \Theta_a^{-M}(z),$$

$$\text{where } \Theta_a^{-M}(z) := \sum_{\ell \in a + 2M\,\mathbb{Z}^s} q^{-\frac{\ell^t M^{-1} \ell}{4}} \prod_{v \in \mathcal{V}(\Gamma)} z_v^{\ell_v}.$$

For example, for the integer homology sphere $Y = \Sigma(2,7,15)$ (and its unique spin^c structure)

$$\widehat{Z}_{\mathfrak{s}_0}(q) = q^{13/2} - q^{23/2} - q^{39/2} + q^{57/2} - q^{179/2} + q^{217/2} + q^{265/2} \pm \cdots$$

The $\widehat{Z}$ invariant of 3-manifolds

- For three-fibred Seifert integer homology spheres (unique spin^c structure), the $\widehat{Z}$-invariant recovers the q-series of Lawrence-Zagier.
- GPPV conjecture that a certain linear combination over spin^c-structures has radial limits equal to WRT invariants (generalizing the result of Lawrence-Zagier).
- Conjecturally $\widehat{Z}$ admits a categorification:

$$\widehat{Z}_{Y,a}(q) = \sum_{i,j} (-1)^i q^j \,\mathrm{rk}\, \mathcal{H}^{i,j}_{\mathrm{BPS}}(Y,a)$$

Conjecture (GPPV) Let Y be a closed 3-manifold with $b_1(Y) = 0$. Set

$$T := \mathrm{Spin}^c(Y)/\,\mathbb{Z}_2\,.$$

There exist invariants

$$\widehat{Z}_a(q) \in 2^{-c} q^{\Delta_a}\,\mathbb{Z}[[q]],$$

with $\widehat{Z}_a(q)$ converging in the unit disk $\{|q| < 1\}$, such that

$$Z_{\mathrm{CS}}(Y; k) = (i\sqrt{2k})^{-1} \sum_{a,b \in T} e^{2\pi i k \cdot \mathrm{lk(a,a)}} |W_b|^{-1} S_{ab} \widehat{Z}_b(q)|_{q \to e^{2\pi i/k}}$$

where

$$S_{ab} = \frac{e^{2\pi i \mathrm{lk(a,b)}} + e^{-2\pi i \mathrm{lk(a,b)}}}{|W_a| \cdot \sqrt{|H_1(Y;\mathbb{Z})|}},$$

Lattice cohomology (Némethi, 2008)

- Given a negative definite plumbed 3-manifold with a spin^c structure $\mathfrak{s}$,

$$\mathbb{H}^*(\Gamma, \mathfrak{s}) = \bigoplus_{i=0}^{\infty} \mathbb{H}^i(\Gamma, \mathfrak{s})$$

  is a $(2\,\mathbb{Z})$-graded $\mathbb{Z}[U]$ module.

- It gives a combinatorial formulation of Heegaard Floer homology HF^+ for a class plumbing trees, by work of Némethi, Ozsváth-Stipsicz-Szabó, Zemke.

  For example, if Γ is almost rational, then as graded $\mathbb{Z}[U]$-modules,

$$\mathbb{H}^i(\Gamma, [k])\,[\text{grading shift}] \cong \begin{cases} HF^+(-Y(\Gamma), [k]) & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

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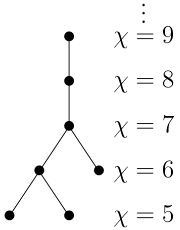
- $\mathbb{H}^0(Y, \mathfrak{s})$ is encoded by the graded root, which was shown by Némethi to be an invariant of $(Y, \mathfrak{s})$.

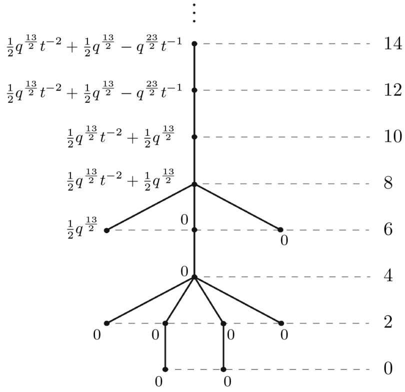
The 0-th lattice cohomology $\mathbb{H}^0(Y, \mathfrak{s})$ is encoded by the graded root, an (infinite) tree which is an invariant of $(Y, \mathfrak{s})$



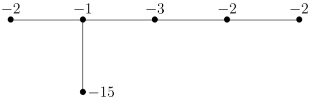
On the other hand, the $\widehat{Z}$-invariant is a q-series.

