LINK GROUPS OF 4-MANIFOLDS

VYACHESLAV KRUSHKAL

Abstract. The notion of a *Bing cell* is introduced, and it is used to define invariants, *link groups*, of 4-manifolds. Bing cells combine some features of both surfaces and 4-dimensional handlebodies, and the link group $\lambda(M)$ measures certain aspects of the handle structure of a 4-manifold M. This group is a quotient of the fundamental group, and examples of manifolds are given with $\pi_1(M) \neq \lambda(M)$. The main construction of the paper is a generalization of the Milnor group, which is used to formulate an obstruction to embeddability of Bing cells into 4-space. Applications to the A-B slice problem and to the structure of topological arbiters are discussed.

1. Introduction

Maps of surfaces and of more general 2-complexes have been classically used to define invariants of topological spaces, for example the fundamental group and the first homology group of a space. More generally, one gets the quotients of $\pi_1 X$ by the terms of its lower central series if one considers based loops in a space X modulo loops bounding maps of certain special 2-complexes, gropes [6]. From this perspective gropes interpolate between surfaces (null-homology) and disks (null-homotopy).

This paper introduces the notion of a *Bing cell*, which may be viewed as a geometric dual to a grope. The origin of this construction is in Milnor's theory of link homotopy [13]. The idea in the definition of a Bing cell is to treat a collection of 2-handles attached to a homotopically essential link on an equal footing with an actual 4-dimensional 2-handle $D^2 \times D^2$, see section 1.2 for a more detailed outline of the construction. A Bing cell is not a 2-complex, rather it is a 4-dimensional handlebody with a 2-dimensional spine where the 4-dimensional thickening plays an important role.

In an analogy with the fundamental group, the *link group* $\lambda(M)$ of a 4-manifold M is defined as based loops in M modulo loops bounding Bing cells. The resulting invariant $\lambda(M)$ reflects certain aspects of the handle structure of a 4-manifold M, and it is not correlated with the homology group $H_1(M)$. Although their definition makes sense in any dimension, the link groups are a non-trivial theory only in dimension 4, and they are a topological but *not* in general a homotopy invariant of a 4-manifold.

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Since the 2-cell D^2 is a trivial example of a Bing cell, the link group $\lambda(M)$ is a quotient of the fundamental group of M. One can easily find examples of 4-manifolds with $\pi_1(M) \neq \lambda(M)$. Bing cells may be given geometric and algebraic gradings: *height* and *nilpotent class*, leading to a two-parameter collection of link groups $\lambda_{i,j}(M)$, where $j > i$. In this notation, the group $\lambda(M)$ above corresponds to $\lambda_{\infty,\infty}$. We show that given a surjection of finitely presented groups $\pi \longrightarrow \lambda$, where π is aspherical, there are 4-manifolds M with $\pi_1(M) \cong \pi$, $\lambda_{1,2}(M) \cong \lambda$.

This work is motivated in part by the question of whether there is a "non-abelian" Alexander duality in dimension 4. This question arises in the analysis of decompositions of the 4-ball in the A, B-slice problem, a reformulation of the 4-dimensional topological surgery conjecture. An application of link groups in this context is given in [10], and it is summarized in section 1.3 below. For another recent application of the theory developed here, to the structure of topological arbiters, see [4] and theorem 1.5 below.

The connection to the A, B-slice problem is provided by the following theorem which is the main result of this paper, showing how Bing cells fit in the framework of Milnor's theory of link homotopy:

Theorem 1.1. *If the components of a link $L \subset S^3 = \partial D^4$ bound disjoint Bing cells in D^4 then L is homotopically trivial.*

This result is based on a generalization of the Milnor group which is developed here in order to formulate an obstruction to embeddability of a disjoint collection of Bing cells in 4-space. We will next give a brief outline of the ideas underlying the construction of Bing cells and of the generalized Milnor group, for more details see sections 4, 6.

1.2. **Outline of the construction.** Consider the 4-dimensional 2-handle $H = D^2 \times D^2$, thought of as a 4-dimensional thickening of its core $D^2 \times \{0\}$. Remove a small disk D^2_ϵ from the core, and consider the corresponding thickening $H_\epsilon = (D^2 \smallsetminus D^2_\epsilon) \times D^2$. H_ϵ has a new part of the boundary, $S^1_\epsilon \times D^2$, which is the boundary of $D^2_\epsilon \times D^2$ that was removed from the handle H. Attach to H_ϵ a pair of zero-framed 2-handles h_1, h_2 along the Bing double of the core of the solid torus $S^1_\epsilon \times D^2$, see figure 1.

This is the basic operation used in the construction of Bing cells, and it may be roughly described as "puncturing" a 2-handle and plugging in the puncture with 2-handles whose attaching curves form an *essential* link in the boundary of the puncture. Here the term "essential" is understood in the context of Milnor's theory of link homotopy, see [13] and section 3.2 in this paper. The most important example (motivating the term *Bing cell*) is the Bing double, or more generally an iterated Bing double, of the core of the solid torus.

Now a *Bing cell of height* 1 is obtained from the 2-handle H by performing this operation in a finite number of distinct locations in the core $D^2 \times \{0\}$. For example,

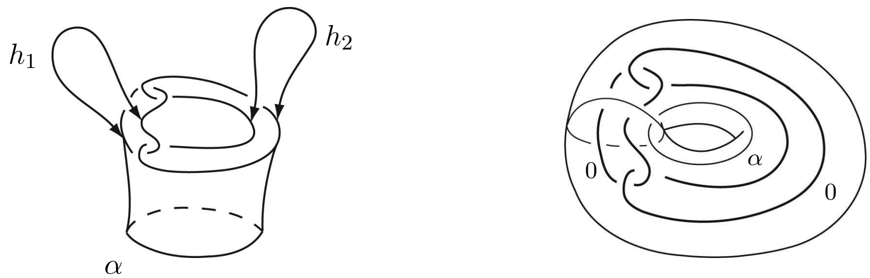

FIGURE 1. A pair of 2-handles attached to the Bing double.

consider the case of two punctures in more detail. Let P denote the pair of pants with boundary components $\gamma, \alpha_1, \alpha_2$, and let C denote $(P \times D^2) \cup$ four zero-framed 2-handles $h_1 \dots, h_4$ attached to the Bing doubles of the curves α_1, α_2, figure 2. The operation above also may be applied to the handles h_i, leading to the construction of a Bing cell of height 2. Iterating this procedure (a finite number of times) yields an inductive construction of a general Bing cell. There is a distinguished curve γ in the boundary of a Bing cell C which is the attaching curve of the original 2-handle H, $\gamma = \partial D^2 \times \{0\}$, and the term Bing cell will refer to a pair (C, γ).

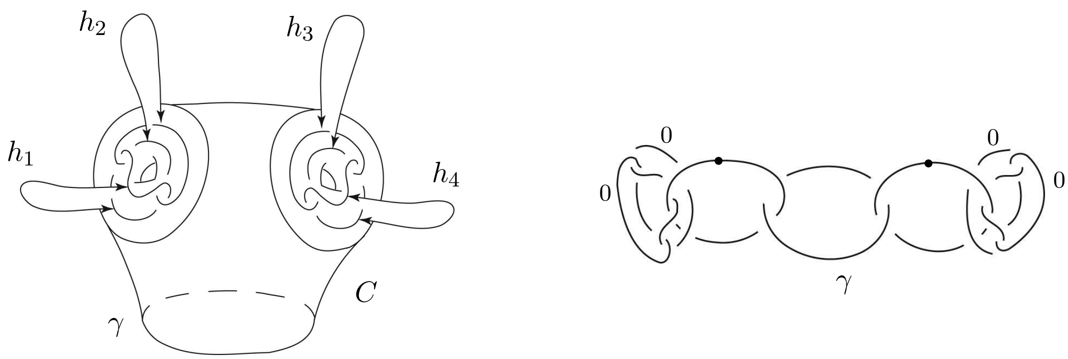

FIGURE 2. The handlebody C (a Bing cell of height 1): a schematic picture and a Kirby diagram.

The defining property in Milnor's theory of link homotopy [13] is that the link components are allowed to move by a homotopy so that *different components stay disjoint* from each other. It is natural to consider Bing cells mapped into 4-manifolds, requiring that the handles attached to different components of the Bing doubles are disjoint in the image. For example, in the case of the Bing cell C in figure 2, for a map $f\colon C \longrightarrow M^4$ the requirement is that $f(h_1)$ is disjoint from $f(h_2)$ and similarly $f(h_3)$ is disjoint from $f(h_4)$.

It is a classical fact [13, 7, 8] that the components of a homotopically essential link in S^3 do not bound disjoint maps of disks in D^4. The 2-handle (a thickening of the disk) may be considered a trivial example of a Bing cell, and the content of theorem 1.1 is the more general fact that the components of a homotopically essential link do not bound disjoint maps of Bing cells into the 4-ball.

One may generalize further and consider surfaces (including those of higher genus) in 4-manifolds with patches replaced by Bing cells. This leads to an interesting theory that shares some of the features of both homology and homotopy. An important open question for applications to the A-B slice problem (see section 1.3) is to what extent this theory satisfies Alexander duality. This question will be pursued in a subsequent paper.

We will next summarize the ideas underlying the definition of the generalized Milnor group and the proof of theorem 1.1. An outline of the argument here will be given to prove that the components (γ', γ'') of the Hopf link do not bound disjoint maps into D^4 of two copies C', C'' of the Bing cell shown in figure 2, where the maps are assumed to satisfy the disjointness requirements discussed above. This example exhibits the main features of the general argument; a complete proof of theorem 1.1 is given in section 9.

Suppose to the contrary that there exist C', C'' whose attaching curves γ', γ'' form the Hopf link in $S^3 = \partial D^4$. Consider meridians m_i' to the 2-handles $\{h_i'\}$ of C' and meridians m_i'' to the 2-handles $\{h_i''\}$ of C'' in D^4. By a meridian we mean a based loop in the complement representing the same homology class as a fiber of the normal circle bundle over a 2-handle. These meridians normally generate the fundamental group $\pi := \pi_1(D^4 \smallsetminus (C' \cup C''))$. Consider the quotient $\overline{M}\pi$ of π by the relations $[(m_i')^x, (m_j')^y]$ and $[(m_i'')^x, (m_j'')^y]$, where $\{i, j\} \neq \{1, 2\}, \{3, 4\}$ and the conjugating elements x, y range over all elements of π. These relations arise from the double points between the various 2-handles in D^4 (they are an algebraic manifestation of the Clifford tori linking the double points, see section 3.8 for further details). The fact that the pairs $\{1, 2\}, \{3, 4\}$ are excluded reflects the fact that these handles are required to be disjoint in the 4-ball. The fact that the conjugates of m_i', m_j'' for any i, j do not commute is due to the fact that C', C'' are disjoint. This is a generalization of the Milnor group $M\pi$ introduced in [13], which in this context is defined as the quotient of π by the relations $[(m_i')^x, (m_i')^y], [(m_i'')^x, (m_i'')^y], i = 1, \ldots, 4$, corresponding to *self*-intersections of the handles. These relations are included among the defining relations of $\overline{M}\pi$.

Let m', m'' denote the meridians to the components of the Hopf link in ∂D^4, formed by the attaching curves γ', γ''. The relation $[m', m''] = 1$ holds in $\pi_1(S^3 \smallsetminus (\gamma' \cup \gamma''))$, therefore it also holds in $\pi_1(D^4 \smallsetminus (C' \cup C''))$. On the other hand, sliding the meridian along arcs in the pair of pants in figure 2 it is easy to see that the equalities $m' = [m_1', m_2'] = [m_3', m_4'], m'' = [m_1'', m_2''] = [m_3'', m_4'']$ hold in $\overline{M}\pi$. Carefully analyzing the second homology of the complement $D^4 \smallsetminus (C' \cup C'')$ one shows that there are no other relations in $\overline{M}\pi$. Then the Magnus expansion can be used to prove that the commutator $[m', m'']$ is in fact non-trivial in $\overline{M}\pi$: phrased differently, there are "not enough" relations in $\pi_1(D^4 \smallsetminus (C' \cup C''))$ to imply the relation $[m', m''] = 1$ which holds in $\pi_1(S^3 \smallsetminus (\gamma' \cup \gamma''))$. This contradiction concludes the outline of the proof that the components of the Hopf link do not bound disjoint Bing cells of height 1 in D^4. A complete proof of theorem 1.1 in section 9 relies on a detailed analysis of the relations in the generalized Milnor group of the complement of Bing cells (of an arbitrary height) in D^4, which forms a central technical part of the paper.

1.3. **Applications: the A-B slice problem, topological arbiters.** The ideas introduced in this paper have been used to prove a number of results in 4-manifold topology. We will now summarize these applications.

The *A-B slice problem* is a reformulation of the 4-dimensional topological surgery conjecture, introduced by Freedman [3] and further developed by Freedman-Lin [5]. In this problem one considers decompositions of the 4-ball, $D^4 = A \cup B$, which extend the standard genus 1 Heegaard decomposition of $S^3 = \partial D^4$. The attaching curves $\alpha \subset \partial A, \beta \subset \partial B$ form the Hopf link in ∂D^4. The problem is then to find out whether there exist decompositions $D^4 = A_i \cup B_i$ and disjoint embeddings of the submanifolds $\{A_i, B_i\}$ into D^4, so that the attaching curves $\{\alpha_i, \beta_i\}$ form a specified link (and its parallel copy) in S^3. The central example of a link in question is the Borromean rings, see [3, 5, 10] for more details.

