Link groups and the A-B slice problem

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Dedicated to the memory of Xiao-Song Lin

The $A-B$ slice problem is a reformulation of the topological $4-$dimensional surgery conjecture in terms of decompositions of the $4-$ball and link homotopy. We show that *link groups*, a recently developed invariant of $4-$manifolds, provide an obstruction for the class of *model decompositions*, introduced by M. Freedman and X.-S. Lin. This unifies and extends the previously known partial obstructions in the $A-B$ slice program. As a consequence, link groups satisfy Alexander duality when restricted to the class of model decompositions, but not for general submanifolds of the $4-$ball.

Keywords: $4-$dimensional surgery, $A-B$ slice problem, Alexander duality, link homotopy, link groups.

1. Introduction

The surgery conjecture, a core ingredient in the geometric classification theory of topological $4-$manifolds, remains an open problem for a large class of fundamental groups. The results to date in the subject: the disk embedding conjecture, and its corollaries – surgery and s-cobordism theorems for *good* groups[1,5,6,11] – show similarities of classification of topological $4-$manifolds with the theory in higher dimensions. On the other hand, it has been conjectured[2] that surgery fails for (non-abelian) free fundamental groups.

The $A-B$ slice problem[3] is a reformulation of the surgery conjecture for free groups which seems most promising in terms of the search for an obstruction. In this approach one considers smooth codimension zero decompositions $D^4 = A_i \cup B_i$ of the $4-$ball, extending the standard genus one Heegaard decomposition of the 3-sphere. (A precise definition is given in section 2, also see figure 1.) Then the problem is formulated in terms of the existence of disjoint embeddings of the submanifolds A_i, B_i in D^4 with a prescribed homotopically essential link in $S^3 = \partial D^4$ as the boundary condition. The central case corresponds to the link equal to the Borromean rings. The problem may be phrased in terms of the existence of a suitably formulated non-abelian Alexander duality in dimension 4. Recently this approach has been sharpened and now there is a precise, axiomatic description of what properties an obstruction, which in this context is an invariant of decompositions of D^4, should satisfy.

The $A-B$ slice formulation of surgery was introduced by Freedman[3] and further extensively studied by Freedman-Lin.[4] In particular, the latter paper introduced a family of *model decompositions* which appear to approximate, in a certain algebraic sense, an arbitrary decomposition $D^4 = A \cup B$. This family of decompositions is defined in section 4. In this paper we use *link groups* of $4-$manifolds, recently introduced by the author,[8] to formulate an obstruction for the family of model decompositions:

Theorem 1.1. *Let L be the Borromean rings, or more generally any homotopically essential link in S^3. Then L is not $A-B$ slice where each decomposition $D^4 = A_i \cup B_i$ is a model decomposition.*

The invariant using link groups formulated in the proof unifies and generalizes the previously known partial obstructions[4,9] in the $A-B$ slice program. The definitions of link groups and the underlying geometric notion of *Bing cells* are given in section 3.

To place this result in the geometric context of link homotopy, it is convenient to introduce the notion of a robust $4-$manifold. Recall that a link L in S^3 is *homotopically trivial*[12] if its components bound disjoint maps of disks in D^4. L is called homotopically essential otherwise. (The Borromean rings is a homotopically essential link with trivial linking numbers.) Let (M, γ) be a pair ($4-$manifold, embedded curve in ∂M). The pair (M, γ) is *robust* if whenever several copies (M_i, γ_i) are properly disjointly embedded in (D^4, S^3), the link formed by the curves $\{\gamma_i\}$ in S^3 is homotopically trivial. The following statement is a consequence of the proof of theorem 1.1:

Corollary 1.2. *Let $D^4 = A \cup B$ be a model decomposition. Then precisely one of the two parts A, B is robust.*

It is interesting to note that there exist decompositions where *neither* of the two sides is robust.[10] The following question relates this notion to the $A-B$ slice problem: *given a decomposition* $D^4 = A \cup B$, *is one of the given embeddings* $A \hookrightarrow D^4$, $B \hookrightarrow D^4$ *necessarily robust?* (The definition of a robust embedding $e\colon\ (M,\gamma) \hookrightarrow (D^4, S^3)$ is analogous to the definition of a robust pair above, with the additional requirement that each of the embeddings $(M_i, \gamma_i) \subset (D^4, S^3)$ is equivalent to e.)

In a certain sense, one is looking in the $A-B$ slice problem for an invariant of 4−manifolds which is more flexible than homotopy (so it satisfies a suitable version of Alexander duality), yet it should be more robust than homology – this is made precise using Milnor's theory of link homotopy. The subtlety of the problem is precisely in the interplay of these two requirements. Following this imprecise analogy, we show that link groups provide a step in construction of such a theory.

