Picture TQFTs, categorification, and localization

Slava Krushkal
(joint work with Ben Cooper)

March 17, 2012

Outline:

- Construction of "Picture TQFTs"

  The Jones-Wenzl projectors

  Evaluation at a root unity

- Categorification

  of the Jones-Wenzl projectors

- Towards categorification of TQFTs and 3-manifold invariants:

  Evaluation at a root of unity as a localization

The Temperley-Lieb algebra TL_n: generators 1 and e_i, $0 < i < n$, satisfying the relations:

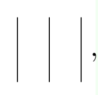
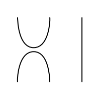
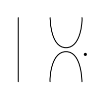
1. $e_i e_j = e_j e_i$ if $|i - j| \geq 2$.
2. $e_i e_{i\pm 1} e_i = e_i$
3. $e_i^2 = [2] e_i$

where

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{-(n-1)} + q^{-(n-3)} + \cdots + q^{n-3} + q^{n-1}$$

Each generator e_i can be pictured as a diagram consisting of n chords between two collections of n points on two horizontal lines in the plane.

Example: $n = 3$:

$1 =$ , $e_1 =$  and $e_2 =$ .

The multiplication is given by vertical composition of diagrams.

Definition

The *Temperley-Lieb space* $\mathrm{TL}(\Sigma)$ of a labelled surface Σ is the set of all $\mathbb{Z}[q, q^{-1}]$ linear combinations of isotopy classes of 1-manifolds ("multi-curves") $F \subset \Sigma$ subject to the local relation: removing a simple closed curve bounding a disk in Σ from a multi-curve is equivalent to multiplying the resulting element by the quantum integer $[2] = q + q^{-1}$.

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$\mathrm{TL}(\Sigma)$ is the 2-dimensional version of the Kauffman skein module of $\Sigma \times [0, 1]$.

The Temperley-Lieb algebra TL_n corresponds to $\Sigma = D^2$

Definition

Let Q be an element of the Temperley-Lieb algebra. The *ideal* $\langle Q \rangle$ generated by Q in $\mathrm{TL}(\Sigma)$ is the smallest submodule containing all elements obtained by gluing Q to B where $\Sigma = D^2 \cup (\Sigma \setminus D^2)$, and B is any element of $\mathrm{TL}(\Sigma \setminus D^2)$.

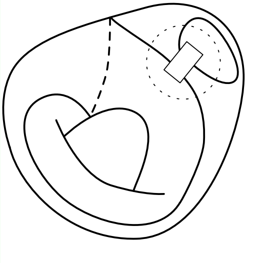

Figure: An element of the ideal $\langle Q \rangle \subset \mathrm{TL}(\Sigma)$

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Theorem (Goodman-Wenzl)
For q a primitive root of unity, $q = e^{2\pi i/n}$, there exists a unique local relation (ideal $\langle Q \rangle$) such that the quotient of $TL(\Sigma)/\langle Q \rangle$ is non-trivial.

In this case $Q =$ Jones-Wenzl projector p_{n-1}, and $TL(\Sigma)/\langle p_{n-1} \rangle$ is the Turaev-Viro theory associated to Σ.

For q not a root of unity, $TL(\Sigma)$ has no non-trivial local relations.

Summary of the construction of the Turaev-Viro theory $TV_n(\Sigma)$:

1. Consider the $\mathbb{Z}[q, q^{-1}]$-module $TL(\Sigma)$,
2. Specialize $q = e^{2\pi i/n}$
   (Algebraically: quotient the coefficient ring by the cyclotomic polynomial $\phi_n(q)$)
3. $TV_n(\Sigma) = TL(\Sigma)/\langle p_{n-1}\rangle$.