In [5] the authors introduced a family of *model decompositions* which appear to approximate, in a certain algebraic sense, an arbitrary decomposition $D^4 = A \cup B$. In [10] the author showed how the idea of link groups of 4-manifolds and theorem 1.1 may be used to formulate an obstruction for the family of model decompositions:

Theorem 1.4. [10] *Let L be the Borromean rings, or more generally any homotopically essential link in S^3. Then L is not A-B slice where each decomposition $D^4 = A_i \cup B_i$ is a model decomposition.*

The proof is based on the observation that given a model decomposition, precisely one of the following two possibilities hold: either α bounds a Bing cell in A or β bounds a Bing cell in B. Note that Bing cells of an arbitrary height, not just height 1, are needed to prove this result. Phrased in terms of link groups, either $\alpha = 1 \in \lambda(A)$ or $\beta = 1 \in \lambda(B)$. The proof then follows as a consequence of theorem 1.1. This proof unified and generalized the previously known partial obstructions in the A-B slice program. The idea based on Bing cells is likely to be central in a solution to this problem.

The notion of a *robust* 4-manifold is useful in putting theorems 1.1, 1.4 in the context of link homotopy. Let (M, γ) be a pair (4-manifold, embedded curve in ∂M). The pair (M, γ) is *robust* if whenever several copies (M_i, γ_i) are properly disjointly embedded in (D^4, S^3), the link formed by the curves $\{\gamma_i\}$ in S^3 is homotopically trivial. Therefore from the perspective of link homotopy theory, robust 4-manifolds act like disks (with self-intersections). It follows from theorem 1.1 that Bing cells are robust. Theorem 1.4 may be rephrased as saying that given a model decomposition $D^4 = A \cup B$, precisely one of the two parts A, B is robust.

We will now discuss an application of the results of this paper to the structure of *topological arbiters*, established in [4]. Given an n-dimensional manifold W, a topological arbiter associates a value 0 or 1 to codimension zero submanifolds of W, subject to natural topological and duality axioms. For example, there is a unique arbiter on $\mathbb{R}P^2$, which reports the location of the essential 1-cycle. The concept of a topolocial arbiter is rooted in Poincaré-Lefschetz duality, indeed homology with field coefficients gives rise to arbiters on projective spaces. A question addressed in [4] is the existence of arbiters not induced by homology.

Theorem 1.5. [4] *There exists an uncountable collection of local topological arbiters in dimension* 4*.*

Theorem 1.1 is an important ingredient in the proof of this result. A local arbiter is a version of a topological arbiter defined on the ball. It is defined on codimension zero submanifolds of D^4 which meet ∂D^4 in a neighborhood of an unknotted circle, and duality in this case is modeled on Alexander duality for homology. Note that homology with various field coefficients can be used to construct only a countable collection of arbiters (in any dimension). Theorem 1.5 contrasts with the situation in dimension 2 where there is a unique local arbiter, and it is induced by homology. A classification of topological arbiters (in dimensions other than 2) remains an open problem; the tools developed in this paper are an important ingredient in analyzing arbiters on D^4.

1.6. **Outline of the paper.** Sections 2, 3 summarize the relevant background material on presentations of nilpotent quotients, Milnor's theory of link homotopy and related results on surfaces in 4-space. Section 4 introduces Bing cells and link groups $\lambda(M)$, and gives examples of 4-manifolds with $\pi_1 \neq \lambda$. Sections 5 – 9 concern embeddings of Bing cells in 4-space, $(C, \gamma) \hookrightarrow (D^4, S^3)$. More specifically, the fundamental group of the complement, $\pi_1(D^4 \smallsetminus C)$, is analyzed in section 5. Section 6 develops a generalization of the Milnor group in the context of Bing cells. It is used, in particular, to define an algebraic grading of Bing cells and link groups $\lambda_{i,j}$. Sections 7, 8, 9 define an obstruction to embeddability of a collection of Bing cells in D^4 with a prescribed boundary, given by a link in S^3.

2. A presentation of nilpotent quotients

The purpose of this section is to describe a presentation of the quotients π/π^q of a group π by the terms of its lower central series, in terms of generators of the first and second homology of π. This technique is well-known (see also [9]), and it will be used often throughout the paper. The lower central series of a group π is defined inductively by $\pi^1 = \pi$, $\pi^2 = [\pi, \pi], \ldots, \pi^q = [\pi, \pi^{q-1}]$.

To state the lemma, fix a group π and suppose that $H_1(\pi;\mathbb{Z})$ is generated by $g_1, \ldots, g_n$, $H_2(\pi;\mathbb{Z})$ is generated by $r_1, \ldots, r_m$, and let $q \geq 2$ be an integer. Then the result of lemma 2.1 is that, roughly speaking, $g_1, \ldots, g_n$ and $r_1, \ldots, r_m$ provide a set of generators and relations respectively in a presentation of π/π^q. To make this precise, consider the quotient homomorhism $\alpha\colon\, \pi/\pi^q \longrightarrow \pi/[\pi,\pi]$ and let $\hat{g}_i \in \pi/\pi^q$ denote some preimage of g_i under α, $i = 1, \ldots, n$. It is a standard fact in nilpotent group theory [16] that $\hat{g}_1, \ldots, \hat{g}_n$ generate π/π^q.

Let $W \longrightarrow K(\pi, 1)$ be a map from the wedge of n circles W, inducing an epimorphism $\beta\colon\, \pi_1(W) \longrightarrow \pi/\pi^q$ and mapping the i-th free generator of $\pi_1(W)$ to $\hat{g}_i$. Let $f_j\colon\, \Sigma_j \longrightarrow K(\pi,1)$ be a map of a surface Σ_j, representing the generator r_j of $H_2(K(\pi,1)) \cong H_2(\pi)$, $j = 1, \ldots, m$. We assume here that each space has a fixed basepoint, and all maps preserve them. The standard basis of $H_1(\Sigma_j)$ pulls back via β to some elements in $\pi_1(W)$; let $\hat{r}_j \in \pi_1(W)$ be a lift via β of the attaching map of the 2-cell of Σ_j. (In particular, if Σ_j is a 2-sphere then the corresponding word $\hat{r}_j$ is trivial.)

Lemma 2.1. *Suppose $H_1(\pi;\mathbb{Z})$ is generated by $g_1, \ldots, g_n$, and $H_2(\pi;\mathbb{Z})$ is generated by $r_1, \ldots, r_m$. Then in the notations as above,*

$$\pi/\pi^q \cong <\hat{g}_1, \ldots, \hat{g}_n \mid \hat{r}_1, \ldots, \hat{r}_m, (F_{\hat{g}_1,\ldots,\hat{g}_n})^q>$$

where $F_{\hat{g}_1,\ldots,\hat{g}_n}$ denotes the free group on generators $\hat{g}_1, \ldots, \hat{g}_n$.

To prove the lemma we need a refinement of Stallings theorem [15] due to Dwyer. Given a space X, the *Dwyer's subspace* $\phi_k(X) \subset H_2(X;\mathbb{Z})$ is defined as the kernel of the composition

$$H_2(X) \longrightarrow H_2(K(\pi_1 X, 1)) = H_2(\pi_1 X) \longrightarrow H_2(\pi_1(X)/\pi_1(X)^{k-1}).$$

Theorem 2.2. [2] *Let k be a positive integer and let $f\colon\, X \longrightarrow Y$ be a map inducing an isomorphism on $H_1(\,.\,;\mathbb{Z})$ and mapping $H_2(X)/\phi_k(X)$ onto $H_2(Y)/\phi_k(Y)$. Then f induces an isomorphism $\pi_1(X)/(\pi_1(X))^k \cong \pi_1(Y)/(\pi_1(Y))^k$.*

Proof of lemma 2.1. Let X be the 2-complex obtained from W by attaching m two-cells along the words $\hat{r}_1, \ldots \hat{r}_m$. The composition $W \longrightarrow K(\pi,1) \longrightarrow K(\pi/\pi^q, 1)$ extends to X, inducing an isomorphism $H_1(X) \cong H_1(\pi) \cong H_1(\pi/\pi^q)$ and an epimorhism on H_2/ϕ_q. Now an application of Dwyer's theorem 2.2 concludes the proof of Lemma 2.1. $\square$

3. The Milnor group: links in S^3 and surfaces in D^4

In this section we recall the relevant results on Milnor groups and $\bar{\mu}$-invariants [13], [14]. This material is used to set up the definition of Bing cells in section 4. Sections 5 – 9 develop a generalization of the Milnor group and of other aspects of the theory in the context of Bing cells in D^4.

3.1. **Links in $\mathbf{S^3}$.** Let $L=(l_1,\dots,l_n)$ be an oriented link in S^3, and consider meridians $m_1,\dots,m_n$ to the components of L. By a meridian m_i we mean a path γ_i in $S^3\smallsetminus L$ from a basepoint to the boundary of a regular neighborhood of the component l_i, followed by a circle (a fiber of the circle normal bundle over l_i) linking l_i once and then followed by γ_i^{-1} back to the basepoint. Observe that $H_1(S^3\smallsetminus L)$ is generated by $m_1,\dots,m_n$, and a set of generators for $H_2(S^3\smallsetminus L)$ is provided by $n-1$ tori: the boundary of a regular neighborhood of $n-1$ components of L. By lemma 2.1, $\pi_1(S^3\smallsetminus L)/(\pi_1(S^3\smallsetminus L))^q$ has a presentation

$$< m_1,\dots,m_n | [m_1,w_1],\dots,[m_{n-1},w_{n-1}],(F_{m_1,\dots,m_n})^q >, \tag{3.1}$$

where $F_{m_1,\dots,m_n}$ denotes the free group generated by $m_1,\dots,m_n$, and w_j is a word in $m_1,\dots,m_n$ representing the untwisted j-th longitude of the link. The Magnus expansion homomorphism $M\colon F_{m_1,\dots,m_n}\longrightarrow \mathbb{Z}\{x_1,\dots,x_n\}$ into the ring of formal non-commutative power series in the indeterminates $x_1,\dots,x_n$ is defined by

$$M(m_i)=1+x_i,\ M(m_i^{-1})=1-x_i+x_i^2\pm\dots$$

for $i=1,\dots,n$. Let

$$M(w_j)=1+\Sigma\mu_L(I,j)x_I$$

be the expansion of w_j, where the summation is taken over all multi-indices $I=(i_1,\dots,i_k)$ with entries between 1 and n, and $x_I=x_{i_1}\cdot\ldots\cdot x_{i_k}$, $k>0$. This expansion defines for each such multi-index I the integer $\mu_L(I,j)$. Let $\Delta_L(i_1,\dots,i_k)$ denote the greatest common divisor of $\mu_L(j_1,\dots,j_s)$ where $j_1,\ \dots,j_s$, $2\le s\le k-1$ range over all sequences obtained by cancelling at least one of the indices $i_1,\dots,i_k$ and permuting the remaining indices cyclically.

Let $\bar{\mu}_L(I)$ denote the residue class of $\mu_L(I)$ modulo $\Delta_L(I)$. Analyzing the indeterminacy caused by the relations in the presentation (3.1), one sees that for each multi-index I of length $|I|\le q$ the residue class $\bar{\mu}_L(I)$ is an *isotopy invariant* of the link L, where $\bar{\mu}_L(I)$ is defined using the quotient $\pi_1(S^3\smallsetminus L)/(\pi_1(S^3\smallsetminus L))^q$. In particular, *the first non-vanishing coefficients $\mu_L(I)$ are well-defined.* (By first non–vanishing coefficients we mean $\mu_L(I)$ such that $\mu_L(J)=0$ for all proper subsets $J\subset I$.)

3.2. **Link homotopy and Milnor groups.** Two n-component links L and L' in S^3 are said to be *link-homotopic* if they are connected by a 1-parameter family of immersions such that different components stay disjoint at all times. L is said to be *homotopically trivial* if it is link-homotopic to the unlink. L is *almost homotopically trivial* if each proper sublink of L is homotopically trivial.

For a group π normally generated by $g_1, \dots, g_k$ its *Milnor group* (with respect to $g_1, \dots, g_k$) $M\pi$ is defined to be the quotient of π by the normal subgroup

$$\ll [g_i, g_i^h] : 1 \le i \le k, \;\; h \in \pi \gg . \tag{3.2}$$

$M\pi$ is nilpotent of class $\le k+1$, in particular it is a quotient of $\pi/(\pi)^{k+1}$, and is generated by the quotient images of $g_1, \dots, g_k$. The Milnor group $M(L)$ of a link L is defined to be $M\pi_1(S^3 \smallsetminus L)$ with respect to its meridians m_i.

Milnor showed in [13] that the Magnus expansion induces a well defined injective homomorphism $MM\colon M(F_{m_1,\dots,m_k}) \longrightarrow R(x_1, \dots, x_k)$ into the ring $R(x_1, \dots, x_k)$ which is the quotient of $\mathbb{Z}\{x_1, \dots, x_k\}$ by the ideal generated by monomials $x_{i_1} \cdot \cdot x_{i_r}$ with some index occuring at least twice. Indeed, every term in the Magnus expansion of each defining Milnor relation (3.2) has repeating variables. Let $\overline{w}_n \in MF_{m_1,\dots,m_{n-1}}$ be a word representing l_n in $M\pi_1(S^3 \smallsetminus (l_1 \cup \ldots \cup l_{n-1}))$. Then $\bar{\mu}$-invariants of L with non-repeating coefficients may also be defined by the equation

$$MM(\overline{w}_n) = 1 + \Sigma \mu_L(I, n) x_I$$

where summation is over all multi-indices I with non-repeating entries between 1 and $n-1$, and $\bar{\mu}_L(I,n)$ is the residue class of $\mu_L(I,n)$ modulo the indeterminacy $\Delta_L(I,n)$, defined above.