Our new invariant unifying lattice cohomology and $\widehat{Z}$ takes the form of a

graded root weighted by 2-variable Laurent polynomials



The weighted graded root associated to the Brieskorn homology sphere $\Sigma(2,7,15)$



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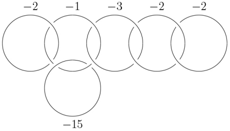
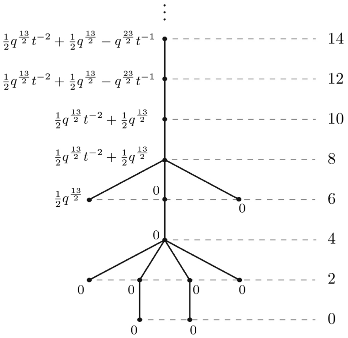
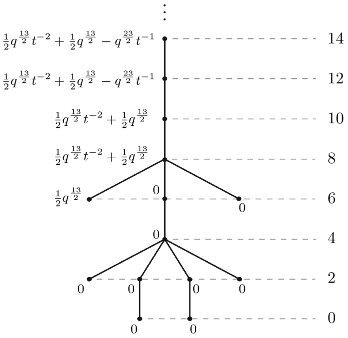


(B) The framed link $\mathcal{L}(\Gamma)$.



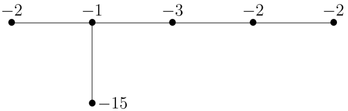
Theorem 1. (Akhmechet-Johnson-K., 2021)
The weighted graded root is an invariant of a 3*-manifold equipped with a* spin^c *structure.*

(Lattice cohomology is recovered by the unlabeled tree.)



Theorem 2. (Akhmechet-Johnson-K.)

- *The sequence of Laurent polynomial weights stabilizes, yielding a* 2*-variable series* $\widehat{\widehat{Z}}_{Y,\mathfrak{s}}(q,t)$.
- *The* 2*-variable series* $\widehat{\widehat{Z}}_{Y,\mathfrak{s}}(q,t)$ *is an invariant of the* 3*-manifold* Y *with a spin*^{c} *structure* $\mathfrak{s}$*, and its specialization at* $t=1$ *equals* $\widehat{Z}_{Y,\mathfrak{s}}(q)$.



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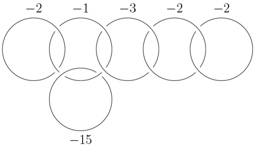


(B) The framed link $\mathcal{L}(\Gamma)$.

Weights (framings) $m\colon \mathcal{V}(\Gamma) \longrightarrow \mathbb{Z}$.

$M\colon\ \mathbb{Z}^s \longrightarrow \mathbb{Z}^s$, $s =$number of vertices of the plumbing graph.

$$\mathrm{spin}^c(Y) \cong \frac{m + 2\,\mathbb{Z}^s}{2M\,\mathbb{Z}^s}.$$

Consider a spin^c representative $k \in m + 2\,\mathbb{Z}^s$.

Define a quadratic function $\chi_k : \mathbb{Z}^s \to \mathbb{Z}$

$$\chi_k(x) = -(k \cdot x + \langle x, x \rangle)/2, \ \text{ where}$$

$$\langle -, - \rangle : \mathbb{Z}^s \times \mathbb{Z}^s \to \mathbb{Z}$$

is the bilinear form associated with M, $\langle x, y \rangle = x^t M y$.

$$\chi_k : \mathbb{Z}^s \to \mathbb{Z}, \quad \chi_k(x) = -(k \cdot x + \langle x, x \rangle)/2$$

