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ADDITIVITY PROPERTIES OF MILNOR'S $\bar{\mu}$-INVARIANTS

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ABSTRACT. We prove that Milnor's $\bar{\mu}$-invariants are additive under the connected sum operation for links, and establish some corollaries of this result. The appendix describes a technique for finding presentations of nilpotent quotients of groups, and a lemma helpful for locating gropes in the complement of surfaces in four-space.

Keywords: $\bar{\mu}$-invariants, connected sum of links, link homotopy, gropes.

1. INTRODUCTION.

J. Milnor introduced in [12], [13] for each link $L = (l_1, \dots, l_n)$ in S^3 a sequence of "higher linking numbers" $\bar{\mu}_L(I)$, very useful for studying link homotopy and link concordance. Here I is a multiindex with entries between 1 and n, and the usual linking number of l_i and l_j corresponds to $\bar{\mu}_L(i,j)$. It is an elementary homological fact that linking numbers are additive under the connected sum operation, and T. Cochran conjectured in [1] that additivity holds for all $\bar{\mu}$-invariants. The difference of this general case from linking numbers is in that one has to consider (non-abelian) nilpotent quotients of link groups, also $\bar{\mu}$-invariants are defined in general only modulo a certain indeterminacy. For two oriented links $L' = (l'_1, \dots, l'_n)$ and $L'' = (l''_1, \dots, l''_n)$ in S^3, separated by a 2-sphere, let $L' \sharp L'' = (l_1, \dots, l_n)$ denote a link the i-th component of which is obtained by taking a connected sum (ambient surgery along an arc) of the components l'_i and l''_i respecting their orientations, $i = 1, \dots, n$, see figure 2. The sum $L' \sharp L''$ depends in general on the choice of bands in S^3, but in each case the choice will be clear from the context. Our main result is the following theorem.

Theorem 1. *Let $L' = (l'_1, \dots, l'_n)$ and $L'' = (l''_1, \dots, l''_n)$ be two oriented links in S^3, separated by a 2-sphere. Then for any choice of connecting bands and for any multiindex I, the indeterminacy $\Delta_{L' \sharp L''}(I)$ is a multiple of $g.c.d.(\Delta_{L'}(I), \Delta_{L''}(I))$, and*

$$\bar{\mu}_{L' \sharp L''}(I) \equiv \bar{\mu}_{L'}(I) + \bar{\mu}_{L''}(I) \ mod \ (g.c.d.(\Delta_{L'}(I), \Delta_{L''}(I))).$$

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This result shows, in particular, that a connected sum of a homotopically essential link and a homotopically trivial link cannot be homotopically trivial, for any choice of connecting bands. The result for the first non-vanishing $\bar{\mu}$-invariants and under the restriction that connecting bands intersect the separating 2-sphere just once has been established in [1], [14]. (By a first non-vanishing $\bar{\mu}$-invariant of a link L we mean $\bar{\mu}_L(i_1, \ldots, i_k)$ such that $\bar{\mu}_L(J) = 0$ for all multiindices J with less than k entries.) In this generality it also follows from the result of V. Turaev [16] which interprets $\bar{\mu}$-invariants in terms of Massey products on the zero-framed surgery of S^3 along the link: the first non-trivial Massey products are defined, integer-valued, and additive.

Section 2 provides background material about $\bar{\mu}$-invariants, gropes and the lower central series of link groups, and section 3 contains the proof of Theorem 1. As an easy consequence of the Grope Lemma [4] and Theorem 1 we establish in section 4 the "grope-concordance" invariance of $\bar{\mu}$-invariants and then we state Corollary 12 which relates $\bar{\mu}$-invariants of links connected in $S^3 \times I$ by disjoint pairs of pants.

The Appendix describes two observations, due to M. Freedman and P. Teichner: Lemma 13 is an application of Dwyer's theorem, and is helpful for finding presentations of nilpotent quotients of groups, in terms of generators of the first and second homology groups. Lemma 14 allows one to "move" arcs in in the complement of surfaces in $\mathbb{R}^4$ by finding singular gropes even in the absence of singular disks. These results are used to formulate the corollaries in section 4.

This work was motivated by the "relative-slice" approach to the four-dimensional topological surgery conjecture, introduced in [3], [4]. The goal is to develop techniques of link homotopy theory, sufficient for determining whether a special family of link pairs in S^3 is relatively slice. See [9], [10] for further study of link homotopy and of the relative-slice problem.

2. Preliminary facts about Milnor's $\bar{\mu}$-invariants, gropes and the lower central series.

The free group on generators $g_1, \ldots, g_k$ will be denoted by $F_{g_1,\ldots,g_k}$. Given a group G, G^q is the q-th lower central subgroup of G, defined inductively by $G^1 = G$, $G^2 = [G, G], \ldots, G^q = [G, G^{q-1}]$.

We briefly review the definition of $\bar{\mu}$-invariants from [13]. Let $L = (l_1, \ldots, l_n)$ be a fixed oriented link in S^3. All links considered in this paper are smooth. Given a positive integer q, the quotient $\pi_1(S^3 \setminus L)/(\pi_1(S^3 \setminus L))^q$ is generated by meridians $m_1, \ldots, m_n$ to the components of L. Let $w_1, \ldots, w_n$ be some words in $m_1, \ldots, m_n$ which represent the untwisted longitudes in this group, then $\pi_1(S^3 \setminus L)/(\pi_1(S^3 \setminus L))^q$ has the presentation

$$< m_1, \ldots, m_n | [m_1, w_1], \ldots, [m_n, w_n], (F_{m_1,\ldots,m_n})^q > .$$

The Magnus expansion homomorphism $M\colon F_{m_1,\ldots,m_n} \longrightarrow \mathbb{Z}\{x_1, \ldots, x_n\}$ into the ring of formal non-commutative power series in the indeterminates $x_1, \ldots, x_n$ is defined by $M(m_i) = 1 + x_i$, $M(m_i^{-1}) = 1 - x_i + x_i^2 \pm \ldots$ for $i = 1, \ldots, n$. Let

$$M(w_j) = 1 + \Sigma \mu_L(I, j) x_I$$

be the expansion of w_j, where the summation is over all multiindices $I = (i_1, \ldots, i_k)$ with entries between 1 and n, and $x_I = x_{i_1} \cdot \ldots \cdot x_{i_k}$, $k > 0$. This expansion defines for each such multiindex I the integer $\mu_L(I, j)$. Let $\Delta_L(i_1, \ldots, i_k)$ denote the greatest common divisor of $\mu_L(j_1, \ldots, j_s)$ where $j_1, \ldots, j_s$, $2 \leq s \leq k-1$ is to range over all sequences obtained by cancelling at least one of the indices $i_1, \ldots, i_k$ and permuting the remaining indices cyclicly.