The Milnor group of L is the largest common quotient of the fundamental groups of all links link-homotopic to L, hence *if L and L' are link homotopic then their Milnor groups are isomorphic.* The next result gives an algebraic criterion for a link to be null-homotopic.

Lemma 3.3. [13, 7, 8] *For an n-component link L, the following conditions are equivalent:*

(i) L is homotopically trivial,
(ii) the components of L bound disjoint immersed disks in D^4,
(iii) $M(L) \cong M(F_{m_1,\dots,m_n})$ with the isomorphism carrying a meridian to l_i to the generator m_i of the free group,
(iv) all $\bar{\mu}$-invariants of L with non-repeating coefficients vanish.

It follows from Lemma 3.3 that L is almost homotopically trivial if and only if all its $\bar{\mu}$-invariants with non-repeating coefficients of length less than n vanish. In particular, if L is almost homotopically trivial then its $\bar{\mu}$-invariants with non-repeating coefficients of length n are well-defined integers.

3.4. **The link composition lemma.** We will now recall the link composition lemma [5] (see also [12]). The result on Bing cells proved in section 9 contains this theorem as a special case. Given a link $\widehat{L} = (l_1, \dots, l_{k+1})$ in S^3 and a link $Q = (q_1, \dots, q_m)$ in the solid torus $S^1 \times D^2$, their "composition" is obtained by replacing the last component of $\widehat{L}$ with Q. More precisely, it is defined as $C = (c_1, \dots, c_{k+m}) := (l_1, \dots, l_k, \phi(q_1), \dots, \phi(q_m))$, where $\phi\colon\ S^1 \times D^2 \hookrightarrow S^3$ is a 0-framed embedding whose image is a tubular neighborhood of l_{k+1}. The meridian $\{1\} \times \partial D^2$ of the solid torus will be denoted by $\wedge$ and we set $\widehat{Q} := Q \cup \wedge$.

Theorem 3.5. *If both $\widehat{L}$ and $\widehat{Q}$ are homotopically essential in S^3 then so is their composition $L \cup \phi(Q)$.*

3.6. **Links in $\mathbf{S^1} \times \mathbf{D^2}$.** Let L be a link in $S^1 \times D^2$. As above denote by $\wedge$ the meridian $\{p\} \times \partial D^2$ and set $\widehat{L} = L \cup \wedge$. Consider $\widehat{L}$ as a link in S^3, using a standard embedding $S^1 \times D^2 \subset S^3$. Links in the solid torus will be used as attaching regions for 2-handles in the definition of Bing cells (see next section), and we need to specify the class of links necessary for the definition. Let $\wedge'$ denote another meridian $\{q\} \times \partial D^2$, $p \neq q$.

Definition 3.7. (Links used in the definition of Bing cells) A link $L = (l_1 \dots, l_n) \subset S^1 \times D^2$ is *essential and (almost trivial)$^+$* if $\widehat{L}$ is homotopically essential, and for each i, $(L \smallsetminus l_i) \cup \wedge \cup \wedge'$ is homotopically trivial.

An important example is given by $L =$Bing double of the core $S^1 \times \{0\}$ (see figure 1), and more generally by $L =$iterated Bing double of the core. The fact that the (iterated) Bing doubles satisfy the conditions in definition 3.7 follows from a computation of the $\bar{\mu}$-invariants of Bing doubles (cf. [13, 1]), see the discussion below. The definition also allows the trivial example: $L =$ core of the solid torus.

The second condition is slightly stronger than just the requirement that $\widehat{L}$ is almost homotopically trivial. We include it since it is technically convenient for the proofs of the properties of Bing cells. We need to reformulate the conditions on L in terms of $\bar{\mu}$ invariants. Consider the solid torus $S^1 \times D^2$ as the complement in S^3 of an unknotted circle and note that

$$\pi_1((S^1 \times D^2) \smallsetminus L)/((\pi_1(S^1 \times D^2) \smallsetminus L))^q \ \cong\ \pi_1(S^3 \smallsetminus (L \cup \wedge'))/\pi_1(S^3 \smallsetminus (L \cup \wedge'))^q.$$

These groups are generated by the meridians $m_1, \dots, m_n$ to the components of L and by the longitude $l = S^1 \times \{x\}$ of the solid torus (respectively the meridian $\bar{m}$ to $\wedge'$ for the second group.) Consider the free group $F_{m_1,\dots,m_n,\bar{m}}$ mapping onto these groups, and the Magnus expansion

$$M\colon\ F_{m_1,\dots,m_n,\bar{m}} \longrightarrow \mathbb{Z}\{x_1, \dots, x_n, y\},\ \ M(m_i) = 1 + x_i,\ M(\bar{m}) = 1 + y. \tag{3.3}$$

Let W be a word representing $\wedge$ in the free group. Assuming that L satisfies the conditions in the definition above, observe that all terms with *non-repeating variables* in the expansion $M(W)$ are either of the form $x_{i_1}\cdots x_{i_n}$ or they contain all variables $x_1,\dots x_n$ and y. Since the link $\widehat{L}$ is homotopically essential, renumbering the components of L if necessary, one can assume that the term $\mu\, x_1\cdots x_n$ in the Magnus expansion $M(W)$ has the coefficient $\mu\neq 0$. It is important to note that there are no terms that contain y and just a proper subset of the variables $x_1,\dots,x_n$.

3.8. **Surfaces in $\mathbf{D^4}$.** Let $\Delta=\cup\Delta_i$ be a collection of immersed disks in $(D^4,\partial D^4)$. By Alexander duality, $H_1(D^4\smallsetminus\Delta)$ is generated by the meridians to the components of Δ, and $H_2(D^4\smallsetminus\Delta)$ is generated by the *Clifford tori* linking the double points of Δ.

More precisely, a local model for the surfaces near a double point is given by $\mathbb{R}^2\times\{0\}\cap\{0\}\times\mathbb{R}^2\subset\mathbb{R}^2\times\mathbb{R}^2$. The Clifford torus is the product of the unit circles $S^1\times S^1$. The linking number of classes $a\in H_1(\Delta)$ and $b\in H_2(D^4\smallsetminus\Delta)$ may be computed as the intersection number of $\Sigma\cdot b$ where $a=\partial\Sigma\subset D^4$. $H_1(\Delta)$ is generated by the double point loops (loops in Δ passing exactly once through a double point). It is clear from the local model that the double point loops are paired up $\delta_{i,j}$ with the Clifford tori.

Suppose the disks Δ_i are disjoint, so all double points are self-intersections. According to lemma 2.1, $\pi_1(D^4\smallsetminus\Delta)/(\pi_1(D^4\smallsetminus\Delta))^q$ is generated by the meridians $m_1,\dots,m_n$ to the components of Δ, and the relations (corresponding to the Clifford tori) are all of the form $[(m_i)^f,(m_j)^g]=1$ for some f,g. In particular, the Milnor group $M\pi_1(D^4\smallsetminus\Delta)$ (with respect to the meridian generators) is isomorphic to the free Milnor group $MF_{m_1,\dots,m_n}$.

This gives a useful perspective on the relation between (i) and (ii) in lemma 3.3: if a link L is homotopically essential then $M(L)$ is not isomorphic to the free Milnor group. This implies that the components of L do not bound disjoint maps Δ of disks in D^4: otherwise the inclusion map $S^3\smallsetminus L\longrightarrow D^4\smallsetminus\Delta$ would induce a homomorphism $M(L)\longrightarrow MF_{m_1,\dots,m_n}$, a contradiction.

4. Bing cells and link groups

There are two kinds of Bing cells (b-cells) defined in this section. First we discuss abstract ("model") Bing cells, then definition 4.4 introduces the notion of a Bing cell in a 4-manifold M which is a map of a model Bing cell into M with only certain types of allowed singularities.

Definition 4.1. A *model Bing cell (b-cell) of height* 1 is a smooth 4-manifold C with boundary and with a specified attaching curve $\gamma\subset\partial C$, defined as follows. Consider a planar surface P with $k+1$ boundary components $\gamma,\alpha_1,\dots,\alpha_k$ $(k\geq 0)$, and set

$\overline{P} = P \times D^2$. Let $L_1, \dots, L_k$ be a collection of links, $L_i \subset \alpha_i \times D^2$, $i = 1, \dots, k$. We assume that for each i, $\widehat{L}_i$ is essential and (almost trivial)$^+$, see definition 3.7. Then C is obtained from $\overline{P}$ by attaching zero-framed 2-handles along the components of $L_1 \cup \dots \cup L_k$.

The surface P (and its thickening $\overline{P}$) will be referred to at the *body* of C, and the 2-handles are the *handles* of C.

A *model b-cell C of height h*, $h > 1$, is obtained from a b-cell of height $h-1$ by replacing its handles with b-cells of height one. The *body* of C consists of all (thickenings of) its surface stages, except for the handles.

The most important example of links L_i in the definition above is given by (iterated) Bing doubles of the core $\alpha_i \times \{0\}$ of the solid torus $\alpha_i \times D^2$. These are the links that appear in applications to the A-B slice problem [10] and to topological arbiters [4]. Figure 2 in the introduction gives an example of a b-cell of height 1: a schematic picture and a precise description in terms of a Kirby diagram. Here P is a pair of pants, and each link L_i is the Bing double of the core of the solid torus $\alpha_i \times D^2$, $i = 1, 2$. The reader is urged to keep this example in mind while reading the paper: the theory already exhibits many of its interesting features in this case.

Remarks. 1. The standard 2-handle $H = D^2 \times D^2$ with $\gamma = \partial D^2 \times \{0\}$ provides a trivial example of a b-cell (of any height) – corresponding to the case $k = 0$ in the definition above. Alternatively, one gets the 2-handle H by considering the links $L =$ cores of the corresponding solid tori. Similarly, a b-cell of height h also satisfies the definition of a b-cell of any height $> h$.

2. One may assume that no body surface of C above the first stage is an annulus: suppose an annulus A is present, $\partial A = \gamma_A \cup \alpha_A$. Then A may be used to deform the attaching maps of handles or higher stages from $\alpha_A \times D^2$ to $\gamma_A \times D^2$. This eliminates A (and increases the number of components of the link one stage below – note that it is still essential and (almost trivial)$^+$, see link composition lemma 3.5 and also section 9). So while technically annuli are allowed by the definition, only planar surfaces with ≥ 3 boundary components above the first stage contribute to the "non-trivial" increase of the height of C.

3. If the links L defining C have at least two components, then C is homotopy equivalent to the wedge of a collection of circles and of a collection of 2-spheres (one for each handle of C). Of course if one considers C up to homotopy then all relevant information (in definition 3.7) about the attaching maps of the 2-handles is lost. This is the reason for the fact that link groups defined further below are a topological but not in general a homotopy invariant of 4-manifolds. Also note that in non-trivial examples of Bing cells C, $\gamma \neq 0 \in H_1(C)$; the link group $\lambda(M)$ and the first homology group $H_1(M)$ are not correlated.

4. In the definition above we used zero framed 2-handles. In fact, in light of definition 4.4 the framing is not going to be important for applications.

5. Recall the assumptions on each link L in definition 4.1: (i) $\widehat{L}$ is homotopically essential, and (ii) $L \cup \wedge \cup \wedge'$ is almost trivial. It is crucial for the applications of b-cells that the link $L \cup \wedge$ is essential – this is made precise using the Magnus expansion $M(\wedge)$, see section 3.6. Therefore the basic requirements on L should be: $\widehat{L}$ is homotopically essential and almost trivial. We made a slightly stronger assumption: L is (almost trivial)$^+$ since this makes the proofs of the properties of b-cells technically easier. It is an interesting question whether this extra condition may be removed in the proof of theorem 1.1.

4.2. **The associated tree.** It is helpful to encode the branching of a b-cell C using an associated tree T_C as follows. Define T_C inductively: suppose C has height 1. Then assign to the body surface P (say with $k+1$ boundary components) of C the cone T_P on $k+1$ points. Consider the vertex corresponding to the attaching circle γ of C as the root of T_P, and the other k vertices as the leaves of T_P. For each handle of C attach an edge to the corresponding leaf of T_P, see figure 3. The leaves of the resulting tree T_C are in 1-1 correspondence with the handles of C.

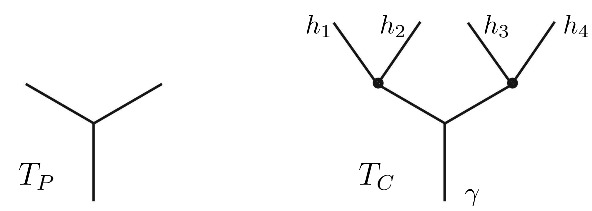

FIGURE 3. The trees T_P, T_C corresponding to the Bing cell in figure 2

Suppose C has height $h > 1$, then it is obtained from a b-cell C' of height $h-1$ by replacing the handles of C' with b-cells $\{C_i\}$ of height 1. Assuming inductively that $T_{C'}$ is defined, one gets T_C by replacing the edges of $T_{C'}$ associated to the handles of C' with the trees corresponding to $\{C_i\}$. Figure 4 gives an example of a b-cell of height 2 and its associated tree.