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An alternative construction: Skip step 2 and define

$$\overline{TV}_n(\Sigma) = TL(\Sigma)/\langle p_{n-1}\rangle$$

The result is closely related to $TV_n(\Sigma)$ (the coefficient ring is divided by $\phi_n(q^2) = \phi_n(q)\phi_n(-q)$ rather than $\phi_n(q)$)

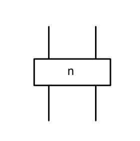
The Jones-Wenzl projectors are elements of the Tempeley-Lieb algebra that are uniquely characterized by two properties:

1. $p_n - 1$ belongs to the subalgebra generated by $\{e_1, e_2, \ldots, e_{n-1}\}$
   In other words, $p_n = 1 + \ldots$
2. $e_i p_n = p_n e_i = 0$ for all $i = 1, \ldots, n-1$ "killed by turnbacks"

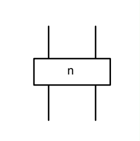
Origins and applications of the Jones-Wenzl projectors:

- Representation theory of $U_q(\mathfrak{sl}_2)$
- A “nice basis” of the Kauffman skein module of annulus$\times[0,1]$
- Building blocks in the definition of quantum spin networks
- Used in constructions of the Reshetikhin-Turaev and Turaev-Viro $SU(2)$ quantum invariants of 3-manifolds
- The colored Jones polynomial

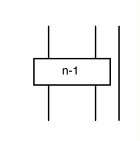
The Jones-Wenzl projectors

$p_n =$ 

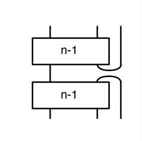
may be defined by the inductive formula:



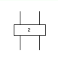
$=$



$- \frac{[n-1]}{[n]}$



Example. The second projector p_2:



$=$  $- \frac{1}{q+q^{-1}}$ 

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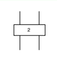


$$= $$

$$- \frac{1}{q+q^{-1}}$$

$$= $$

$$+ \sum_{i=1}^{\infty} (-1)^i q^{2i-1}$$

Categorification:

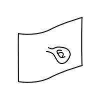
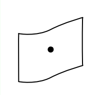
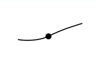
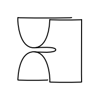
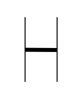
Bar-Natan's formulation of the Khovanov categorification of the Temperlay-Lieb algebra:

Additive category $\mathrm{Pre\text{-}Cob}(n)$:

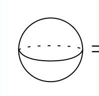
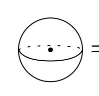
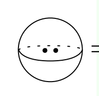
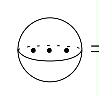
Objects: isotopy classes of formally q-graded Temperley-Lieb diagrams with $2n$ boundary points.

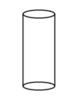
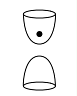
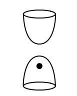
Morphisms: the free $\mathbb{Z}$-module spanned by isotopy classes of orientable cobordisms bounded in $\mathbb{R}^3$ between two planes containing diagrams.

Notation:

 $=\ 2$  $=\ 2$  and  $=$ 

Form a new category $\mathrm{Cob}(n) = \mathrm{Cob}^3_{\cdot/l}(n)$ obtained as a quotient of the category $\text{Pre-Cob}(n)$ by the relations

 $= 0$  $= 1$  $= 0$  $= \alpha$

 $=$  $+$ 

$\Sigma_3 = 8\alpha$ is a parameter.

Recall: the Jones-Wenzl projectors are elements of the Tempeley-Lieb algebra that are uniquely characterized by

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Categorification of the Jones-Wenzl projectors:

Theorem: (Cooper - K.) For each $n > 0$, there exists a chain complex P_n (positively graded with degree zero differential) such that

1. The identity diagram appears only in homological degree zero and only once.
2. The chain complex P_n is contractible "under turnbacks": for any generator $e_i \in \mathrm{TL}_n$, $0 < i < n$, $P_n \otimes e_i \simeq 0$, $e_i \otimes P_n \simeq 0$