Let $\bar{\mu}_L(I)$ denote the residue class of $\mu_L(I)$ modulo $\Delta_L(I)$. For each multiindex I of length $|I| \leq q$ the residue class $\bar{\mu}_L(I)$ is an *isotopy invariant* of the link L, where $\bar{\mu}_L(I)$ is defined using the quotient $\pi_1(S^3 \setminus L)/(\pi_1(S^3 \setminus L))^q$.

Lemma 2. ([13]) *Let L' be a link obtained by replacing each component of L by a collection of untwisted parallel copies. Suppose that the i-th component of L' corresponds to the $h(i)$-th component of L. Then $\bar{\mu}_{L'}(i_1, \ldots, i_s) = \bar{\mu}_L(h(i_1), \ldots, h(i_s))$.*

2.1. **Link homotopy and Milnor groups.** Two n-component links L and L' in S^3 are said to be *link-homotopic* if they are connected by a 1-parameter family of immersions such that different components stay disjoint at all times. L is said to be *homotopically trivial* if it is link-homotopic to the unlink. L is *almost homotopically trivial* if each proper sublink of L is homotopically trivial.

For a group π normally generated by $g_1, \ldots, g_k$ its *Milnor group* (with respect to $g_1, \ldots, g_k$) $M\pi$ is defined to be the quotient of π by its subgroup $\ll [g_i, g_i^h] : 1 \leq i \leq k,\ h \in \pi \gg$. $M\pi$ is nilpotent of class $\leq k+1$, in particular it is a quotient of $\pi/(\pi)^{k+1}$, and is generated by the quotient images of $g_1, \ldots, g_k$. The Milnor group $M(L)$ of a link L is defined to be $M\pi_1(S^3 \setminus L)$ with respect to its meridians m_i.

Milnor showed in [12] that the Magnus expansion induces a well defined injective homomorphism $MM\colon M(F_{m_1,\ldots,m_k}) \longrightarrow R(x_1, \ldots, x_k)$ into the ring $R(x_1, \ldots, x_k)$ which is the quotient of $\mathbb{Z}\{x_1, \ldots, x_k\}$ by the ideal generated by monomials $x_{i_1} \cdots x_{i_r}$ with some index occuring at least twice. Let $\overline{w}_n \in MF_{m_1,\ldots,m_{n-1}}$ be a word representing l_n in $M\pi_1(S^3 \setminus (l_1 \cup \ldots \cup l_{n-1}))$. Then $\bar{\mu}$-invariants of L with non-repeating coefficients may also be defined by the equation

$$MM(\overline{w}_n) = 1 + \Sigma \mu_L(I, n) x_I$$

where summation is over all multiindices I with non-repeating entries between 1 and $n-1$, and $\bar{\mu}_L(I, n)$ is the residue class of $\mu_L(I, n)$ modulo the indeterminacy $\Delta_L(I, n)$, defined above.

The Milnor group of L is the largest common quotient of the fundamental groups of all links link-homotopic to L, hence one has the following result.

Theorem 3. (Invariance under link homotopy [12]) *If L and L' are link homotopic then their Milnor groups are isomorphic. In particular, for any multiindex I with non-repeating entries $\bar{\mu}_L(I) = \bar{\mu}_{L'}(I)$.*

Isotopy of links is a special kind of *concordance*, and it is a result of Stallings that Milnor's invariants are preserved under this more general equivalence relation.

Theorem 4. (Concordance invariance [15]) *If L and L' are concordant then all their $\bar{\mu}$-invariants coincide. In fact, if $L \subset S^3 \times \{0\}$ and $L' \subset S^3 \times \{1\}$ are connected in $S^3 \times I$ by disjoint immersed annuli then L and L' are link-homotopic* ([6], [7], [11]).

The next result gives an algebraic reformulation of the notion of a homotopically trivial link.

Lemma 5. ([12]) *For an n-component link L, the following conditions are equivalent:*
(i) L is homotopically trivial,
(ii) the components of L bound disjoint immersed disks in B^4,
(iii) $M(L) \cong M(F_{m_1,\dots,m_n})$ with the isomorphism carrying a meridian to l_i to the generator m_i of the free group,
(iv) all $\bar{\mu}$-invariants of L with non-repeating coefficients vanish.

It follows from Lemma 5 that L is almost homotopically trivial if and only if all its $\bar{\mu}$-invariants with non-repeating coefficients of length less than n vanish. In particular, if L is almost homotopically trivial then its $\bar{\mu}$-invariants with non-repeating coefficients of length n are well-defined integers.

2.2. **Gropes and the lower central series.** A *grope* is a special pair (2-complex, circle). A grope has a *class* $k = 1, 2, \dots, \infty$. For $k = 1$ a grope is defined to be the pair (circle, circle). For $k = 2$ a grope is precisely a compact oriented surface Σ with a single boundary component. For k finite a *k-grope* is defined inductively as follow: Let $\{\alpha_i, \beta_i, i = 1, \dots, \text{genus}\}$ be a standard symplectic basis of circles for Σ. For any positive integers p_i, q_i with $p_i + q_i \geq k$ and $p_{i_0} + q_{i_0} = k$ for at least one index i_0, a k-grope is formed by gluing p_i-gropes to each α_i and q_i-gropes to each β_i.

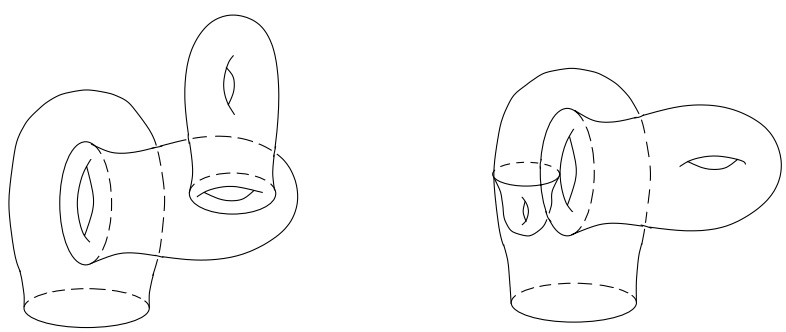

FIGURE 1. Two gropes of class 4.

The proof of the following lemma, and additional properties of gropes may be found in [5], [10].