It is convenient to divide the vertices of T_C into two types: the cone points corresponding to body surfaces are *unmarked*; the rest of the vertices are *marked* and are represented in figures by a wider dot. Therefore the valence of an unmarked vertex equals the number of boundary components of the corresponding body surface. The marked vertices are in 1-1 correspondence with the links L defining C, and the valence of a marked vertex is the number of components of L plus 1. It is convenient to consider the 1-valent vertices of T_C (its root and leaves, corresponding to the handles of C) as unmarked. This terminology is useful in defining the maps of

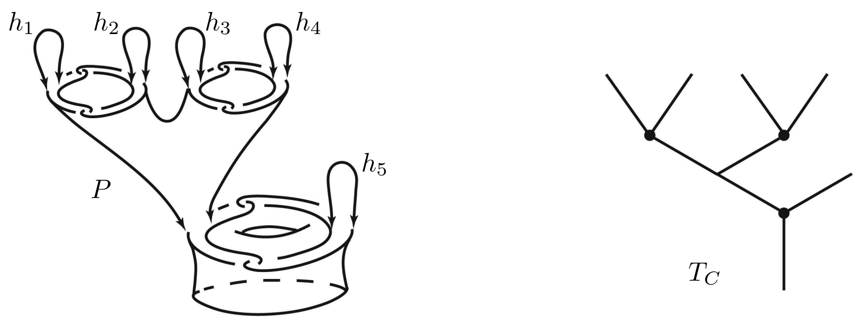

FIGURE 4. A schematic picture of a Bing cell C of height 2 and the associated tree. Note that the bottom stage planar surface in C is an annulus, giving rise to an unmarked 2-valent vertex which is not indicated in the bottom edge in T_C.

b-cells into 4-manifolds below. The height of a b-cell C may be read off from T_C as the maximal number of marked vertices along a geodesic joining a leaf of T_C to its root.

4.3. **Convention:** Recall from section 3.6 that for each link L in the definition of a Bing cell there is an ordering $l_1, \ldots, l_n$ of its components so that the coefficient of the monomial $x_1 \cdots x_n$ in the expansion $M(\wedge)$ is non-trivial. Fix a specific planar embedding of T_C reflecting this order, so that the clockwise ordering of the edges coincides with the ordering $1, \ldots, n$ of the link components. This applies to marked vertices; there is a flexibility in the planar embedding of the tree at its unmarked vertices.

Definition 4.4. A *Bing cell in a* 4*-manifold* M is an embedding $\overline{C} \subset M$, where C is a model Bing cell and $\overline{C}$ is the result of a finite number of self-plumbings and plumbings among the handles and body surfaces of C, subject to the following disjointness requirement:

• Let A, B be either handles or body stages of C, and let a, b denote the corresponding vertices in T_C. (For body surfaces this is the corresponding unmarked cone point, for handles this is the associated leaf.) Consider the geodesic joining a, b in T_C, and look at its vertex c closest to the root of T_C. In other words, c is the first common ancestor of a, b. If c is a marked vertex then no plumbings are allowed between A and B.

In particular, self-plumbings of any handle and body surface are allowed. In the example shown in figure 4, the handle h_1 is required to be disjoint from h_2, h_3 is disjoint from h_4; h_1-h_4 and P are disjoint from h_5. Abusing the notation, throughout the paper we will denote Bing cells in 4-manifolds by $C \subset M^4$, meaning the embedding of the plumbed version $\overline{C}$ into M.

Note that a Bing cell C is a thickening of a 2-dimensional spine, and in particular the solid tori $\alpha \times D^2$ which serve as the attaching regions for higher stages are thickenings of circles. From this perspective, given any map $f\colon\, C \longrightarrow M^4$, it may be perturbed so that all singularities are plumbings (thickenings of double points between handles and body surfaces), and solid tori $\alpha \times D^2$ discussed above are embedded and disjoint from everything else. The essential restriction in definition 4.4 is that handles and higher stages attached to different components of each link L_i defining a Bing cell are disjoint. It is straightforward to see that omitting this restriction (i.e. allowing plumbing of arbitrary handles and body surfaces) would yield a trivial theory, since any homotopically essential link may be unlinked by a suitable homotopy. On the other hand, there is no disjointness requirement on handles/surfaces attached to different boundary components of a body surface.

Definition 4.5. Let M be a 4-manifold with a basepoint. Given $n \geq 1$, the n-th *link group* $\lambda_n(M)$ is defined as $\{$based loops in $M\}/\!\sim$. The equivalence relation $\sim$ on based loops in M is defined as follows: $\gamma \sim \gamma'$ if there is a based homotopy from $\gamma(\gamma')^{-1}$ to a based loop which bounds a Bing cell of height n in M.

Proposition 4.6. *The relation $\gamma \sim \gamma'$ in definition 4.5 is an equivalence relation, and moreover it is preserved by the product structure on loops.*

Proof. Consider the first part of the statement, specifically the implication $\gamma_1 \sim \gamma_2, \gamma_2 \sim \gamma_3 \Rightarrow \gamma_1 \sim \gamma_3$. Assume $\gamma_1(\gamma_2)^{-1}$ is homotopic to a loop bounding a b-cell C' and $\gamma_2(\gamma_3)^{-1}$ is homotopic to a loop bounding C'', then $\gamma_1(\gamma_3)^{-1}$ is homotopic to a loop bounding the wedge $(C_1, \alpha_1) \vee_p (C_2, \alpha_2)$ of two b-cells of height n, where the identification point p is the base point. Using a boundary connected sum of the bottom-stage surfaces, $C' \vee C''$ is converted into a b-cell of height n. Using isotopy, the attaching regions of the form $\alpha \times D^2$ for higher-stage surfaces and handles of C', C'' are made disjoint from each other, since they are thickening of 1-manifolds in M^4. The intersections between arbitrary handles and body surfaces of C' and those of C'' are allowed, since there is no disjointness requirement on handles/surfaces attached to different boundary components of a body surface in definition 4.4.

To prove the second part of the proposition, one needs to verify that if $\gamma_1 \sim \gamma_1'$ and $\gamma_2 \sim \gamma_2'$ then $\gamma_1\gamma_2 \sim \gamma_1'\gamma_2'$. This follows from the equivalences $\gamma_1\gamma_2 \sim \gamma_1'\gamma_2 \sim \gamma_1'\gamma_2'$. □

Remarks. 1. In light of remark 1 following definition 4.1, it follows that $\pi_1(M)$ surjects onto $\lambda_1(M)$. Moreover, since a b-cell of height n satisfies the definition of a b-cell of height $n+1$, $\lambda_n(M)$ maps onto $\lambda_{n+1}(M)$. In section 6 we introduce an additional grading on b-cells, leading to a two-parameter family of groups $\lambda_{i,j}(M)$.

2. Note that the definition of $\lambda_n(M)$ makes sense for a manifold of any dimension (and in fact for any topological space), but the theory is non-trivial only in dimension

4. If $dim\,M \geq 5$ then the disjointness requirement is satisfied by general position. If $dim\,M < 4$ then one doesn't expect it to hold due to the dimension count.

It is easy to find examples of 4-manifolds with $\pi_1 \neq \lambda_1$. Consider an example of M^4 with $\pi_1 M \cong \mathbb{Z}$ and $\lambda_1(M) = 0$:

Example 4.7. Consider $M = (S^1 \times D^2 \times I) \cup_L$ 2-handles where L is the Bing double of the core of the solid torus $S^1 \times D^2 \times \{1\}$, see figure 1. Clearly $\pi_1 M \cong \mathbb{Z}$ and $\lambda_1(M) = 0$. On the other hand, it is not difficult to see that $N = S^1 \times D^3$ provides an example where $\lambda_1(N) \cong \pi_1(N) \cong \mathbb{Z}$. See lemma 6.5 for a more detailed discussion.

5. Bing cells in 4-space

We begin the section by showing that any b-cell (C, γ) has a realization in $(D^4, \partial D^4)$. The main purpose of the section is to analyze the fundamental group of the complement, $\pi_1(D^4 \smallsetminus C)$. In particular, we will use the technique presented in section 2 to find a presentation of the nilpotent quotients $\pi_1(D^4 \smallsetminus C)/(\pi_1(D^4 \smallsetminus C))^q$. These results will be used in sections 6 – 8 to formulate invariants which depend only on the underlying model b-cell C and not on its particular realization in the 4-ball. To fix the notation, recall that a model b-cell (C, γ) of height 1 is determined by the following data:

- the number of boundary components of the body surface P: $\partial P = \gamma \cup \alpha_1 \cup \ldots \cup \alpha_k$,
- A collection of links $L_1 \ldots, L_k$ where $L_i \subset \alpha_i \times D^2$.

Lemma 5.1. *Let (C, γ) be a model Bing cell. Then there is a realization of $(C, \gamma) \subset (D^4, \partial D^4)$.*

Strictly speaking, the claim of the lemma is that there is an embedding $(\overline{C}, \gamma) \subset (D^4, \partial D^4)$ as in definition 4.4. Abusing the notation we refer to the plumbed version of the Bing cell as C as well.

Proof. of lemma 5.1. The proof is inductive, starting with the base surface of C and moving up. Start with an unknotted circle γ in S^3 and let $\gamma \times D^2$ bound a 2-handle $D^2 \times D^2$ in D^4. Puncture the core of the handle to get an embedding of the first stage planar surface P. Note that for each i, $\alpha_i \times D^2$ bounds a (just removed) 2-handle H_i in the complement of P in D^4. The link $L_i \subset \alpha_i \times D^2$ is homotopically trivial, so it bounds disjoint immersed disks $\{\Delta\}$ in H_i. The self-intersections of handles and of body surfaces are allowed in the definition of b-cells. If the height of C is greater than 1, repeat the construction. (The disks Δ are converted into second stage surfaces by puncturing them in the complement of double points, etc.) $\square$

5.2. **A presentation of $\boldsymbol{\pi_1}(\mathbf{D^4} \smallsetminus \mathbf{C})/(\boldsymbol{\pi_1}(\mathbf{D^4} \smallsetminus \mathbf{C}))^{\mathbf{q}}$.**

For the remainder of this section fix $q \geq 2$. Given $(C, \gamma) \subset (D^4, \partial D^4)$, let m denote a meridian to γ in S^3. First assume C has height one and P is a pair of pants, see figure 2 in the introduction. Fix the notations:

$$\partial P = \gamma \cup \alpha_1 \cup \alpha_2,\ \overline{P} = P \times D^2,\ C = \overline{P} \cup_{L_1 \cup L_2} 2\text{−handles},$$

where $L_i \subset \alpha_i \times D^2$ are links satisfying the conditions in definition 4.1. Let $I_1,\ I_2$ be the index sets for the components of L_1, L_2 respectively. $\wedge_1, \wedge_2$ will denote the meridional curves $\{p_i\} \times \partial D^2$ of the solid tori $\alpha_i \times D^2$.

$H_1(D^4 \smallsetminus C)$ is generated by $\mathcal{M} = \{m_i\}$: the meridians to the handles of C (in the sense of section 3.8). The index sets I_1, I_2 also parametrize the handles of C, and to be specific, divide the set of meridians $\mathcal{M}$ into two subsets: $\mathcal{M}_{I_1}, \mathcal{M}_{I_2}$. Denote by $F_{\mathcal{M}} = F_{M_{I_1}, M_{I_2}}$ the free group generated by the elements of $\mathcal{M}$, and consider the Magnus expansion M:

$$\pi_1(D^4 \smallsetminus C)/(\pi_1(D^4 \smallsetminus C))^q \xleftarrow{p} F_{\mathcal{M}} = F_{M_{I_1}, M_{I_2}} \xrightarrow{M} \mathbb{Z}\{X\} = \mathbb{Z}\{X_{I_1}, X_{I_2}\} \tag{5.1}$$

where as in section 3.1, $M(m_i) = 1 + x_i$, $i \in I_1 \cup I_2$. We need to fix a specific word in $F_{\mathcal{M}}$ representing the meridian m. Observe that $\wedge_1$ and m cobound a cylinder in $D^4 \smallsetminus C$: the circle normal bundle of P in D^4, restricted to a path in P joining two points in α_1 and γ. Therefore $m, \wedge_1$ are conjugate in $\pi_1(D^4 \smallsetminus C)$. Consider $\wedge_1$ in $\pi_1(\alpha_1 \times D^2 \smallsetminus L_1)$ and consider the commutative diagram of Magnus expansions, induced by the inclusion map $i\colon\ S^1 \times D^2 \smallsetminus L_1 \subset D^4 \smallsetminus C$:

$$\begin{array}{ccccc}
\pi_1(S^1 \times D^2 \smallsetminus L_1)/(\pi_1(S^1 \times D^2 \smallsetminus L_1))^q & \longleftarrow & F_{\mathcal{M}_1 \cup \bar{m}} & \xrightarrow{M_1} & \mathbb{Z}\{X_{I_1}, y\} \\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_\sharp} & & \downarrow \\
\pi_1(D^4 \smallsetminus C)/(\pi_1(D^4 \smallsetminus C))^q & \longleftarrow & F_{\mathcal{M}} & \xrightarrow{M} & \mathbb{Z}\{X\}
\end{array} \tag{5.2}$$

where $\bar{m}$ and y are as in section 3.6 and specifically in (3.3). The homomorphism $i_\sharp$ maps m_j to m_j for each $j \in \mathcal{M}_1$, and it maps $\bar{m}$ to some fixed pullback of $i_*(\bar{m})$ in $F_{\mathcal{M}}$.