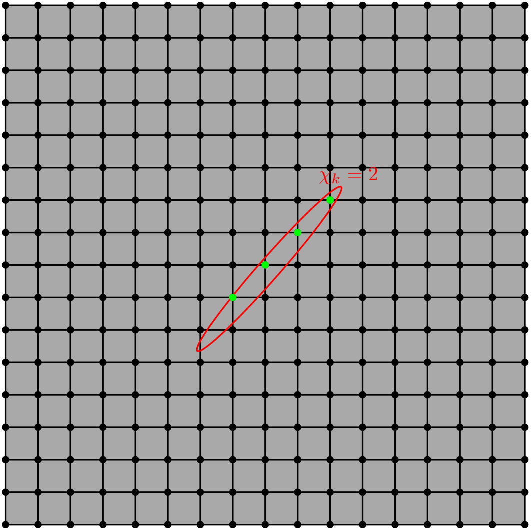
Consider the standard cubulation of $\mathbb{R}^s$ (with vertices in $\mathbb{Z}^s$), and extend χ_k to a function on cells (cubes) $\square$ of any dimension:

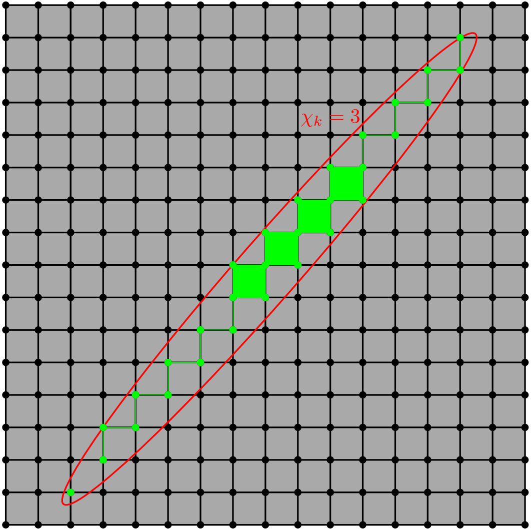
$$\chi_k(\square) := \max\{\chi_k(v) \mid v \text{ is a 0-cell of } \square\}$$

Let $S_j \subset \mathbb{R}^s$ denote the *sublevel set* $\chi_k \leq j$:

S_j is a (compact) subcomplex of the cubulation consisting of cells $\square$ such that $\chi_k(\square) \leq j$.

(Recall that the intersection form $\langle -, - \rangle$ is negative definite!)


$\chi_k = 2$


$\chi_k = 3$

Definition of the graded root (R_k, χ_k), following Némethi:

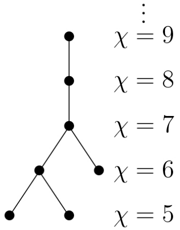
Consider the connected components of each sublevel set:

$$S_j = C_{j,1} \sqcup \cdots \sqcup C_{j,n_j}$$

The vertices of the graded root R_k consist of connected components among all the S_j.

The grading χ_k is given by $\chi_k(C_{j,\ell}) = j$.

Edges of R_k correspond to inclusions of connected components: there is an edge connecting $C_{j,\ell}$ and $C_{j+1,\ell'}$ if $C_{j,\ell} \subseteq C_{j+1,\ell'}$.



Némethi, 2008:
The graded root is an invariant of $(Y, [k])$, and encodes the structure of $\mathbb{H}^0(Y, [k])$.

Next: the new invariant, weighted graded root

A rough idea:

Given a function

$$F_{\Gamma,k} : \mathbb{Z}^s \to \mathcal{R}$$

valued in some ring $\mathcal{R}$, each vertex v in the graded root (R_k, χ_k) can be given a *weight* by taking the sum of $F_{\Gamma,k}$ over lattice points in the connected component C representing v :

$$F_{\Gamma,k}(C) := \sum_{x \in C \cap \mathbb{Z}^s} F_{\Gamma,k}(x).$$

Subtlety: find a function $F_{\Gamma,k}$ so the weights of the graded root are invariant under Neumann's moves on the plumbing trees.