2. Surgery and the $A-B$ slice problem.

The 4−dimensional topological surgery exact sequence (cf [FQ], Chapter 11), as well as the 5−dimensional topological s-cobordism theorem, are known to hold for a class of *good* fundamental groups. In the simply-connected case, this followed from Freedman's disk embedding theorem[1] allowing one to represent hyperbolic pairs in $\pi_2(M^4)$ by embedded spheres. Currently the class of good groups is known to include the groups of subexponential growth[6,11] and it is closed under extensions and direct limits. There is a specific conjecture for the failure of surgery for free groups:[2]

Conjecture 2.1. *There does not exist a topological* 4−*manifold* M, *homotopy equivalent to* $\vee^3 S^1$ *and with* ∂M *homeomorphic to* $\mathcal{S}^0(Wh(Bor))$, *the zero-framed surgery on the Whitehead double of the Borromean rings.*

In fact, this is one of a collection of canonical surgery problems with free fundamental groups, and solving them is equivalent to the unrestricted surgery theorem. The $A-B$ slice problem, introduced in ref. 3, is a reformulation of the surgery conjecture, and it may be roughly summarized as follows. Assuming on the contrary that the manifold M in the conjecture above exists, consider the compactification of the universal cover $\widetilde{M}$, which is homeomorphic to the 4−ball.[3] The group of covering transformations (the free group on three generators) acts on D^4 with a prescribed action on the boundary, and roughly speaking the $A-B$ slice problem is a program for finding an obstruction to the existence of such actions. Recall the definition of an $A-B$ slice link.[3,4]

Definition 2.1. A *decomposition* of D^4 is a pair of smooth compact codimension 0 submanifolds with boundary $A, B \subset D^4$, satisfying conditions (1) $-$ (3) below. (Figure 1 gives a 2$-$dimensional example of a decomposition.) Denote

$$\partial^+ A = \partial A \cap \partial D^4, \;\; \partial^+ B = \partial B \cap \partial D^4, \;\; \partial A = \partial^+ A \cup \partial^- A, \;\; \partial B = \partial^+ B \cup \partial^- B.$$

(1) $A \cup B = D^4$,
(2) $A \cap B = \partial^- A = \partial^- B$,
(3) $S^3 = \partial^+ A \cup \partial^+ B$ is the standard genus 1 Heegaard decomposition of S^3.

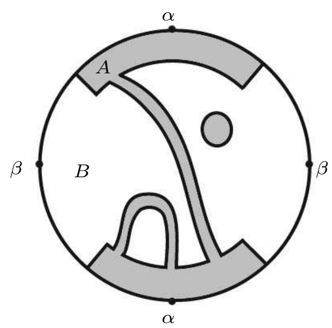

Fig. 1. A 2$-$dimensional analogue of a decomposition (A, α), (B, β): $D^2 = A \cup B$, A is shaded; (α, β) are linked 0$-$spheres in ∂D^2.

Definition 2.2. Given an $n-$component link $L = (l_1, \ldots, l_n) \subset S^3$, let $D(L) = (l_1, l_1', \ldots, l_n, l_n')$ denote the $2n-$component link obtained by adding an untwisted parallel copy L' to L. The link L is $A - B$ *slice* if there exist decompositions $(A_i, B_i), i = 1, \ldots, n$ of D^4 and self-homeomorphisms α_i, β_i of D^4, $i = 1, \ldots, n$ such that all sets in the collection $\alpha_1 A_1, \ldots, \alpha_n A_n, \beta_1 B_1, \ldots, \beta_n B_n$ are disjoint and satisfy the boundary data: $\alpha_i(\partial^+ A_i)$ is a tubular neighborhood of l_i and $\beta_i(\partial^+ B_i)$ is a tubular neighborhood of l_i', for each i.

The surgery conjecture holds for all groups if and only if the Borromean Rings (and the rest of the links in the canonical family of links) are $A - B$ slice.[3] Conjecture 2.1 above can therefore be reformulated as saying that the Borromean Rings are not $A - B$ slice.

As an elementary example, note that if a link L is $A-B$ slice where for each i the decomposition $D^4 = A_i \cup B_i$ consists of $A_i = 2-$handle $D^2 \times D^2$, and $B_i =$ the collar on $\partial^+ B_i$, then L is actually *slice*.

Of course the Borromean Rings is not a slice (or homotopically trivial) link. However to show that a link is not $A-B$ slice, one needs to eliminate *all* choices for decompositions (A_i, B_i).

3. Link groups and Bing cells.

In this section we recall the definition of Bing cells and link groups of $4-$manifolds, denoted $\lambda(M^4)$, introduced in Ref. 8, in order to formulate the invariant I_λ used in the proof of theorem 1.1. The definition is inductive.