Denote by W_1 some word representing $\wedge_1$ in the free group $F_{\mathcal{M}_1 \cup \bar{m}}$, then $W := i_\sharp(W_1)$ represents m in $F_{\mathcal{M}}$. Recall (see section 3.6) that each term with non-repeating variable in the expansion $M_1(W_1)$ contains all of the variables $x_1, \ldots, x_n$, where $I_1 = \{1, \ldots, n\}$. According to the commutative diagram above, this is also true for $M(W)$. It is important to remember (see last paragraph in section 3.6) that specifically $M(W)$ contains the non-trivial term $\mu\, x_1 \cdots x_n$.

Given an element $g \in \pi_1(D^4 \smallsetminus C)/(\pi_1(D^4 \smallsetminus C))^q$, consider a word w representing it in $F_{\mathcal{M}}$. As in the classical case of Milnor's invariants of links, discussed in section 3, the coefficients of the Magnus expansion $M(w)$ in general are not well-defined invariants of g. This is due to the choice of the meridians generating the group, and due to the fact that the kernel of the surjection from $F_{\mathcal{M}}$ is non-trivial. In the present context, compared to the classical situation, the kernel involves more relations in $\pi_1(D^4 \smallsetminus C)/(\pi_1(D^4 \smallsetminus C))^q$ reflecting the topology of Bing cells.

According to lemma 2.1, to see the relations in $\pi_1(D^4 \smallsetminus C)/(\pi_1(D^4 \smallsetminus C))^q$ we need to analyze the generators of $H_2(D^4 \smallsetminus C)$. By Alexander duality, $H_2(D^4 \smallsetminus C) \cong H_1(C, \gamma)$. Note that $H_1(C, \gamma)$ is generated (1) by double point loops corresponding to the intersections among the handles and body surfaces, subject to the disjointness requirement in definition 4.4, and (2) by $H_1(P, \gamma)$. (Here we assume the non-trivial case: each link L_i consists of at least two components, so C is homotopy equivalent to the wedge of two circles with a collection of 2-spheres, one 2-sphere for each handle.) We will divide the corresponding dual generators of $H_2(D^4 \smallsetminus C)$ into four types, $(R_1) - (R_4)$, and analyze the resulting indeterminacy in the coefficients of the Magnus expansion (5.1).

$(\mathbf{R_1})$ Clifford tori for the self-intersections of any handle H_i of C, $i \in I_1 \cup I_2$.

The corresponding relations are of the form $[(m_i)^f, (m_i)^g] = 1$, $i \in I_1 \cup I_2$, $f, g \in \pi_1(D^4 \smallsetminus C)$ (see section 3.8), and are familiar from the study of link homotopy and the classical Milnor group (see 3.2). Pulling back the relations to $F_{\mathcal{M}}$, consider the ideal $\mathcal{I}_1$ generated by their images in $\mathbb{Z}\{X\}$. Observe that each term (besides 1) of any element in the ideal $\mathcal{I}_1$ has repeating variables.

More precisely, note that for any $a \in F_{\mathcal{M}}$ the Magnus expansions $M(a^{-1} m_i a)$ and $M(a^{-1}(m_i)^{-1} a)$ are of the form 1+terms containing x_i (where $M(m_i) = 1 + x_i$.) The commutator $[(m_i)^f, (m_i)^g]$ is a product of $(m_i)^g$ conjugated by $(m_i)^f$ and $(m_i^{-1})^g$, therefore $M([(m_i)^f, (m_i)^g]) = 1+$terms containing at least two entries of x_i. Hence the monomials with non-repeating variables are invariant under multiplication by a conjugate of the relation (R_1).

According to definition 4.4, any handle attached to L_1 can intersect any handle attached to L_2. The corresponding generators of H_2 are

$(\mathbf{R_2})$ Clifford tori for the intersections between the 2-handles H_{i_1} and H_{i_2}, where $i_1 \in I_1, i_2 \in I_2$.

These tori give relations $[(m_{i_1})^f, (m_{i_2})^g] = 1$. Each term of any element in the ideal generated by the Magnus expansion of these relations has both variables x_{i_1} and x_{i_2}, where $i_1 \in I_1, i_2 \in I_2$.

$(\mathbf{R_3})$ Clifford tori for the intersections of any handle H_i with the body surface P, and Clifford tori for the self-intersections of P.

These generators of H_2 impose the relations of the form $[m_i^f, m^g]$, and of the form $[m^f, m^g]$. Here m_i is a meridian to a handle H_i, $i \in I_1 \cup I_2$ and m is a meridian to P. Recall from the discussion at the beginning of section 5.2 that each term in the expansion $M(m)$ contains each of the variables X_{I_1}. If i above is an element of I_1 then all terms in the expansion of $[m_i^f, m^g]$ contain a repeating variable (one of the X_{I_1}). If i is an element of I_2 then each term in the expansion of $[m_i^f, m^g]$ contains both variables x_{i_1} and x_{i_2} for some $i_1 \in I_1, i_2 \in I_2$. In either case, the indeterminacy has already appeared as a result of relations $(R_1), (R_2)$.

There is another type (R_4) of generators of $H_2(D^4 \smallsetminus C)$, Alexander dual to $H_1(P, \partial P \cap S^3) \cong \mathbb{Z}$. Since we assumed each link L_i has at least two components, the meridian $\wedge_i = \{p_i\} \times \partial D^2$ of the solid torus $\alpha_i \times D^2$ bounds a surface S_i in $(\alpha_i \times D^2) \smallsetminus L_i$. (Consider the disk $\{p_i\} \times D^2$. Since $\wedge_i$ has the trivial linking number with each component of L_i, the disk may be converted into a surface disjoint from the link.)

A geometric representative for this class of $H_2(D^4 \smallsetminus C)$ is given by the surface $S_1 \cup \text{annulus} \cup S_2$. Here the annulus is cobounded by $\wedge_1$ and $\wedge_2$, and is the circle normal bundle of P in D^4, restricted to a path in P joining two points in α_1, α_2. As above, denote by W_1, W_2 some words in the free group representing $\wedge_1, \wedge_2$. Then the corresponding relation is

$$(\mathbf{R_4}) \qquad (W_1)^g\,(W_2)^{-1} = 1.$$

Now consider the *general height* $= 1$ *case:* $\partial P = \gamma \cup \alpha_1 \cup \ldots \cup \alpha_n$. The relations are directly analogous to those described above; in particular there are $n-1$ relations of type (R_4): $(W_1)^{g_1}(W_2)^{-1} = 1, \ldots, (W_{n-1})^{g_{n-1}}(W_n)^{-1} = 1$.

The general case (height ≥ 1). Denote by $\mathcal{M}$ the collection of meridians $\{m_i\}$ to the handles of C, and by X a corresponding collection of variables $\{x_i\}$. The double points of C occur as intersections of handles and body surfaces, subject to the disjointness assumption in definition 4.4. More precisely, the general relations of types $(\mathbf{R_1}) - (\mathbf{R_3})$ are represented by the Clifford tori for self-intersections of each handle and body surface of C, and for intersections of any two handles and/or body surfaces, such that the first common ancestor of the corresponding vertices in T_C is an unmarked vertex. Recall that the generators of $H_1(D^4 \smallsetminus C)$, and also the variables X are in 1-1 correspondence with the handles of C and also with the leaves of T_C. The analysis directly analogous to the above implies that *each term of any element in the ideal generated by the Magnus expansions of the relations* $(R_1) - (R_3)$ *either contains repeating variables, or it contains variables* x_i *and* x_j *whose first common ancestor in* T_C *is unmarked.*

There is also a collection of relations $(\mathbf{R_4})$ for the body surfaces of C. Each generator of $H_1(\text{body of } C, \gamma)$ contributes a relation of type $(W_1)^g(W_2)^{-1}$ as above.

6. The generalized Milnor group.

Starting with a Bing cell $(C,\gamma)\subset(D^4,\partial D^4)$ we will derive invariants of (C,γ) *independent of the embedding* into D^4. This feature of the invariants is particularly important for applications to the A-B slice problem [10]. Recall from lemma 3.3 that if a link $L\subset S^3$ is homotopically trivial, its components bound disjoint immersed disks Δ in D^4, and the Milnor group $M\pi_1(D^4\smallsetminus\Delta)$ is isomorphic to the free Milnor group. In particular (see section 3.2) the coefficients in the Magnus expansion $M\pi_1(D^4\smallsetminus\Delta)\longrightarrow R[X]$ are well-defined. In our setting $M\pi_1(D^4\smallsetminus C)$ is *not* the free Milnor group. The goal is to analyze the indeterminacy and to extract useful invariants.

Recall the notation: we fix a collection $\mathcal{M}$ of meridians $\{m_i\}$ to the handles of C, one for each handle. Then the elements of $\mathcal{M}$ generate any nilpotent quotient of $\pi_1(D^4\smallsetminus C)$.

Definition 6.1. The *generalized Milnor group* $GM(C)$ denotes $\pi_1(D^4\smallsetminus C)$ modulo the normal closure of all elements of the form

$$(6.1)\quad [m^f,m^g],\ \text{ and }\ [m_1^f,m_2^g],\ \text{ where }\ f,g\in\pi_1(D^4\smallsetminus C),\ m,m_1,m_2\in\mathcal{M},\ \text{ and}$$

the first common ancestor of m_1,m_2 is unmarked (see definition 4.4).

In particular, $GM(C)$ is a quotient of the classical Milnor group $M\pi_1(D^4\smallsetminus C)$ defined using the set $\mathcal{M}$ of normal generators. Consequently, $GM(C)$ is nilpotent, and so is generated by the elements of $\mathcal{M}$.

For example, consider a realization in D^4 of the b-cell in figure 4 in section 4. Denoting by m_i a meridian to the handle h_i, $i=1,\dots,5$, the relations in the definition of $GM(C)$ are:

$$[m_i^f,m_i^g]=1,\ \ i=1,\dots,5,\ \ [m_1^f,m_3^g]=[m_1^f,m_4^g]=[m_2^f,m_3^g]=[m_2^f,m_4^g]=1,$$

where $f,g\in\pi_1(D^4\smallsetminus C)$. The definition of $M(C)$ incorporates the relations $(R_1)-(R_3)$ in $\pi_1(D^4\smallsetminus C)$, discussed in the previous section. In the classical Milnor's theory, the free Milnor group has a well-defined representation into (the units of) the ring of polynomials where the terms have non-repeating variables. In the next section we describe the analogous representation for $GM(C)$. In the present setup there is also an additional indeterminacy, due to the relations (R_4), and this is analyzed in section 8. It is convenient to define, analogously to the classical case, the free Milnor group:

Definition 6.2. The *free* generalized Milnor group $GM(F_{\mathcal{M}})$ is defined to be the free group $F_{\mathcal{M}}$ modulo the relations of the form (6.1).

It follows that $GM(C)$ is the quotient of $GM(F_{\mathcal{M}})$ by the relations (R_4). Analogously to the classical case, $MC(G)$ has the following property.

Proposition 6.3. *Given a model b-cell C, there exists a realization $\overline{C} \subset D^4$ of C such that $\pi_1(D^4 \smallsetminus \overline{C}) \cong GM(\overline{C})$.*

Proof. Consider any realization $C' \subset D^4$ of C. $GM(C')$ is nilpotent and finitely generated, and is therefore finitely presented. That is, $MC(G)$ is isomorphic to $\pi_1(D^4 \smallsetminus C')$ modulo a finite number of relations (6.1). It is a standard observation that these relations may be introduced by finger moves yielding plumbings and self-plumbings of C' of the allowed type. This gives $\overline{C}$ satisfying the proposition. □

6.4. **Grading of Bing cells.** Given $(C, \gamma) \subset (D^4, \partial D^4)$, let m denote a meridian to γ in S^3. There is no relation, in general, between the height of C and how deep m is in the lower central series of $\pi_1(D^4 \smallsetminus C)$, or of $GM(C)$. For example, let (C_1, γ_1) be a b-cell of height k where each link is the Bing double of the core of the corresponding solid torus (and the body surfaces are arbitrary – to be specific consider pairs of pants.) Also consider (C_2, γ_2) of height 1 where the body surface is an annulus and the link is the k-iterated Bing double of the core. Then C_1, C_2 have different heights, while for both $i = 1, 2$, m_i is in the 2^k-th term of the lower central series of $\pi_1(D^4 \smallsetminus C_i)$.

Define the *nilpotency class* of C to be the least k such that the k-th term of the lower central series $GM(C)^k$ is trivial. Assuming that each link in the definition of C has at least two components, it is clear that the nilpotency class of a b-cell of height k is at least $k+1$. Refining definition 4.5, consider $\lambda_{i,j}(M) = \{$based loops in $M\}$ modulo loops bounding b-cells of height i and having nilpotency class j. There is a commutative diagram of surjections

$$\begin{tikzcd}
\pi_1(M) \arrow[r] & \lambda_{1,2}(M) \arrow[r] & \lambda_{1,3}(M) \arrow[r] \arrow[d] & \lambda_{1,4}(M) \arrow[r] \arrow[d] & \dots \\
 & & \lambda_{2,3}(M) \arrow[r] & \lambda_{2,4}(M) \arrow[r] \arrow[d] & \dots \\
 & & & \vdots &
\end{tikzcd}$$

Lemma 6.5. *Let π, λ be finitely presented groups, where π is aspherical, and suppose π maps onto λ. Then there are 4-manifolds M with $\pi_1(M) \cong \pi$ and $\lambda_{1,2}(M) \cong \lambda$.*

Proof. Consider an aspherical 2-complex K with $\pi_1 K \cong \pi$. Replacing the cells of K by 0-, 1- and 2-handles, one gets a 4-manifold N with boundary. Observe that $\pi_1(N) \cong \lambda_{1,2}(N)$: suppose there is a loop $\gamma \subset N$ trivial in $\lambda_1(N)$ but not in $\pi_1(N)$. Then γ is homotopic to a loop γ' which bounds a b-cell C of height 1. Denote the body surface of C by P, $\partial P = \gamma' \cup \alpha_1 \cup \ldots \cup \alpha_n$. It follows that $\alpha_i \neq 1 \in \pi_1(N)$ for some i. The link $L_i \subset \alpha_i \times D^2$ has two components. Consider the 2-spheres S_1, S_2 formed by the cores of the handles H_1, H_2 of C attached to the components of L_i, capped off by the null-homotopies of the components of L in $\alpha_i \times D^2$. Due to the assumptions on the link, and since the handles H_1, H_2 are disjoint, the intersection of S_1, S_2 is non-trivial in $\mathbb{Z}\pi_1(N)$. This is a contradiction with the asphericity assumption.