Lemma 6. (Lemma 2.1 in [5]) *For a space X, a loop γ lies in $\pi_1(X)^k, 1 \leq k < \omega$, if and only if γ bounds a map of some k-grope. Moreover, the class of a grope (G, γ) is the maximal k such that $\gamma \in \pi_1(G)^k$.*

If the components of a link L bound disjoint immersed disks in B^4 then L is homotopically trivial, see Lemma 5. Grope Lemma shows that the same conclusion holds if instead of immersed disks one has immersed gropes of a sufficiently large class.

Theorem 7. (Grope Lemma [4], see also [10]) *Let L be an n-component link in S^3. Suppose the components of L bound disjoint maps of gropes of class n in B^4. Then L is homotopically trivial.*

Given a surface S, an *S-like grope* of class k is a 2-complex obtained by replacing a 2-cell in S with a k-grope. For example, one has *annulus-like* k-gropes; sphere-like gropes are sometimes also referred to as *closed* gropes. Given a space X, the *Dwyer's subgroup* $\phi_k(X)$ of $H_2(X;\mathbb{Z})$ is the set of all homology classes represented by maps of closed gropes of class k into X.

Theorem 8. (Dwyer's Theorem [2]; see also Lemma 2.3 in [5]) *Let k be a positive integer and let $f\colon X \longrightarrow Y$ be a map inducing an isomorphism on $H_1(\,.\,;\mathbb{Z})$ and mapping $H_2(X)/\phi_k(X)$ onto $H_2(X)/\phi_k(Y)$. Then f induces an isomorphism $\pi_1(X)/(\pi_1(X))^k \cong \pi_1(Y)/(\pi_1(Y))^k$.*

A *Grope* is a special "untwisted" 4-dimensional thickening of a grope (G,γ); it has a preferred solid torus (around the base circle γ) in its boundary. This "untwisted" thickening is obtained by first embedding G in $\mathbb{R}^3$ and taking its thickening there, and then crossing it with the interval $[0,1]$. The definition of a 0-framed grope is independent of the chosen embedding of G in $\mathbb{R}^3$. Similarly, one defines sphere- and annulus-like Gropes, the capital letter indicating a 4-dimensional thickening of the corresponding 2-complex.

3. Additivity of Milnor's $\bar{\mu}$-invariants.

This section contains the proof of Theorem 1. The idea is to connect L', L'' and $L'\sharp L''$ by disjoint pairs of pants in $S^3 \times I$ and to compare the link groups with the fundamental group of the complement of surfaces in $S^3 \times I$. This idea of comparison with the 4-dimensional complement has proved useful in link homotopy theory – see the proof of concordance invariance 4 in [15], and the proof of lemma 5 in [5].

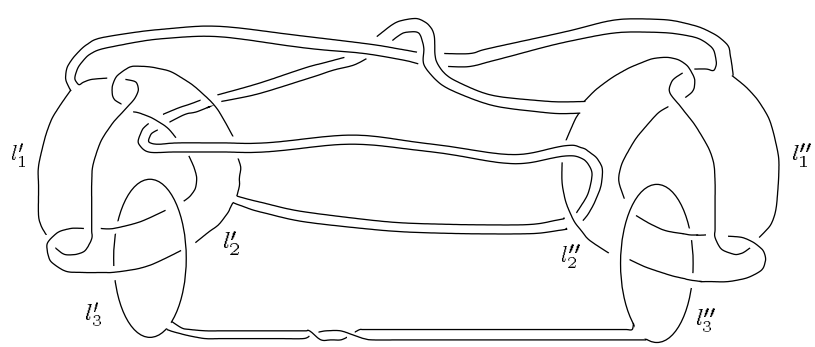

Figure 2. A connected sum of links.

Before proving Theorem 1 we will give a geometric argument to establish the following lemma, corresponding to the case of non-repeating indices and trivial indeterminacies; it illustrates the proof of Theorem 1 in this easier case.

Lemma 9. *Let $L' = (l'_1, \dots, l'_n)$ and $L'' = (l''_1, \dots, l''_n)$ be two almost homotopically trivial oriented links in S^3, separated by a 2-sphere. Then for any choice of connecting bands $L' \sharp L'' = (l_1, \dots, l_n)$ is also almost homotopically trivial, and*

$$\bar{\mu}_{L'\sharp L''}(1, \dots, n) = \bar{\mu}_{L'}(1, \dots, n) + \bar{\mu}_{L''}(1, \dots, n).$$

Proof. Considering $L' \sharp L''$ as a link in $S^3 \times \{0\}$ and L', L'' in $S^3 \times \{1\}$, one may construct disjoint properly embedded pairs of pants $P_1, \dots, P_n$ in $S^3 \times [0,1]$ with $\partial P_i = l_i \cup l'_i \cup l''_i$, $i = 1, \dots, n$. Precisely, P_i is formed by starting with l_i at time 0, shrinking the band in the middle so that l_i surgers into two components at time $1/2$, and an isotopy of these two components to l'_i and l''_i between $1/2$ and 1.

We will first show that $L' \sharp L''$ is almost homotopically trivial. Consider the link $(l_1, \dots, \hat{l}_i, \dots, l_n) \subset S^3 \times \{0\}$ (the i-th component is deleted) and the pairs of pants $P_1, \dots, \hat{P}_i, \dots, P_n \subset S^3 \times [0,1]$. The link $(l'_1, \dots, \hat{l}'_i, \dots, l'_n, l''_1, \dots, \hat{l}''_i, \dots, l''_n) \subset S^3 \times \{1\}$ is homotopically trivial and bounds disjoint immersed disks in $S^3 \times [1,2]$. Now $(l_1, \dots, \hat{l}_i, \dots, l_n)$ bounds in $S^3 \times [0,2]$ disjoint immersed disks, so by Lemma 5 it is homotopically trivial. Since this conclusion holds for each $i = 1, \dots, n$, $L' \sharp L''$ is almost homotopically trivial.