Example: the second projector:

$$p_2 =$$

$$- \frac{1}{q + q^{-1}}$$

$$=$$

$$+ \sum_{i=1}^{\infty} (-1)^i q^{2i-1}$$


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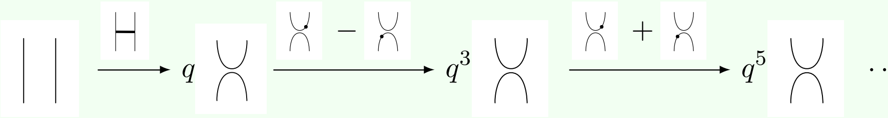
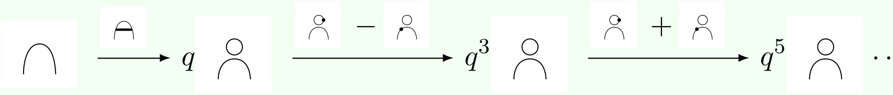
Categorified second projector P_2:



Contractibility under turnbacks:



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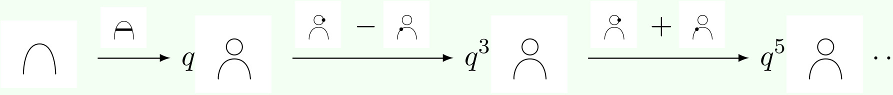

Applications of categorified projectors:

- Categorification of spin networks
- In particular, $6j$-symbols
- The colored Jones polynomial
- The $SO(3)$ Kauffman polynomial
- The hromatic polynomial of planar graphs

Algebra.

Temperley - Lieb algebra: TL_n

$p_n \in \mathrm{TL}_n$

$p_n \cdot p_n = p_n$

p_n is unique

Category.

Khovanov - Bar-Natan Category

$P_n \in \mathrm{Kom}(n)$, $K_0(P_n) = p_n$

$P_n \otimes P_n \simeq P_n$

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Alternative constructions:
Lev Rozansky, Igor Frenkel - Catharina Stroppel - Joshua Sussan

More recent developments:
Ben Cooper - Matt Hogancamp: generalized projectors (for $\mathfrak{sl}_2$)
David Rose: $\mathfrak{sl}_3$
Sabin Cautis: $\mathfrak{sl}_n$
Matt Hogancamp: functoriality under cobordisms

Recall:

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Categorify this construction?

$TL(\Sigma) \rightsquigarrow Kom(Cob(\Sigma))$ $\qquad$ $p_{n-1} \rightsquigarrow P_{n-1}$

Categorical analogue of taking a quotient $TL(\Sigma)/\langle p_{n-1}\rangle$:
Localization

Definition

Given P in $Ho(Kom(D^2)$, consider $<< P >>$: the smallest full subcategory of $Ho(Kom(\Sigma))$ which contains all objects obtained by gluing P to B where $\Sigma = D^2 \cup (\Sigma \setminus D^2)$, B is any object of $Ho(Kom(\Sigma \setminus D^2))$, and which is closed under cones and grading shifts.

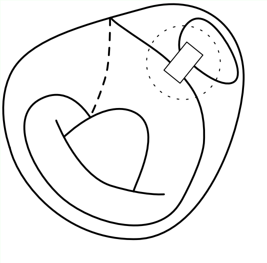

Figure: Anobject in $<< P >$

In a category $\mathcal{C}$, localized at an object P, the cone on any morphism $P \longrightarrow Q$ should be isomorphic to Q.

The classical Verdier localization is defined as the "quotient" of the category $\mathcal{C}$ by the smallest thick subcategory containing $<< P >>$.

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It turns out that the smallest thick subcategory containing $<< P_n >>$ is the entire category $Ho(Kom(\Sigma))$. (The chain complex for the trace of the projector is chain-homotopic to its homology). Therefore the Verdier localization is trivial.

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Towards 3-manifold invariants: we categorify the "magic element" ω at low levels, and show that the localized construction is invariant under handle slides.