Fix a commutative ring $\mathcal{R}$. A family of functions $F = \{F_n : \mathbb{Z} \to \mathcal{R}\}_{n \geq 0}$ is *admissible* if

1. $F_2(0) = 1$ and $F_2(r) = 0$ for all $r \neq 0$.
2. For all $n \geq 1$ and $r \in \mathbb{Z}$,

$$F_n(r+1) - F_n(r-1) = F_{n-1}(r).$$

Note that not only F_2, but also F_0 and F_1 are uniquely determined by conditions 1 and 2:

$$F_1(r) = \begin{cases} 1 & \text{if } r = -1, \\ -1 & \text{if } r = 1, \\ 0 & \text{otherwise.} \end{cases} \qquad F_0(r) = \begin{cases} 1 & \text{if } r = \pm 2, \\ -2 & \text{if } r = 0, \\ 0 & \text{otherwise.} \end{cases}$$

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A key example ($n \geq 3$):

$$\widehat{F}_n(r) = \begin{cases} \frac{1}{2}\operatorname{sgn}(r)^n \binom{\frac{n+|r|}{2} - 2}{n-3} & \text{if } |r| \geq n-2 \text{ and } r \equiv n \bmod 2 \\ 0 & \text{otherwise.} \end{cases}$$

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A characterization of admissible families $\mathrm{Adm}(\mathcal{R})$:

There is a bijection $\mathrm{Adm}(\mathcal{R}) \cong (\mathcal{R} \times \mathcal{R})^{\mathbb{N}}$

(the set of all sequences with entries in $\mathcal{R} \times \mathcal{R}$.)

$F \ \mapsto \ (F_{n+2}(0), F_{n+2}(1))_{n \geq 1}$ is a bijection.

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For an admissible family $F = \{F_n\}_{n \geq 0}$, define $F_{\Gamma,k} : \mathbb{Z}^s \to \mathcal{R}$ by

$$F_{\Gamma,k}(x) = \prod_{i=1}^{s} F_{\delta_i}\left((2Mx + k - m - \delta)_i\right),$$

where δ is the degree vector of the plumbing graph, and $(-)_i$ denotes the i-th component.

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Lemma: *The graded root with vertex weights*

$$F_{\Gamma,k}(C) := \sum_{x \in L(C)} F_{\Gamma,k}(x)$$

is an invariant of $(Y, [k])$.

Finally, a formulation of the new invariant:

For an admissible family $F = \{F_n\}_{n \geq 0}$, define $F_{\Gamma,k} : \mathbb{Z}^s \to \mathcal{R}$ by

$$F_{\Gamma,k}(x) = \prod_{i=1}^{s} F_{\delta_i}\left((2Mx + k - m - \delta)_i\right),$$

To each $x \in \mathbb{Z}^s$ assign a Laurent polynomial weight

$$F_{\Gamma,k}(x) q^{\varepsilon_k(x)} t^{\langle x,u \rangle}$$

where $\varepsilon_k(x) = \Delta_k + 2\chi_k(x) + \langle x, u \rangle$.

Here Δ_k is an overall normalization used to eliminate dependence on the choice of spin^c representative and is similar in form to the d-invariant from Heegaard Floer homology.

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$$F_{\Gamma,k}(x) = \prod_{i=1}^{s} F_{\delta_i}\left((2Mx + k - m - \delta)_i\right),$$

Theorem: *The graded root with vertex weights*

$$P_{F,k}(C) = \sum_{x\in L(C)} F_{\Gamma,k}(x) q^{\varepsilon_k(x)} t^{\langle x,u\rangle},$$

is an invariant of $(Y,[k])$.

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The above weights can be interpreted geometrically as follows.

For $n \in \mathbb{Z}$, the coefficient of t^n in $P_k(C)$ is given by summing $F_{\Gamma,k}(x) q^{\Delta_k + 2\chi_k(x) + n}$ over all $x \in \mathbb{Z}^s$ which lie on the intersection of C with the hyperplane $\{y \in \mathbb{R}^s \mid \langle y, u \rangle = n\}$.