Let $l_{n,1}$ and $l_{n,2}$ in $S^3 \times \{0\}$ denote the result of surgery on l_n which cuts the connecting band; the corresponding surgery on P_n gives two annuli A_1, $A_2 \subset S^3 \times [0,1] \setminus (P_1 \cup \dots \cup P_{n-1})$ with $\partial A_1 = l_{n,1} \cup l'_n$, $\partial A_2 = l_{n,2} \cup l''_n$. Let L_1 denote the link $(l_1, \dots, l_{n-1}, l_{n,1})$. We will construct in $S^3 \times [0,2]$ a singular concordance between $L_1 \subset S^3 \times \{0\}$ and $L' \subset S^3 \times \{2\}$. It consists of $P_1, \dots, P_{n-1}, A_1$ in $S^3 \times [0,1]$, the product $L' \times [1,2]$ and a disjoint from it nullhomotopy of $(l''_1, \dots, l''_{n-1})$ in $S^3 \times [1,2]$. By singular concordance invariance 4, $\bar{\mu}_{L_1}(1, \dots, n) = \bar{\mu}_{L'}(1, \dots, n)$. Analogously we get $\bar{\mu}_{L_2}(1, \dots, n) = \bar{\mu}_{L''}(1, \dots, n)$ for the link $L_2 = (l_1, \dots, l_{n-1}, l_{n,2})$.

Recall that a longitude of l_n (also denoted by l_n) consists of an arc γ from the basepoint, followed by an untwisted parallel copy of l_n and then by γ^{-1}. For completeness of the proof we include here, in the case of trivial indeterminacy, the argument given in [13] to show that $\bar{\mu}$-invariants are well-defined with respect to the choice of longitudes. Let m_i denote a meridian to the component l_i, $i = 1, \dots, n$. Since $L' \sharp L''$ is almost homotopically trivial, the Magnus expansion

$$M(l_1 \cup \dots \cup l_{n-1}) \cong M(F_{m_1, \dots, m_{n-1}}) \xrightarrow{MM} R(x_1, \dots, x_{n-1})$$

of l_n is of the form $1 + \omega_n$ where

$$\omega_n = \mu_{L'\sharp L''}(1, \dots, n) x_1 \cdots x_{n-1} + \text{terms obtained by permutations from } x_1 \cdots x_{n-1}.$$

The Magnus expansion of $m_i l_n m_i^{-1}$ is then of the form

$$(1 + x_i)(1 + \omega_n)(1 - x_i \pm \dots) = 1 + \omega_n \pm \text{terms containing both } \omega_n \text{ and } x_i;$$

clearly the coefficient of $x_1 \dots x_{n-1}$ in this expansion is the same as in ω_n.

This allows us to choose longitudes $l_{n,1}$, $l_{n,2}$ (keeping the same notations for the longitudes as for the components), so that $l_n = l_{n,1} \cdot l_{n,2} \in \pi_1(S^3 \times \{0\} \setminus (l_1 \cup \dots \cup l_{n-1}))$. Then $MM(l_n) = MM(l_{n,1}) \cdot MM(l_{n,2})$, and the coefficient $\bar{\mu}_{L'\sharp L''}(1, \dots, n)$ of $x_1 \cdots x_{n-1}$ in $MM(l_n)$ is equal to $\bar{\mu}_{L_1}(1, \dots, n) + \bar{\mu}_{L_2}(1, \dots, n) = \bar{\mu}_{L'}(1, \dots, n) + \bar{\mu}_{L''}(1, \dots, n)$. This concludes the proof of Lemma 9. $\square$

Proof of Theorem 1. Considering $L'\sharp L''$ as a link in $S^3\times\{0\}$ and L', L'' in $S^3\times\{1\}$, one has disjoint properly embedded pairs of pants $P_1,\dots,P_n$ in $S^3\times[0,1]$ with $\partial P_i = l_i\cup l_i'\cup l_i''$, $i=1,\dots,n$. P_i is formed by starting with l_i at time 0, shrinking the band in the middle so that l_i surgers into two components at time 1/2, and an isotopy of these two components to l_i' and l_i'' between 1/2 and 1. Choosing the appropriate trivializations of the normal bundles of $P_1,\dots,P_n$, one can find for each i a parallel copy $\overline{P}_i$ of P_i, such that $\partial\overline{P}_i = \bar{l}_i\cup\bar{l}_i'\cup\bar{l}_i''$ with $lk(l_i,\bar{l}_i) = lk(l_i',\bar{l}_i') = lk(l_i'',\bar{l}_i'') = 0$. After taking a sufficient number of parallel copies of $P_1,\dots,P_n$, we can assume by Lemma 2 that I is a multiindex with non-repeating entries. Let P denote the union $P_1\cup\ldots\cup P_{n-1}$.

Let $\Delta(I)$ denote the greatest common divisor of $\Delta_{L'}(I)$ and $\Delta_{L''}(I)$ and let $k=|I|$, the number of entries in I. We may assume that n is the last index in I, so that $I=(J,n)$.

Let $l_{n,1}$ and $l_{n,2}$ denote the result of surgery on l_n which cuts the connecting band; the corresponding surgery on P_n gives two annuli A_1, $A_2\subset S^3\times I\smallsetminus P$ with $\partial A_1 = l_{n,1}\cup l_n'$, $\partial A_2 = l_{n,2}\cup l_n''$. Consider links $L_1 := (l_1,\dots,l_{n-1},l_{n,1})$ and $L_2 := (l_1,\dots,l_{n-1},l_{n,2})$ in $S^3\times\{0\}$. Similarly to the proof of Lemma 9, we will show that

$$\bar{\mu}_{L_1}(I)\equiv\bar{\mu}_{L'}(I)\ mod\ \Delta(I)\ \text{ and }\ \bar{\mu}_{L_2}(I)\equiv\bar{\mu}_{L''}(I)\ mod\ \Delta(I).$$

We fix basepoints $b_0\in S^3\times\{0\}$, $b_1\in S^3\times\{1\}$, and introduce the following short notations for the fundamental groups:

$$\pi_0 := \pi_1(S^3\times\{0\}\smallsetminus(l_1\cup\ldots\cup l_{n-1}),\, b_0),\quad \pi := \pi_1(S^3\times[0,1]\smallsetminus P,\, b_0),$$

$$\pi_1 := \pi_1(S^3\times\{1\}\smallsetminus(l_1'\cup\ldots\cup l_{n-1}'\cup l_1''\cup\ldots\cup l_{n-1}''),\, b_1).$$