Consider a collection of elements $\alpha = \{\alpha_1, \dots, \alpha_k\}$ in $\pi_1 K$ such that the quotient of π_1 by the normal closure of α is isomorphic to λ. Represent α by embedded curves in ∂N, then the 4-manifold $M = N \cup_\alpha$ 2-handles, where the handles are attached to Bing doubles of the cores of $\alpha_i \times D^2 \subset \partial N$, satisfies the proposition. $\square$

Examples of 4-manifolds M for which $\lambda_{i,j}(M) \neq \lambda_{i,j+1}(M)$ are considered in [10]. It is an interesting question whether there are manifolds for which vertical maps are not isomorphisms either.

7. Representations of $GM(C)$.

The purpose of this section is to analyze the indeterminacy of the Magnus expansion due to the relations in the generalized Milnor group $GM(C)$. Consider a set $\mathcal{M} = \{m\}$ of generators of $H_1(D^4 \smallsetminus C)$ provided by meridians to the handles of C. The elements of $\mathcal{M}$ are in 1-1 correspondence with the leaves of the associated tree T_C, and are parametrized by multi-indices $I = (i_1, j_1, \dots, i_n, j_n)$ where n is the height of C, the indices i_k correspond to the branching of the planar surface stages, and the indices j_l correspond to the components of the attaching links L. Phrased in terms of the associated tree, the indices i_k (respectively j_l) correspond to branching of the tree T_C at unmarked (respectively marked) vertices.

Definition 7.1. Consider the set $X = \{x\}$ whose elements are in 1-1 correspondence with the elements of $\mathcal{M}$. Let $R[C]$ denote the quotient of the free associative ring $\mathbb{Z}\{X\}$ generated by X by the ideal generated by the monomials $M = x_{I_1} \cdots x_{I_k}$ such that

- either M contains repeating variables, or
- M contains variables x_I, $x_{I'}$ whose first common ancestor in T_C is unmarked (compare with definition 4.4).

The second condition may be rephrased as follows: let $I = (i_1, \dots, j_n)$, $I' = (i'_1, \dots, j'_n)$ be two multi-indices as above. Consider the first index where these sequences differ: if it is one of the j's then any monomial containing x_I, $x_{I'}$ is in the ideal.

Proposition 7.2. *The Magnus expansion $F_{\mathcal{M}} \longrightarrow \mathbb{Z}\{X\}$ induces a well-defined homomorphism $GM(F_{\mathcal{M}}) \longrightarrow R[C]$, which abusing the notation is also denoted M:*

$$\begin{tikzcd} F_{\mathcal{M}} \arrow[r] \arrow[d] & GM(F_{\mathcal{M}}) \arrow[r] \arrow[d, "M"] & GM(C) \\ \mathbb{Z}\{X\} \arrow[r] & R[C] & \end{tikzcd}$$

Proof. The kernel of $F_{\mathcal{M}} \longrightarrow GM(F_{\mathcal{M}})$ is normally generated by the relations (6.1). Note that every every term (besides 1) in the expansion $M([m^f, m^g])$ has a repeating variable, x, corresponding to m. Similarly, every term in the expansion $M([m_1^f, m_2^g])$ contains both variables x_1, x_2. Therefore the expansion of each relation is in the ideal defining $R[C]$. $\square$

Definition 7.3. Let v be a vertex of T_C. Assign to it an additive subgroup $\widetilde{R}_v \subset R[C]$ as follows. Denote by T_v the subtree of T_C rooted at v, and let X_v denote all variables corresponding to the leaves of T_v.

At each unmarked vertex of T_v keep exactly one branch and erase the rest. Denote the resulting subtrees rooted at v by $\{T_v^\alpha\}$. Then $\widetilde{R}_v$ is defined to be the span of the monomials read off, clockwise starting at the left-most leaf, from all possible planar embeddings of T_v^α, for all α.

Suppose C has height n and without loss of generality assume all branches have uniform length (insert extra stages = annuli if necessary). Set $\widetilde{R}_k(C) = \oplus_v \widetilde{R}_v \subset R[C]$, where the direct sum is taken over all vertices v at height k. Denote

$$\widetilde{R}(C) := \widetilde{R}_0(C) = \widetilde{R}_{\text{root of } T_C}.$$

For example, consider the Bing cell in figure 4. Then there are two subtrees entering the definition of $\widetilde{R}(C)$, shown in figure 5. There are a total of 8 planar embeddings of these subtrees, giving the monomials $\{x_1x_2x_5,\ x_2x_1x_5,\ x_5x_1x_2,\ x_5x_2x_1,\ x_3x_4x_5,\ x_4x_3x_5,\ x_5x_3x_4,\ x_5x_4x_3\}$. Some of the terms, for example $x_1x_5x_2$, do not appear since they do not arise from a subtree.

We will be interested in the subring $1 + \widetilde{R}(C)$ of $R[C]$. It follows from definition of $R[C]$ and from the assumptions in definition 3.7 on the links forming the Bing cell C that all monomials in $\widetilde{R}(C)$ have "maximal length". That is, if X_I is a monomial in $\widetilde{R}(C)$ then for any variable $x \in X$, inserting x anywhere in X_I gives a trivial element of $R[C]$. Observe that the product in $1 + \widetilde{R}(C)$ is given by

$$(1 + \sum_I \alpha_I X_I)(1 + \sum_I \beta_I X_I) = 1 + \sum_I (\alpha_I + \beta_I) X_I.$$

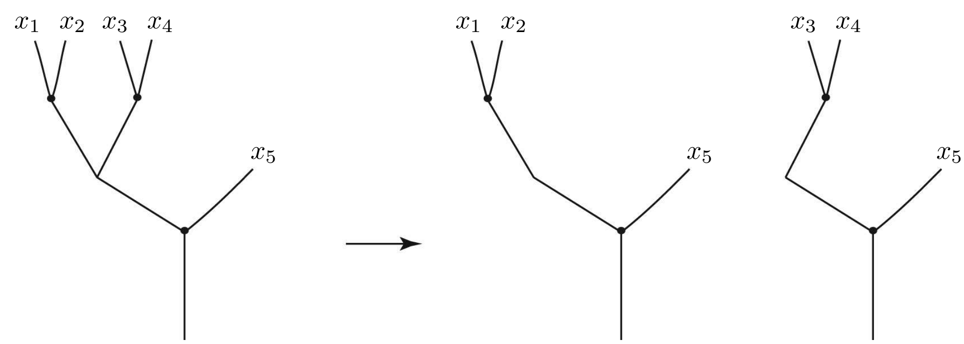

FIGURE 5.

Definition 7.4. For each vertex v of T_C consider the subring

$$S_v = 1 + \widetilde{R}_v + \text{higher order terms}$$

of $R[C]$. By higher order terms we mean all terms of the form

$$T = f_1\, x_1\, f_2\, x_2\, \dots\, f_m\, x_m\, f_{m+1}$$

where the monomial $x_1 \dots x_m$ (obtained from T by deleting the f's) is in $\widetilde{R}_v$, and at least one of the monomials $f_1, \dots, f_{m+1} \in \mathbb{Z}\{X\}$ is not equal to 1. Similarly, set $S_k = 1 + \widetilde{R}_k +$higher order terms. Observe that $S_0(C) = 1 + \widetilde{R}(C)$: the monomials in $\widetilde{R}(C)$ already have maximal length, so there are no higher order terms.

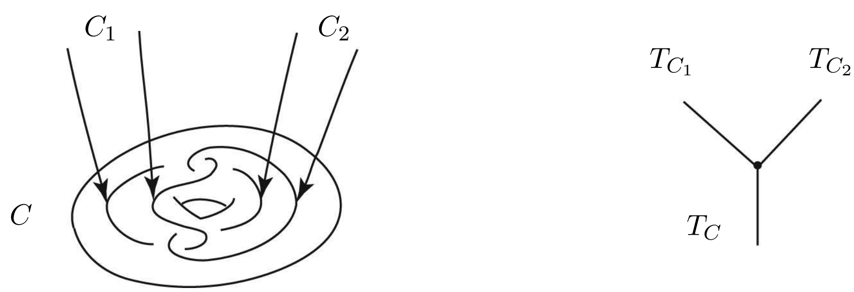

FIGURE 6. Raising the height: step 1.

7.5. It is useful to note an inductive construction of the representation $S_0(C) = 1 + \widetilde{R}(C)$. A Bing cell of height k is assembled from a bottom stage planar surface P, $\partial P = \gamma \cup \alpha_1 \cup \ldots \cup \alpha_n$, and Bing cells of height $k-1$ attached to the components of the links L_i, $L_i \subset \alpha_i \times D^2$. This assembly may be decomposed into two steps. Step one (figure 6) corresponds to attaching Bing cells of height $k-1$ to a single link

L. Supposing for simplicity of notation that L consists of just two components, it follows from definition 7.3 that in this case

$$\widetilde{R}(C) \cong (\widetilde{R}(C_1) \otimes \widetilde{R}(C_2)) \oplus (\widetilde{R}(C_2) \otimes \widetilde{R}(C_1)),$$

with the obvious generalization for links L with more than two components. Here the map $\widetilde{R}(C_i) \otimes \widetilde{R}(C_j) \longrightarrow \widetilde{R}(C)$ is defined on generators by $X_i \otimes X_j \longmapsto X_i \cdot X_j$, the product of monomials. Step two (figure 7) combines the results of step one which are attached to an arbitrary planar surface. In this case $\widetilde{R}(C) \cong \widetilde{R}(C_1) \oplus \widetilde{R}(C_2)$.

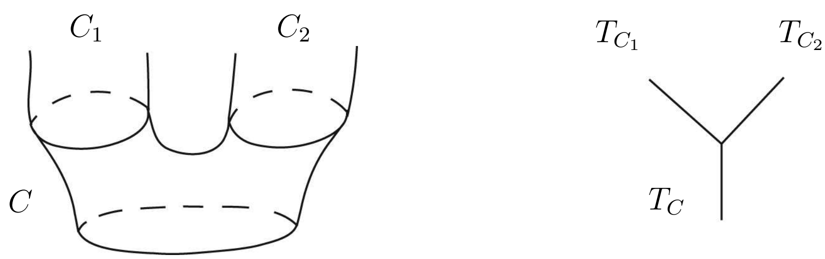

FIGURE 7. Raising the height: step 2.

Lemma 7.6.
1. Let m be a meridian to a body surface of C, and let v be the corresponding vertex in T_C. Then there exists a word $w \in F_{\mathcal{M}}$ representing it so that $M(w) \in S_v$.

2. In particular, let m_0 denote a meridian to the bottom stage of C in D^4 (for example, a meridian to γ in S^3.) Then there exists a word w_0 representing it in $F_{\mathcal{M}}$ such that $M(w_0) \in S_0(C) = 1 + \widetilde{R}(C)$.

Proof. The proof is inductive, moving from the handles down. If m is a meridian to a handle of C then $M(v) = 1 + x$ and the statement is obviously true. Suppose the statement holds for the meridians to all body surfaces at height $k+1$, and let m be a meridian to a surface P at height k. Note that the statement is independent of a choice of the meridians: if one of the meridians is replaced by a conjugate, the Magnus expansion would also satisfy the condition. Denote, as usual, $\partial P = \gamma \cup \alpha_1 \cup \ldots \cup \alpha_n$; the surfaces at height $k+1$ are attached to $P \times D^2$ along the links L_i, $L_i \subset \alpha_i \times D^2$. For each i, the meridian m is conjugate to the curve $\wedge_i$ (connected to the basepoint). Therefore for the inductive step it suffices to consider only step one of the height raising discussed above. In other words, one can assume that P is an annulus, and there is only one link $L \subset \alpha \times D^2$.

Consider the map $\pi_1(\alpha \times D^2 \smallsetminus L) \longrightarrow \pi_1(D^4 \smallsetminus C)$. The map is obtained by pushing $\alpha \times D^2 \smallsetminus$ (a thickening of L) slightly into the complement of C in D^4. Let $L = (l_1, \ldots, l_n)$; denote the corresponding Bing cells attached to them by $C_1, \ldots, C_n$, as in figure 6. To distinguish them from the meridians to the handles of C, denote the meridians to the components of L in the solid torus by $m'_1, \ldots, m'_n$, and let $z_1, \ldots, z_n$ be the corresponding variables for the Magnus expansion. Denote the longitude of the torus, $\{p\} \times \partial D^2$, by l, and the corresponding variable by y.