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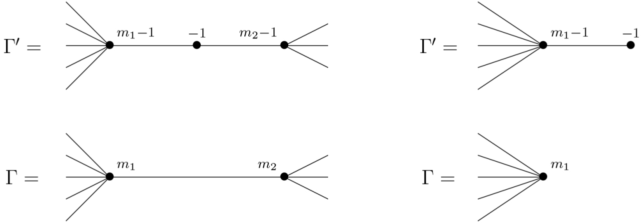
The proof shows invariance under the Neumann moves:



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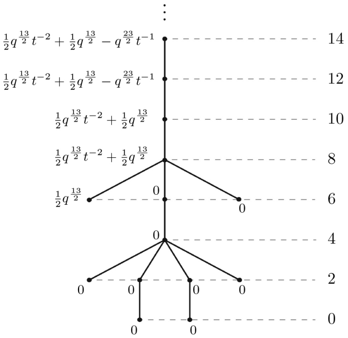

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A new feature: behavior under conjugation of spin^c structures:

Theorem 3. (Akhmechet-Johnson-K.)

$$\widehat{\widehat{Z}}_{Y,\mathfrak{s}}(q,t) = \widehat{\widehat{Z}}_{Y,\bar{\mathfrak{s}}}(q,t^{-1}).$$

In contrast, both lattice cohomology and the $\widehat{Z}$ q-series are known to be invariant under conjugation of the spin^c structure.

In fact, in some examples conjugate spin^c structures may be distinguished by their weighted graded roots.

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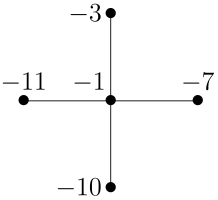

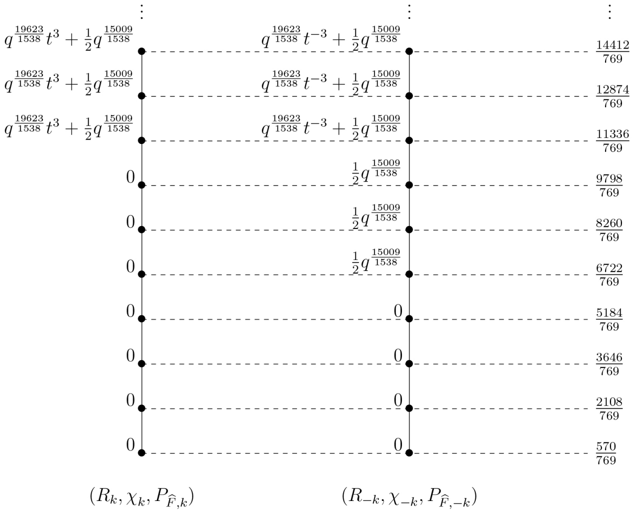

P. Johnson: program for computing the weighted graded root

Summary

- A new connection between quantum topology and Floer theory: the weighted graded root unifies lattice cohomology and the $\widehat{Z}$ invariant.
- A 2-variable series $\widehat{\widehat{Z}}_{Y,\mathfrak{s}}(q,t)$ specializing to $\widehat{Z}_{Y,\mathfrak{s}}(q)$.
- A new feature: distinguishes conjugate spin^c structure.
- Our construction is more general than $\widehat{Z}$: a weighted graded root is built for any choice of *admissible functions* $F = \{F_n : \mathbb{Z} \to \mathcal{R}\}_{n\geq 0}$ where $\mathcal{R}$ is a commutative ring.

Open problems

- Categorification of quantum 3-manifolds invariants?

$$\widehat{Z}_{Y,a}(q) = \sum_{i,j} (-1)^i q^j \,\mathrm{rk}\, \mathcal{H}^{i,j}_{\mathrm{BPS}}(Y,a)$$

When $Y = S^3$,

$$\widehat{Z}_0(q) = q^{-1/2}(-2+2q)$$

The Poincaré series is conjectured to be

$$-2q^{-1/2}\big(1+tq+(t+t^2)q^2+(t+2t^2+t^3)q^3+(t+2t^2+2t^3+t^4)q^4+\dots\big)$$

Open problems

- Categorification of quantum 3-manifolds invariants?

$$\widehat{Z}_{Y,a}(q) = \sum_{i,j} (-1)^i q^j \operatorname{rk} \mathcal{H}^{i,j}_{\mathrm{BPS}}(Y,a)$$

- Modular properties of $\widehat{\widehat{Z}}_{Y,\mathfrak{s}}(q,t)$?
- Extension to a 2-variable series F_K of Gukov-Manolescu for knot complements and knot lattice homology?
- Other homology spheres? $b_1 > 0$?