Let $i_o\colon\ \pi_0\longrightarrow\pi$ and $i_1\colon\ \pi_1\longrightarrow\pi$ be maps induced by inclusions; the definition of i_1 requires choosing a path joining b_0 and b_1 in $S^3\times I\smallsetminus P$, which will be fixed for the rest of the proof. Fix some meridians m_i', m_i'', based at b_1, to l_i', l_i'' respectively in $S^3\times\{1\}$ and a meridian m_i, based at b_0, to l_i in $S^3\times\{0\}$, $i=1,\dots,n$. Since $H_1(\pi)$ is generated by $m_1,\dots,m_{n-1}$, the quotient π/π^k is also generated by these meridians, see Lemma 13. For each $1\le i\le n$ let w_i' (respectively w_i'') be a word in $m_1,\dots,m_{n-1}$ representing $i_1(l_i')$ (respectively $i_1(l_i'')$) in $\pi/(\pi)^k$; also let u_i', u_i'' be words in $m_1,\dots,m_{n-1}$ representing $i_1(m_i')$, $i_1(m_i'')$. By Alexander duality the homology classes of the 2-tori $-$ boundaries of the normal bundles in $S^3\times\{1\}$ of $l_1',\dots,l_{n-1}',l_1'',\dots,l_{n-1}''$ $-$ generate $H_2(S^3\times[0,1]\smallsetminus P;\mathbb{Z})$, and it follows from Lemma 13 that

$$\pi/(\pi)^k\cong< m_1,\dots,m_{n-1}|[u_i',w_i'],[u_i'',w_i''],i=1,\dots,n-1;(F_{m_1,\dots,m_{n-1}})^k>.$$

Since m_i, $i_1(m_i')$ and $i_1(m_i'')$ are conjugate in π, we can use the identity $[a^c,b]=[a,b^{(c^{-1})}]^c$ to get the presentation

$$\pi/(\pi)^k\cong<m_1,\dots,m_{n-1}|[m_i,(w_i')^{g_i}],[m_i,(w_i'')^{h_i}],i=1,\dots,n-1;(F_{m_1,\dots,m_{n-1}})^k>$$

where g_i,h_i are some elements of $F_{m_1,\dots,m_{n-1}}$. For each $1\le i\le n$ and each multiindex J with entries between 1 and $(n-1)$ let $\nu'(J,i)$ (respectively $\nu''(J,i)$) denote the coefficient of x_J in $M(w_i')$ (respectively $M(w_i'')$), where M is the Magnus expansion $M\colon\ F_{m_1,\dots,m_{n-1}}\longrightarrow\mathbb{Z}\{x_1,\dots,x_{n-1}\}$.

Proposition 10. *For each $1 \le i \le n$ and each multiindex J with entries between 1 and $(n-1)$ the coefficient $\nu'(J,i)$ of x_J in the Magnus expansion $M(w_i')$ is well-defined modulo $\Delta(J,i)$. Precisely, its residue class is well defined with respect to*

(i) conjugation of m_j, $j = 1, \dots, n-1$,
(ii) conjugation of a word w_i' representing l_i' in π/π^k,
(iii) multiplication of w_i' by conjugates of the relations $[m_j, (w_j')^{g_j}]$, $[m_j, (w_j'')^{h_j}]$
(iv) multiplication of w_i' by an element of $(F_{m_1,\dots,m_{n-1}})^k$.

Analogously the residue class of the coefficient $\nu''(J,i)$ in $M(w_i'')$ modulo $\Delta(J,i)$ is also well-defined.

Proof. This proposition is different from the theorem [13] that $\bar{\mu}$-invariants are well-defined modulo their indeterminacy in that we have more relations in the presentation for $\pi/(\pi)^k$ and as a result the larger indeterminacy. We will now adapt the proof of Theorem 5 in [13] to this setting.

In the ring $\mathbb{Z}\{x_1, \dots, x_{n-1}\}$ let D_i denote the ideal

$$D_i = \{\Sigma\sigma_J\, x_J \mid |J| \ge k, \text{ or } \sigma_J \equiv 0 \; mod \; \Delta(J,i) \text{ if } |J| < k\}.$$

To prove that two words w_i and w_i' give rise to the same residue classes of $\nu'_{J,i}$ mod $\Delta(J,i)$ it suffices to show that $M(w_i - w_i') \in D_i$.

(1) *D_i is a two-sided ideal.*

Proof. Let $\sigma_J\, x_J \in D_i$ and let $\lambda \cdot x_H$ be an arbitrary monomial. Notice that either $|J| \ge k$ or $\sigma_J \equiv 0 \; mod \; \Delta(J,i)$ and $\Delta(J,i) \equiv 0 \; mod \; \Delta(H,J,i)$, hence $\lambda \cdot \sigma_J \equiv 0 \; mod \; \Delta(H,J,i)$. This implies $\lambda \cdot \sigma_J\, x_H \cdot x_J \in D_i$ and proves that D_i is a left ideal. Similarly it can be shown that D_i is a right ideal.

This argument proves a more general fact

(1′) *Let $\sigma_J\, x_J \in D_i$. If one or more new factors x_j are inserted anywhere in $\sigma_J\, x_J$ then the resulting term is also in D_i.*

(2) *Let $\nu'(J,i)x_J$ be any term in the expansion $M(w_i)$. If one or more new factors x_j are inserted anywhere in $\nu'(J,i)x_J$ then the resulting term is congruent to zero modulo D_i.*

Proof. Let $J = (j_1, \dots, j_t)$. The congruence

$$\nu'(j_1, \dots, j_t, i) \equiv 0 \; mod \; \Delta(j_1, \dots, j_s, j, j_{s+1}, \dots, j_t, i)$$

implies that

$$\nu'(j_1, \dots, j_t, i) x_{j_1} \cdots x_{j_s} x_j x_{s+1} \cdots x_{j_t} \equiv 0 \; mod \; D_i.$$

(3) *Let $M(w_j') = 1 + \omega_j'$. Then $\omega_j' x_j \equiv x_j \omega_j' \equiv 0 \; mod \; D_i$ for any i, and the analogous congruencies hold for w_j''.*

Proof. Let $\nu'(J,j)x_J$ be any term of ω_j'. Now the statement follows from the congruences $\nu'(J,j) \equiv 0 \; mod \; \Delta(J,j,i)$, $\nu'(J,j)x_J x_j \equiv 0 \; mod \; D_i$ and the analogous congruences with J and j switched.

Proof of (i). Suppose that m_j is replaced by $\overline{m}_j = m_h m_j m_h^{-1}$, $M(m_j) = 1 + x_j$, $M(\overline{m}_j) = 1 + \bar{x}_j$. Then $m_j = m_h^{-1}\overline{m}_j m_h$ and

$$x_j = (1 - x_h + x_h^2 \pm \dots)\bar{x}_j(1 + x_h) = \bar{x}_j + \text{ terms involving } x_h\bar{x}_j \text{ or } \bar{x}_j x_h.$$

Each time the factor x_j occurs in the Magnus expansion of w_i' it is to be replaced by this expression. An application of (2) concludes the proof of (i).