The meridians m'_j to the components of L may be viewed as meridians to the bottom surface stages of C_j. By the inductive assumption, there are preimages w_j of $i_*(m'_j)$ in $F_{\mathcal{M}}$ such that the Magnus expansion $M(w_j)$, composed with the projection to $R[C]$, is in $S_{v_j} = 1 + \widetilde{R}_{v_j}$+higher order terms. In the following diagram, the map ϕ between the free groups is defined on generators by taking the preimage w_j of $i_*(m'_j)$ in $F_{\mathcal{M}}$. Similarly, $\phi(l)$ is defined as a pullback of $i_*(l)$ in $F_{\mathcal{M}}$. Then $\psi(z_j)$ is defined as $M(\phi(m'_j)) - 1 = M(w_j) - 1$.

$$\begin{tikzcd}
\pi_1(\alpha \times D^2 \smallsetminus L)/(\pi_1(\alpha \times D^2 \smallsetminus L))^q \arrow[d, "i_*"] & F_{m'_1,\ldots,m'_n,l} \arrow[l] \arrow[r, "M'"] \arrow[d, "\phi"] & \mathbb{Z}\{z_1, \ldots, z_n, y\} \arrow[d, "\psi"] & \\
\pi_1(D^4 \smallsetminus C)/(\pi_1(D^4 \smallsetminus C))^q & F_{\mathcal{M}} \arrow[l] \arrow[r, "M"] & \mathbb{Z}\{X\} \arrow[r, "\pi"] & R[C]
\end{tikzcd}$$

Recall from the discussion preceding this lemma that

$$\widetilde{R}_v \cong \oplus[\widetilde{R}_{v_{i_1}} \otimes \ldots \otimes \widetilde{R}_{v_{i_n}}]$$

where the direct sum is taken over all permutations of $\{1, \ldots, n\}$, and the inclusion $\widetilde{R}_{v_{i_1}} \otimes \ldots \otimes \widetilde{R}_{C_{i_n}} \longrightarrow \widetilde{R}_v$ is defined on the additive generators by multiplication of the monomials. Let w be a word representing $\wedge$ (the meridian of the torus $\alpha \times D^2$) in the free group $F_{m'_1,\ldots,m'_n,l}$. We will use the assumptions 3.7 on the links L in the definition of Bing cells 4.1. In particular, every term with non-repeating variables in the expansion $M'(w)$ contains each of the variables $z_1, \ldots, z_n$ (and in addition it may also contain y.) The expansion $M(\phi(w))$ is obtained from $M'(w)$ by replacing each z_i and y with $\psi(z_i)$, $\psi(y)$. The proof is completed by the observation that

$$\pi(\prod_{j=1}^{n} \psi(z_{i_j})) \text{ and } \pi[\psi(z_{i_1}) \ \ldots \ \psi(z_{i_k}) \ \psi(y) \ \psi(z_{i_{k+1}}) \ \ldots \ \psi(z_{i_n})]$$

are elements of S_v, provided that for each j, $\pi(\psi(z_{i_j})) \in S_{v_{i_j}}$. The expansion $M'(w) \in \mathbb{Z}\{z_1, \ldots, z_n, y\}$ may contain a *proper* subset of the variables $\{z_1, \ldots, z_n\}$, provided that at least one of them, say z_i, is *repeated*. However by assumption $\psi(z_i) \in S_{v_i}$, so according to definition 7.3 every term of $\psi(z_i)$ contains all of the variables associated to a subtree $T^{\alpha}_{v_i}$. Then to analyze $\psi(z_i) \cdots \psi(z_i)$ consider the product of any two such terms. Either they correspond to the same tree T^{α} and then the product contains repeated variables and so is trivial in $R[C]$, or they correspond to different subtrees T^{α}, T^{β}, and then the product is again trivial in $R[C]$, by the second condition in definition 7.1. $\square$

To define invariants of Bing cells in the next section, we need to fix a more specific subspace of $\widetilde{R}_v$, for each v, containing precisely the monomials with non-trivial $\bar{\mu}$-invariants of the links $\widehat{L}$ in the definition of b-cells (see definition 4.1 and the discussion at the end of section 3.6.) The definition is similar to that of $\widetilde{R}_v$ but it involves only a specific order of the variables X.

Definition 7.7. Let v be a vertex of T_C. Consider the subtrees T^{α}_v of T_C whose root is v, as in definition 7.3. Then Q_v is the additive subgroup of $R[C]$ spanned by the monomials read off, clockwise, from the *fixed* planar embedding, defined in 4.3, of T^{α}_v, for all α. (Therefore $Q_v \subset \widetilde{R}_v$.) Set $Q_k(C) = \oplus_v Q_v \subset R[C]$, where the summation is taken over all vertices v at height k. Also denote $Q(C) = Q_0(C) = Q_r$ where r is the root of T_C.

In the example in figures 4, 5, $Q(C)$ is spanned by the monomials $x_1x_2x_5$, $x_3x_4x_5$. (Compare with the computation of $\widetilde{R}(C)$ in this example, following definition 7.3.)

We will also use an alternative, inductive, description of $Q(C)$, analogous to that of $\widetilde{R}(C)$ (see 7.5). For each leaf l of T_C, the corresponding Q_l is the subgroup ($\cong \mathbb{Z}$) of $R[C]$ spanned by x_l. Suppose Q_v is defined for vertices of T_C at height $> k$, and let v be an (unmarked) vertex at height k. Moving down the Bing cell from height $k+1$ to height k may be decomposed into steps, illustrated in figures 6, 7. The first step (corresponding to $P =$annulus) gives $Q \cong Q_1 \otimes Q_2$. The second step (figure 5) gives $Q \cong Q_1 \oplus Q_2$. To combine these two steps, denote $\partial P = \gamma \cup \alpha_1 \cup \ldots \cup \alpha_n$; surfaces at height $k+1$ are attached along the links $L_i \subset \alpha_i \times D^2$. Let I_i be the (ordered) index set for the components of L_i. Then

$$Q = \bigoplus_{i} \bigotimes_{j \in I_i} Q_j \tag{7.1}$$

Remark. The structure of $Q(C)$ may be read off from the tree T_C associated to C: the "generators" correspond to the leaves of T_C; then form a tensor product for each marked vertex of the tree and a direct sum for each unmarked vertex.

7.8. **The ring structure.** For each v, S_v is a subring of $R[C]$. Consider $1+\widetilde{R}_v$ as the quotient of S_v by the ideal generated by the higher order terms (see definition 7.4), and let $p_1\colon\ S_v \twoheadrightarrow 1+\widetilde{R}_v$ denote the projection. Similarly, $1+Q_v$ is the quotient of $1+\widetilde{R}_v$ by the ideal generated by all monomials which do not respect the fixed order of the variables, $p_2\colon\ 1+\widetilde{R}_v \twoheadrightarrow 1+Q_v$. The product in $1+\widetilde{R}_v$, $1+Q_v$ is given by

$$(1+\sum_I \alpha_I X_I)(1+\sum_I \beta_I X_I) = 1+\sum_I(\alpha_I+\beta_I)X_I.$$

Let m be a meridian to the bottom stage of C, then by lemma 7.6 there exists a word w representing it in the free group whose Magnus expansion $M(w)$ is an element of

$S(C)$. Consider its image in $1+Q(C)$:

$$p_2(p_1(M(w))) = 1 + \sum_I \alpha_i X_I, \tag{7.2}$$

where the summation is over all subtrees with a prescribed planar embedding, as discussed above. The coefficients α_I are well-defined with respect to the relations $(R_1)-(R_3)$ of section 5. (That is, with respect to multiplying w by a conjugate of one of the relations $(R_1)-(R_3)$.) The next section introduces an invariant well-defined with respect to (R_4) as well.

8. An invariant Φ of Bing cells.

The purpose of this section is to prove the following statement. The main content is in the proof, which will be generalized from knots to the setting of links in section 9.

Lemma 8.1. *Let K be a knot in S^3, suppose C is a Bing cell in D^4 bounded by K, and fix $q\geq 2$. If g is an element of $\pi_1(S^3\smallsetminus K)$ whose image is non-trivial in $H_1(S^3\smallsetminus K)$, then $i_*(g)\neq 1\in \pi_1(D^4\smallsetminus C)/(\pi_1(D^4\smallsetminus C))^q$. Here i_* is the map induced by the inclusion $i\colon S^3\smallsetminus K\subset D^4\smallsetminus C$.*

8.2. *Notation.* Given $g\in\pi_1(S^3\smallsetminus\gamma)/\pi_1(S^3\smallsetminus\gamma)^q$, according to lemma 7.6 there is a word w representing it in the free group $F_{\mathcal{M}}$ whose Magnus expansion $M(w)$ is an element of the subring $S(C)$ of $R[C]$. Denote by $\overline{M}(w)$ the image of $M(w)$ under the projection $S(C)\longrightarrow 1+Q(C)$, so $\overline{M}(w)=p_2(p_1(M(w)))$ in the notation of (7.2).

8.3. *Definition of Φ in the height $=1$ case.* First consider the special case when the first stage planar surface P is a pair of pants, $\partial P=\gamma\cup\alpha_1\cup\alpha_2$. We will follow the notation of section 5.2, and we will use the Magnus expansion (5.1). In particular, the set X of the variables corresponding to the meridians to the handles of C in D^4 is divided into two subsets X_{I_1}, X_{I_2}, where the indices reflect the components of the links $L_i\subset\alpha_i\times D^2$ that the handles are attached to.

Let Y_i be a monomial with non-repeating variables of maximal length in the variables X_{I_i}, $i=1,2$, respecting the preferred order (see 4.3). Note that $Q(C)$ in this case is 2-dimensional, spanned by the monomials Y_1, Y_2. Denoting by W_i a word representing the curve $\wedge_i$ in the free group, given by the commutative diagram 5.2, note that $\overline{M}(W_i)=\mu_i Y_i$, where $\mu_i\neq 0$, $i=1,2$.

Proposition 8.4. *Given an element $g\in\pi_1(S^3\smallsetminus L)/\pi_1(S^3\smallsetminus L)^q$, let w be a word representing it as in 8.2, and consider its expansion in $1+Q(C)$:*

$$\overline{M}(w)=1+\alpha_1Y_1+\alpha_2Y_2$$

for some α_1, α_2. Then $\Phi(g):=\mu_2\alpha_1+\mu_1\alpha_2\in\mathbb{Z}$ is an invariant of g.

Proof. The coefficients α_i are well-defined with respect to the relations $(R_1)-(R_3)$, see the discussion following equation (7.2). The relation (R_4) is given by $(W_1)^g(W_2)^{-1}$, and its expansion is of the form

$$\overline{M}((W_1)^g(W_2)^{-1}) = 1 + \mu_1 Y_1 - \mu_2 Y_2.$$

Let w' be w multiplied by a conjugate of $(W_1)^g(W_2)^{-1}$, $\overline{M}(w') = 1 + \alpha'_1 Y_1 + \alpha'_2 Y_2$. Then $\alpha'_1 = \alpha_1 + \mu_1$, $\alpha'_2 = \alpha_2 - \mu_2$. Therefore $\Phi(w') = \Phi(w)$. □

Consider the *general height* 1 *case*: $\partial P = \gamma \cup \alpha_1 \cup \ldots \cup \alpha_k$. As above, let Y_j be the preferred monomial in the variables X_{I_j}, and $\overline{M}(W_j) = 1 + \mu_j Y_j$, $\mu_j \neq 0$, $j = 1, \ldots, k$. Define $\mu'_j = \prod_{i \neq j} \mu_i$. The proof of the following statement is a direct generalization of the proof in the pair of pants case.

Proposition 8.5. *Given an element $g \in \pi_1(S^3 \smallsetminus L)/\pi_1(S^3 \smallsetminus L)^q$, as in proposition 8.4 consider the expansion in $1+Q(C)$: $\overline{M}(w) = 1 + \sum_j \alpha_j Y_j$. Then $\Phi(g) := \sum_j \alpha_j \mu'_j$ is an invariant of g.*

Remark. In fact there is a collection of $I_1! \cdots I_k!$ invariants Φ, parametrized by the monomials in non-repeating variables $X_{I_1}, \ldots, X_{I_k}$. We chose a specific Φ, reflecting a particular choice of non-trivial $\bar{\mu}$-invariants of the homotopically essential links $\widehat{L}_j$.

8.6. *Definition of the invariant Φ in the general case.* The definition is inductive. Suppose the homomorphism $\Phi\colon (Q(C), +) \longrightarrow (\mathbb{Z}, +)$ is defined for b-cells of height $< h$, and let (C, γ) be a b-cell of height h. C is obtained from $\overline{P} = P \times D^2$ by attaching b-cells $\{C_j\}_{j \in I_i}$ of height $h-1$ to the components of links L_i, $L_i \subset \alpha_i \times D^2$. Here I_i is the (ordered) index set for the components of L_i. As above, let μ_i be the non-trivial $\bar{\mu}$-invariant of $\widehat{L}_i$ in the expansion of $\wedge_i$, with the given order of the components of L_i. Let $\Phi_j\colon Q(C_j) \longrightarrow \mathbb{Z}$ denote the inductively defined invariant of C_j. Recall from (7.1) that

$$Q(C) = \oplus_i \ \otimes_{j \in I_i} \ Q(C_j).$$

Denoting $\mu'_j = \prod_{i \neq j} \mu_i$, define

$$\Phi\colon \ Q(C) \longrightarrow \mathbb{Z} \ \text{ by } \ \Phi = \sum_i \mu'_i (\otimes_{j \in I_i} \Phi_j).$$

Proposition 8.7. *Given $g \in \pi_1(S^3 \smallsetminus K)/(\pi_1(S^3 \smallsetminus K))^q$, let w be a word representing it in the free group, as in 8.2. Then $\Phi(\overline{M}(w))$ is well-defined, and will be denoted $\phi(g)$.*

Proof. The proof is inductive. The statement is true for b-cells of height 1 by proposition 8.5. Suppose the statement is true for b-cells of height $< h$, and let C be a b-cell of height h. Assembling C from b-cells of height $h-1$ will be separated into two steps: (1) attaching them to a link in a solid torus, and (2) attaching the results of step (1) to a (planar surface)$\times D^2$, see section 7.5 and figures 6, 7 in section 7.