Proof of (ii). Suppose that w_i' is replaced by $m_j w_i' m_j^{-1}$, $M(w_i') = 1 + \omega_i'$. Then it follows from (1) and (2) that

$$M((w_i')^{m_j}) = 1 + (1 + x_j)\omega_i'(1 - x_j + x_j^2 \pm \dots) =$$
$$1 + \omega_i' + \text{ terms involving } x_j\omega_i' \text{ or } \omega_i' x_j \equiv \omega_i \ mod\ D_i.$$

The statement about w_j'' is proved analogously.

Proof of (iii). Assume first that $g_j = h_j = 1$. Extend the Magnus expansion homomorphism to the group ring $\mathbb{Z}F_{m_1,\dots,m_{n-1}}$, then

$$M(m_j w_j' - w_j' m_j) = (1 + x_j)(1 + \omega_j') - (1 + \omega_j')(1 + x_j) = x_j\omega_j' - \omega_j' x_j,$$

$$M([m_j, w_j']) = M(1 + (m_j w_j' - w_j' m_j)m_j^{-1}(w_j')^{-1})$$

together with (3) imply that $M([m_j, w_j']) \equiv 1 \ mod\ D_i$.

For arbitrary g_j, h_j let $M(w_j') = 1 + \omega_j'$, $M((w_j')^{g_j}) = 1 + \bar{\omega}_j'$. Since

$$x_j\omega_j' \equiv \omega_j' x_j \equiv 0 \ mod\ D_i,$$

$$M(m_j(w_j')^{g_j} - (w_j')^{g_j} m_j) = (1 + x_j)(1 + \bar{\omega}_j') - (1 + \bar{\omega}_j')(1 + x_j) = x_j\bar{\omega}_j' - \bar{\omega}_j' x_j,$$

it follows by remark $(1')$ and the proof of (ii) that

$$M(m_j(w_j')^{g_j} - (w_j')^{g_j} m_j) \equiv 1 \ mod\ D_i$$

for any i. This implies that

$$M([m_j, (w_j')^{g_j}]) = M(1 + (m_j(w_j')^{g_j} - (w_j')^{g_j} m_j)m_j^{-1}(w_j')^{-g_j}) \equiv 1 \ mod\ D_i.$$

The statement about w_j'' is proved analogously.

Part (iv) is a well-known fact [8]. This concludes the proof of Proposition 10. $\square$

To finish the proof of Theorem 1 consider the commutative diagram:

$$\begin{array}{ccccc}
\pi_1/(\pi_1)^k & \xleftarrow{p_1} & F_{m_1',\dots,m_{n-1}',m_1'',\dots,m_{n-1}''} & \xrightarrow{M_1} & \mathbb{Z}\{x_1',\dots,x_{n-1}',x_1'',\dots,x_{n-1}''\} \\
{\scriptstyle i_1}\downarrow & & {\scriptstyle \phi_1}\downarrow & & {\scriptstyle \psi_1}\downarrow \\
\pi/(\pi)^k & \xleftarrow{p} & F_{m_1,\dots,m_{n-1}} & \xrightarrow{M} & \mathbb{Z}\{x_1,\dots,x_{n-1}\} \\
{\scriptstyle i_0}\uparrow & & {\scriptstyle \phi_0}\uparrow & & {\scriptstyle \psi_0}\uparrow \\
\pi_0/(\pi_0)^k & \xleftarrow{p_0} & F_{m_1,\dots,m_{n-1}} & \xrightarrow{M_0} & \mathbb{Z}\{x_1,\dots,x_{n-1}\}
\end{array}$$

where the maps i_0 and i_1 are induced by inclusions,

p, p_0 and p_1 are the obvious epimorphisms,

$\phi_0 = id,\ \phi_1(m_i') = (m_i)^{a_i'},\ \phi_1(m_i'') = (m_i)^{a_i''}$ for some $a_i', a_i'' \in F_{m_1,\dots,m_{n-1}}$ and

$\psi_0 = id,\ \psi_1(x_i') = (1+\alpha_i')(1+x_i)(1+\bar{\alpha}_i')-1,\ \psi_1(x_i'') = (1+\alpha_i'')(1+x_i)(1+\bar{\alpha}_i'')-1$
where $\alpha_i' = M(a_i'),\ \bar{\alpha}_i' = M((a_i')^{-1}),\ \alpha_i'' = M(a_i''),\ \bar{\alpha}_i'' = M((a_i'')^{-1})$.

Let $\lambda_{n,1} \in F_{m_1,\dots,m_{n-1}}$ and $\lambda_n' \in F_{m_1',\dots,m_{n-1}',m_1'',\dots,m_{n-1}''}$ be such that $p_0(\lambda_{n,1}) = l_{n,1},\ p_1(\lambda_n') = l_n'$. We may assume that λ_n' is a word in just $m_1', \dots, m_{n-1}'$. Then

$$M(\phi_0(\lambda_{n,1})) = M_0(\lambda_{n,1}) = \Sigma\mu_{L_1}(J,n)\,x_J,$$

$$M_1(\lambda_n') = \Sigma\mu_{L'}(J,n)\,x_J' \text{ and}$$

$$M(\phi_1(\lambda_n')) = \Sigma\nu'(J,n)\,x_J.$$

Since $l_{n,1}$ and l_n' cobound the annulus $A_1 \subset S^3 \times I \setminus P$, $i_0(l_{n,1})$ and $i_1(l_n')$ are conjugate in π and by Proposition 10, $\mu_{L_1}(J,n) \equiv \nu'(J,n)\ mod\ \Delta(J,n)$. Since the meridians m_i and $i_1(m_i')$ are conjugate in π, $i = 1, \dots, n-1$, Proposition 10 (i) implies that $\mu_{L'}(J,n) \equiv \nu'(J,n)\ mod\ \Delta(J,n)$, and we conclude that

$\bar{\mu}_{L_1}(J,n) \equiv \bar{\mu}_{L'}(J,n)\ mod\ \Delta(J,n)$; analagously $\bar{\mu}_{L_2}(J,n) \equiv \bar{\mu}_{L''}(J,n)\ mod\ \Delta(J,n)$.

Choosing the longitudes $l_{n,1}$ and $l_{n,2}$ in $S^3 \times \{0\}$ appropriately, we may assume $l_n = l_{n,1} \cdot l_{n,2} \in \pi_0$. If $\lambda_{n,1},\ \lambda_{n,2} \in F_{m_1,\dots,m_{n-1}}$ are such that $p_0(\lambda_{n,1}) = l_{n,1}$, $p_0(\lambda_{n,2}) = l_{n,2}$, then $p_0(\lambda_{n,1} \cdot \lambda_{n,2}) = l_n$.

The coefficient of x_J in the expansion $M_0(\lambda_{n,1} \cdot \lambda_{n,2})$ is equal to

$$\mu_{L'\sharp L''}(J,n) = \Sigma\mu_{L_1}(J_1,n) \cdot \mu_{L_2}(J_2,n) \equiv \Sigma\mu_{L'}(J_1,n) \cdot \mu_{L''}(J_2,n) \equiv$$

$$\equiv \mu_{L'}(J,n) + \mu_{L''}(J,n)\ mod\ \Delta(J,n)$$

where the summations are over all multiindices $J_1,\ J_2$ with $(J_1, J_2) = J$.

An inductive argument now shows that the indeterminacy $\Delta_{L'\sharp L''}(J,n)$ is a multiple of $\Delta(J,n)$. This concludes the proof of Theorem 1. □

4. Corollaries.

As a generalization of concordance invariance 4 and of Grope Lemma 7, we state

Corollary 11. (Grope-concordance invariance) *Let $L = (l_1, \dots, l_n)$ and $L' = (l_1', \dots, l_n')$ be two links in $S^3 \times \{0\}$ and $S^3 \times \{1\}$ respectively.*

(i) Suppose there are disjoint immersed annulus-like gropes $A_1, \dots, A_n$ of class n in $S^3 \times [0,1]$ with $\partial A_i = l_i \cup l_i'$, $i = 1, \dots, n$. Then for any multiindex I with non-repeating entries $\bar{\mu}_L(I) = \bar{\mu}_{L'}(I)$.

(ii) Suppose there are disjoint embedded annulus-like Gropes $A_1, \dots, A_n$ of class k in $S^3 \times [0,1]$ with $\partial A_i = l_i \cup l_i'$, $i = 1, \dots, n$. Then for any multiindex I with $|I| \le k$, $\bar{\mu}_L(I) = \bar{\mu}_{L'}(I)$.

Proof of (i). L and L' may be thought of as two links separated by a 2-sphere in S^3, and connected by disjoint gropes $A_1, \dots, A_n$ in B^4. For each $1 \le i \le n$ let α_i

be an arc in the first stage of A_i connecting its two boundary components, l_i and l_i'. By Alexander duality $H_1(B^4 \setminus (A_1 \cup \ldots \cup A_n); \mathbb{Z})$ is generated by meridians to the first stages of the gropes $A_1, \ldots, A_n$, and the proof of Lemma 14 extends without changes to this setting. By Lemma 14, for each i there exists an arc β_i in S^3 such that $\alpha_i \cup \beta_i$ bounds an immersed grope G_i of class n in the complement of other A_i's. Now a singular surgery of A_i along G_i for each i exhibits a connected sum $L \sharp L'$ as the boundary of disjoint immersed gropes of class n in B^4. By Theorem 1 and Grope Lemma 7, $0 = \bar{\mu}_{L \sharp L'}(I) = \bar{\mu}_L(I) + \bar{\mu}_{L'}(I)$. The difference of signs with the statement of Corollary 11 is due to the change of orientations between the original setting and considering both links in the same S^3.

By Lemma 2, part (ii) follows from (i) after taking a sufficient number of parallel copies of $A_1, \ldots, A_n$. □

Remark. For a different proof of Corollary 11 see Theorem 2 and Corollary 4.2 in [10].

Corollary 12. *Let $L = (l_1, \ldots, l_n)$ be a link in $S^3 \times \{0\}$, and $L' = (l_1', \ldots, l_n')$, $L'' = (l_1'', \ldots, l_n'')$ be two links separated by a 2-sphere in $S^3 \times \{1\}$. Assume there are disjoint embedded pairs of pants $P_1, \ldots, P_n$ in $S^3 \times [0,1]$ with $\partial P_i = l_i \cup l_i' \cup l_i''$, $i = 1, \ldots, n$. Given some orientations of the pairs of pants, let the links be oriented as their boundaries. Then for any multiindex I the indeterminacy $\Delta_L(I)$ is a multiple of $g.c.d.(\Delta_{L'}(I), \Delta_{L''}(I))$, and*

$$\bar{\mu}_L(I) \equiv \bar{\mu}_{L'}(I) + \bar{\mu}_{L''}(I) \ mod \ (g.c.d.(\Delta_{L'}(I), \Delta_{L''}(I))).$$

Note that the statement is well-defined with respect to the choice of orientations of the pairs of pants: if one of them is reversed, each term in the formula changes its sign.

Proof. Choosing the appropriate trivializations of the normal bundles of $P_1, \ldots, P_n$, one can find for each i a parallel copy $\bar{P}_i$ of P_i, $\partial \bar{P}_i = \bar{l}_i \cup \bar{l}_i' \cup \bar{l}_i''$ with $lk(l_i, \bar{l}_i) = lk(l_i', \bar{l}_i') = lk(l_i'', \bar{l}_i'') = 0$. After taking a sufficient number of parallel copies of $P_1, \ldots, P_n$, we can assume by Lemma 2 that I is a multiindex with non-repeating entries. Using Lemma 14 the pairs of pants may be surgered into disjoint immersed annulus-like gropes of arbitrarily large class, cobounded by $L \subset S^3 \times \{0\}$ and $L' \sharp L'' \subset S^3 \times \{1\}$ for some bands connecting L' and L'' in $S^3 \times \{1\}$, and 12 is reduced to Theorem 1 by grope-concordance invariance, Corollary 11 (i). □

Remarks. It immediately follows from the proof of Theorem 1 that, in fact, Theorem 1 is equivalent to Corollary 12.

The conclusion of Corollary 12 is easily seen to hold if $P_1, \ldots, P_n$ are (pair of pants)-like gropes of class k, for multiindices I with $|I| \leq k$. The proof of Lemma 14 is unchanged if surfaces $\Sigma_1, \ldots, \Sigma_n$ in its formulation are replaced by k-gropes.