Step (1). Consider a b-cell C of height h such that $C = S^1 \times D^2 \times I \cup (C_1 \cup C_2)$ where the b-cells C_i have height $h-1$ and are attached along the components of a link $L = (l_1, l_2) \subset S^1 \times D^2 \times \{1\}$. For simplicity of notation, we assume L has two components; the proof for a larger number of components is directly analogous. Given a relation r of type (R_4), let I denote the ideal in $R[C]$ generated by the Magnus expansion $M(W)-1$, where W is a word representing r. It suffices to prove that the intersection $I \cap Q(C)$ is in the kernel of $\Phi\colon\ Q(C) \longrightarrow \mathbb{Z}$. The representation $Q = Q(C)$ decomposes as $Q_1 \otimes Q_2$ where $Q_i = Q(C_i)$, and $\Phi = \Phi_1 \otimes \Phi_2\colon\ Q \longrightarrow \mathbb{Z}$, so

$$ker\, \Phi = (ker\, \Phi_1) \otimes Q_2 + Q_1 \otimes (ker\, \Phi_2).$$

Since the bottom stage surface of C is the annulus, there are no relations (R_4) at height 1. Therefore the relation r corresponds to a body surface in either C_1 or C_2, say in C_1.

First we impose an *additional assumption* that, in the context of definition 3.7, for each link L defining the b-cell C there is a word W representing $\wedge$ in the free group such that W involves only the variables $m_1, \ldots, m_n$, and not the longitude l of the solid torus. For example, this assumption is satisfied in the central case $L =$(iterated) Bing double. After giving a proof in this restricted setting, we show how the argument goes through in the general case. The assumption above implies that each relation r of type (R_4) has a word representing it in the free group, whose Magnus expansion is an element of either $R[C_1]$ or $R[C_2]$.

Let $r \in R[C_1] \subset R[C]$ be a relation, and denote by I_1 and I the ideals generated by r in $R[C_1]$, $R[C]$ respectively. Observe that $I \cap Q(C) = I \cap (Q(C_1) \otimes Q(C_2)) = (I \cap Q(C_1)) \otimes Q(C_2)$. Since $I_1 \subset ker\, \Phi_1$, $I \cap Q(C) \subset ker\, \Phi$, and the proof is complete.

Now consider the general case, i.e. we remove the extra assumption imposed in the paragraph above. The difference with that case is that even though r is a relation corresponding to C_1, one cannot assume that r is an element of the subring $R[C_1]$ of $R[C]$. However (see end of section 3.6) $\wedge$ has a word representing it whose expansion is of the form $1 + x_{i_1} \cdots x_{i_n} +$higher order terms. That is, all *first non-vanishing terms* with non-repeating variables in its Magnus expansion are elements of $R[C_1]$. The proof is completed by the observation that only first non-vanishing terms contribute to $I \cap Q(C)$.

Step (2), see figure 7. Now C equals $(P \times D^2) \cup C_1 \cup C_2$, where P is a planar surface, the b-cells C_i have height h and whose bottom stage surfaces are annuli. For simplicity of notation we assume P is a pair of pants; the case of a planar surface with more boundary components is treated analogously. Denoting $\partial P = \gamma \cup \alpha_1 \cup \alpha_2$, C_i is attached along $\alpha_i \times D_2$, $i = 1, 2$. In this case

$$R[C] \cong R[C_1] \oplus R[C_2], \;\; Q(C) \cong Q(C_1) \oplus Q(C_2).$$

As above, given a relation r of type (R_4), we need to show

$$I \cap Q(C) \subset ker(\Phi\colon\; Q(C) \longrightarrow \mathbb{Z}).$$

We have $\Phi = \mu_2\Phi_1 \oplus \mu_1\Phi_2$. There are two cases to consider: r corresponds to a surface in C at height >1, or it is a new relation corresponding to P. In the first case, one may assume $r \in R[C_1]$. Denote by I_1, I the ideals generated by r in $R[C_1]$, $R[C]$. Then since $R[C]$ is a direct sum of rings $R[C_1] \oplus R[C_2]$, $I = I_1 \subset R[C_1] \subset R[C]$. Clearly then $I \subset ker(\Phi)$.

Consider the second case: r is a new relation, corresponding to the bottom stage surface P of C. Denote the meridians to L_1 by $m'_1, \ldots, m'_k$ and the meridians to L_2 by $m''_1, \ldots, m''_l$; let $\{x'_i\}, \{x''_j\}$ be the corresponding variables. Then the Magnus expansion of r is of the form

$$M(r) = 1 + \mu_1 x'_1 \cdots x'_k - \mu_2 x''_1 \cdots x''_l + \text{higher order terms}.$$

Consider the image of r in $R[C]$. Note that the first term $\mu_1 x'_1 \cdots x'_k$ is in $R[C_1]$, the second term $\mu_2 x''_1 \cdots x''_l$ is in $R[C_2]$, and in fact all higher order terms vanish in $R[C]$, since the first non-vanishing terms already have maximal length. Any element of $R[C]$ of the form $\mu_1 Y + \mu_2 Z$, where $Y \in R[C_1]$, $Z \in R[C_2]$, is in the kernel of Φ. Therefore $r \in ker(\Phi)$, and any other element in the ideal generated by r is longer and vanishes in $R[C]$ (so in fact $I = \{r\} \subset ker(\Phi)$.) This concludes the proof of proposition 8.7. □

Proposition 8.7 constructs a homomorphism $\phi\colon\; \pi_1(S^3 \smallsetminus K) \longrightarrow \mathbb{Z}$. In particular, $\phi(g)$ is well-defined with respect to multiplication by elements of the relation subgroup in $F_{\mathcal{M}}$, so $\phi(g) \neq 0$ implies $g \neq 1 \in \pi_1(D^4 \smallsetminus C)/(\pi_1(D^4 \smallsetminus C))^q$. It suffices to prove lemma 8.1 for g equal to a meridian m to the knot K in S^3. The fact that $\phi(m) \neq 0$ is proved by inspection: at each surface stage P of C, $\partial P = \gamma_P \cup_i \alpha_i$, the meridian to P is conjugate to the $\wedge$-curve corresponding to the solid torus $\alpha_i \times D^2$, for any given i. Applying the analysis at the end of section 3.6 inductively to the meridians to the surface stages of C, moving up from the meridian m to the bottom stage, one observes that there is a word w representing m in $F_{\mathcal{M}}$ such that $\overline{M}(w)$ is a generating monomial for $Q(C)$. Due to the tensor decompositions of $Q(C)$ and Φ, $\phi(m) = \Phi(\overline{M}(w)) \neq 0$. This concludes the proof of lemma 8.1. □

9. Applications to link homotopy: proof of theorem 1.1

This section shows how the theory of Bing cells fits in the framework of Milnor's theory of link homotopy. We generalize the invariant Φ defined in the previous section to a collection of Bing cells to prove theorem 1.1.

Proof of theorem 1.1. Let $L=(l_1,\ldots,l_n)$ and suppose the components of L bound disjoint Bing cells $C_1\ldots,C_n$ in D^4. Denote $C=\cup_i C_i$. Suppose L is homotopically essential, and without loss of generality one may assume L is almost homotopically trivial, so there is a well-defined and non-trivial μ-invariant with non-repeating coefficients of length n. Order the components of L so that $\mu_{1\ldots n}(L)\neq 0$.

The results of sections 5, 6, 7 and 8 generalize from the setting of a single b-cell as follows. Let $\mathcal{M}_i$ denote a set of meridians to the handles of C_i. By Alexander duality $H_1(D^4\smallsetminus C)$ is generated by $\mathcal{M}=\sqcup_i\mathcal{M}_i$. Denote the corresponding variables for the Magnus expansion by X_i, $X=\sqcup X_i$. Again by Alexander duality, the relations in $\pi_1(D^4\smallsetminus C)/(\pi_1(D^4\smallsetminus C))^q$ are all of types $(R_1)-(R_4)$ (see section 5.2), contributed by the b-cells C_i. Each relation of type $(R_1)-(R_3)$ involves only variables in a single set $\mathcal{M}_i$. The assumptions on the links defining the b-cells in section 3.6 imply that all first non-vanishing terms in the Magnus expansion of any relation of type (R_4) also involve the variables in a single X_i. Variables from other sets X_j may be present, but only in higher-order terms.

Define $GM(C)$ as the free group $F_{\mathcal{M}}$ modulo relations (6.1), where all of the meridians m,m_1,m_2 involved in the commutators in (6.1) are elements of the same $\mathcal{M}_i$, for any given $1\leq i\leq n$. Define $R[C]$ as the quotient of $\mathbb{Z}\{X\}$ by the ideal introduced in definition 7.1 where the variables x_I, $x_{I'}$ are elements of X_i for the same i. Consider the Magnus expansion in the following diagram, analogous to that in proposition 7.2:

$$\begin{tikzcd}
F_{\mathcal{M}} \arrow[r] \arrow[d] & GM(F_{\mathcal{M}}) \arrow[r] \arrow[d] & GM(C) \\
\mathbb{Z}\{X\} \arrow[r] & R[C] &
\end{tikzcd}$$

Following definitions 7.3, 7.4, 7.7, introduce $\widetilde{R}(C)$, $S(C)=1+\widetilde{R}(C)$. Define $Q(C)$ using the order on the components of L reflecting a non-trivial μ-invariant (see above):

$$Q(C)=Q(C_1)\otimes\ldots\otimes Q(C_n).$$

The proof of lemma 7.6 goes through, in particular given any element $g\in\pi_1(S^3\smallsetminus L)/(\pi_1(S^3\smallsetminus L))^q$, there is a word w_0 representing it in $F_{\mathcal{M}}$ such that $M(w_0)\in S(C)$. Denote by $\overline{M}$ the composition of the Magnus expansion M with the projection $S(C)\longrightarrow 1+Q(C)$. Denoting by Φ_i the homomorphism $Q(C_i)\longrightarrow\mathbb{Z}$ defined in 8.6, consider

$$\Phi=\otimes_i\Phi_i\colon\ Q(C)=\otimes_i Q(C_i)\longrightarrow\mathbb{Z}.$$

Given $g\in\pi_1(S^3\smallsetminus L)/(\pi_1(S^3\smallsetminus L))^q$, $\Phi(\overline{M}(w_0))$ is a well-defined integer. Moreover, if $\Phi(\overline{M}(g))\neq 0$, then $i_*(g)\neq 1\in\pi_1(D^4\smallsetminus C)/(\pi_1(D^4\smallsetminus C))^q$. Consider the commutative diagram

$$\begin{tikzcd}
\pi_1(S^3 \smallsetminus L)/(\pi_1(S^3 \smallsetminus L))^q \arrow[r, "i_*"] & \pi_1(D^4 \smallsetminus C)/(\pi_1(D^4 \smallsetminus C))^q \\
F_{m_1,\dots,m_n} \arrow[u, "p_1"] \arrow[r, "\alpha"] \arrow[d, "M_1"'] & F_{\mathcal{M}} = F_{\mathcal{M}_1,\dots,\mathcal{M}_n} \arrow[u, "p_2"'] \arrow[d, "M"] \\
\mathbb{Z}\{x_1,\dots,x_n\} \arrow[r, "\beta"] & \mathbb{Z}\{X\} = \mathbb{Z}\{X_1,\dots,X_n\}
\end{tikzcd}$$

Recall from the proof of lemma 8.1 at the end of section 8 that each meridian m_i has a word w_i representing it in $F_{\mathcal{M}}$ such that $\overline{M}(w_i)$ is a generating monomial for $Q(C_i)$, and $\Phi_i(\overline{M}(w_i)) \neq 0$. In the diagram above α is defined by setting $\alpha(m_i) = w_i$. Then β is given by $\beta(x_j) = M(\alpha(m_j)) - 1$.

Since L is homotopically essential, there is a relation $[m_i, l_i]$ in $\pi_1(S^3 \smallsetminus L)/(\pi_1(S^3 \smallsetminus L))^q$ such that the Magnus expansion M_1 of a word W representing it in $F_{m_1,\dots,m_n}$ is of the form $1 + \mu x_1 \cdots x_n + \dots$ where $\mu \neq 0$. However the projection of $\beta(x_1 \cdots x_n)$ onto $Q(C)$ is a product of generating monomials, one for each $Q(C_i)$, and it follows from the definition of Φ that $\Phi(\alpha(W)) \neq 0$. Since $\Phi(\overline{M}(w_0))$ is an invariant of $g \in \pi_1(S^3 \smallsetminus L)/(\pi_1(S^3 \smallsetminus L))^q$, where $p_1(w_0) = g$, $p_1(W) \neq 1 \in \pi_1(S^3 \smallsetminus L)/(\pi_1(S^3 \smallsetminus L))^q$. But $p_1(W) = [m_i, l_i]$ is a relation in that group. This contradiction concludes the proof of theorem 1.1. $\square$

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Department of Mathematics, University of Virginia, Charlottesville, VA 22904

E-mail address: krushkal@virginia.edu