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Appendix. Two useful lemmas: finding presentations of nilpotent quotients, and pulling up arcs in B^4. (After M. H. Freedman and P. Teichner)

To state the first lemma, fix a group π and suppose that $H_1(\pi;\mathbb{Z})$ is generated by $g_1,\dots,g_n$, $H_2(\pi;\mathbb{Z})$ is generated by $r_1,\dots,r_m$, and let $q\geq 2$ be an integer. Then the result of lemma 13 is that, roughly, $g_1,\dots,g_n$ and $r_1,\dots,r_m$ provide the set of generators and relations respectively in a presentation of π/π^q. To make this precise, consider the quotient homomorhism $\alpha\colon \pi/\pi^q \longrightarrow \pi/[\pi,\pi]$ and let $\hat{g}_i\in\pi/\pi^q$ denote some preimage of g_i under α, $i=1,\dots,n$. It is a standard fact in nilpotent group theory [17] that $\hat{g}_1,\dots,\hat{g}_n$ generate π/π^q: let N denote π/π^q and let H be the subgroup generated by $\hat{g}_1,\dots,\hat{g}_n$. Since $\alpha(H)=\pi/[\pi,\pi]$, we have $N=H\cdot[N,N]$, and this condition implies $H=N$. (The proof is by induction on the nilpotency class of N, using the fact that if $x\equiv g \ mod\ N^q$, $y\equiv h\ mod\ N^q$ then $[x,y]\equiv[g,h]\ mod\ N^{q+1}$.)

Let $W\longrightarrow K(\pi,1)$ be a map from the wedge of n circles W, inducing an epimorphism $\beta\colon \pi_1(W)\longrightarrow\pi/\pi^q$ and mapping the i-th free generator of $\pi_1(W)$ to $\hat{g}_i$. Let $f_j\colon \Sigma_j\longrightarrow K(\pi,1)$ be a map of a surface Σ_j, representing the generator r_j of $H_2(K(\pi,1))\cong H_2(\pi)$, $j=1,\dots,m$. We assume here that each space has a fixed basepoint, and all maps preserve them. The standard basis of $H_1(\Sigma_j)$ pulls back via β to some elements in $\pi_1(W)$; let $\hat{r}_j\in\pi_1(W)$ be a lift via β of the attaching map of the 2-cell of Σ_j. (In particular, if Σ_j is a 2-sphere then the corresponding word $\hat{r}_j$ is trivial.)

Lemma 13. *Suppose $H_1(\pi;\mathbb{Z})$ is generated by $g_1,\dots,g_n$, and $H_2(\pi;\mathbb{Z})$ is generated by $r_1,\dots,r_m$. Then in the notations as above,*

$$\pi/\pi^q \cong <\hat{g}_1,\dots,\hat{g}_n \mid \hat{r}_1,\dots,\hat{r}_m,(F_{\hat{g}_1,\dots,\hat{g}_n})^q>$$

where $F_{\hat{g}_1,\dots,\hat{g}_n}$ denotes the free group on generators $\hat{g}_1,\dots,\hat{g}_n$.

Proof. Let X be the 2-complex obtained from W by attaching m two-cells along the words $\hat{r}_1,\dots\hat{r}_m$. The composition $W \longrightarrow K(\pi,1) \longrightarrow K(\pi/\pi^q,1)$ extends to X, inducing an isomorphism $H_1(X) \cong H_1(\pi) \cong H_1(\pi/\pi^q)$ and an epimorhism on H_2/ϕ_q. Now an application of Dwyer's theorem 8 concludes the proof of Lemma 13. □

Lemma 14. *Let $\Sigma = \Sigma_1 \cup \dots \cup \Sigma_n$ be a collection of properly immersed disjoint compact connected surfaces in B^4 with $\partial\Sigma_i \neq \emptyset$ for each $i = 1,\dots,n$. Let $(\alpha,\partial\alpha)$ be an arc in $(B^4 \smallsetminus \Sigma, S^3 \smallsetminus \partial\Sigma)$, and let k be a positive integer. Then there exists an arc $\beta \subset S^3 \smallsetminus \partial\Sigma$ with $\partial\beta = \partial\alpha$ such that $\alpha \cup \beta$ bounds an immersed grope G of class k in $B^4 \smallsetminus \Sigma$.*

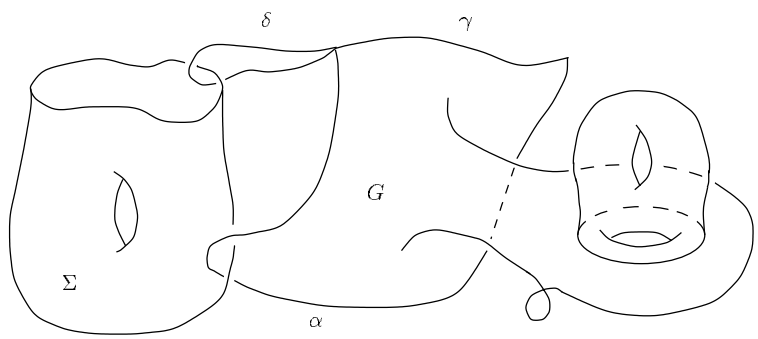

FIGURE 3

Proof. Let $\gamma \subset S^3 \smallsetminus \partial\Sigma$ be any arc with $\partial\gamma = \partial\alpha$. For each $1 \leq i \leq n$ fix a meridian m_i to a component of $\partial\Sigma_i$. $H_1(B^4 \smallsetminus \Sigma;\mathbb{Z})$ is generated by $m_1,\dots,m_n$, hence by lemma 13, $\pi_1(B^4 \smallsetminus \Sigma)/(\pi_1(B^4 \smallsetminus \Sigma))^k$ is also generated by these meridians. Similarly $\pi_1(S^3 \smallsetminus \partial\Sigma)/(\pi_1(S^3 \smallsetminus \partial\Sigma))^k$ is generated by meridians to all components of $\partial\Sigma$. Hence the homomorphism

$$i\colon \pi_1(S^3 \smallsetminus \partial\Sigma)/(\pi_1(S^3 \smallsetminus \partial\Sigma))^k \longrightarrow \pi_1(B^4 \smallsetminus \Sigma)/(\pi_1(B^4 \smallsetminus \Sigma))^k$$

induced by inclusion is surjective, and there exists a loop $\delta \subset S^3 \smallsetminus \partial\Sigma$ such that $i(\delta) = i(\alpha \cdot \gamma)$. (Here $\cdot$ denotes the composition of paths). Then $i(\alpha \cdot \gamma \cdot \delta^{-1}) \in (\pi_1(B^4 \smallsetminus \Sigma))^k$, and the arc $\beta := \gamma \cdot \delta^{-1}$ satisfies the conclusion of Lemma 14. □

Remark. Lemma 14 shows that immersed gropes are much easier to find than immersed disks: it is not difficult to construct an example of surfaces Σ and an arc α such that $\alpha \cup \beta$ does not bound an immersed *disk* for any choice of β.