Definition 3.1. A *model Bing cell of height* 1 is a smooth 4-manifold C with boundary and with a specified attaching curve $\gamma \subset \partial C$, defined as follows. Consider a planar surface P with $k+1$ boundary components $\gamma, \alpha_1, \ldots, \alpha_k$ $(k \geq 0)$, and set $\overline{P} = P \times D^2$. Let $L_1, \ldots, L_k$ be a collection of links, $L_i \subset \alpha_i \times D^2$, $i = 1, \ldots, k$. Here for each i, L_i is the (possibly iterated) Bing double of the core α_i. Then C is obtained from $\overline{P}$ by attaching zero-framed 2-handles along the components of $L_1 \cup \ldots \cup L_k$.

The surface S (and its thickening $\overline{S}$) will be referred to at the *body* of C, and the 2-handles are the *handles* of C.

A *model Bing cell C of height h* is obtained from a model Bing cell of height $h-1$ by replacing its handles with Bing cells of height one. The *body* of C consists of all (thickenings of) its surface stages, except for the handles.

Figures 2, 3 give an example of a Bing cell of height 1: a schematic picture and a precise description in terms of a Kirby diagram. Here P is a pair of pants, and each link L_i is the Bing double of the core of the solid torus $\alpha_i \times D^2$, $i = 1, 2$.

Remark 3.1. To avoid a technical discussion, the definition presented here involves only the links L which are Bing doubles. To reflect this difference, we reserve for these objects the term *Bing cells* rather than the more general *flexible cells* discussed in Ref. 8. The definition in Ref. 8 involves more general homotopically essential links, however just the Bing doubles suffice for the applications in this paper.

Bing cells in a $4-$manifold M are defined as maps of model Bing cells in M, subject to certain crucial disjointness requirements. (In particular, this

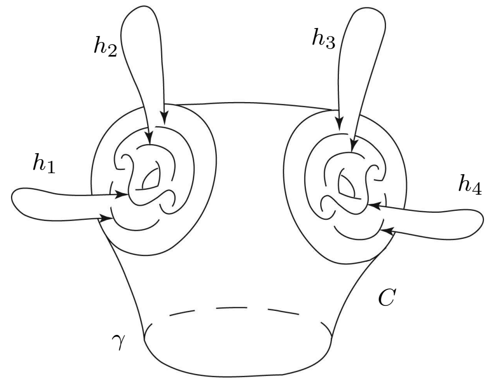

Fig. 2. Example of a model Bing cell of height 1: a schematic picture

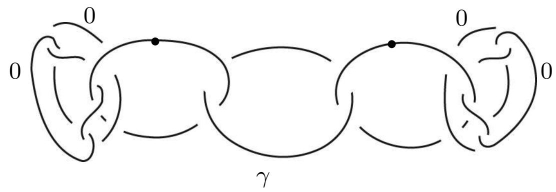

Fig. 3. A Kirby diagram of the model Bing cell in figure 1

will be important for the discussion of model decompositions in section 4.) Roughly speaking, objects attached to different components of any given link L_i in the definition are required to be disjoint in M. To formulate this condition rigorously, recall the definition of the tree associated to a given Bing cell.

3.1. *The associated tree*

Given a Bing cell C, define the tree T_C inductively: suppose C has height 1. Then assign to the body surface P (say with $k+1$ boundary components) of C the cone T_P on $k+1$ points. Consider the vertex corresponding to the attaching circle γ of C as the root of T_P, and the other k vertices as the leaves of T_P. For each handle of C attach an edge to the corresponding leaf

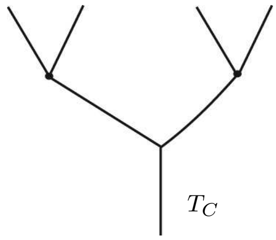

Fig. 4. The tree T_C associated to the Bing cell C in figures 2, 3.

of T_P. The leaves of the resulting tree T_C are in $1-1$ correspondence with the handles of C.

Suppose C has height $h > 1$, then it is obtained from a Bing cell C' of height $h-1$ by replacing the handles of C' with Bing cells $\{C_i\}$ of height 1. Assuming inductively that $T_{C'}$ is defined, one gets T_C by replacing the edges of $T_{C'}$ associated to the handles of C' with the trees corresponding to $\{C_i\}$. Figure 4 shows the tree associated to the Bing cell in figure 2.

Divide the vertices of T_C into two types: the vertices ("cone points") corresponding to body (planar) surfaces are *unmarked*; the rest of the vertices are *marked*. Therefore the valence of an unmarked vertex equals the number of boundary components of the corresponding planar surface. The marked vertices are in $1-1$ correspondence with the links L defining C, and the valence of a marked vertex is the number of components of L plus 1. It is convenient to consider the $1-$valent vertices of T_C: its root and leaves (corresponding to the handles of C) as unmarked. This terminology is useful in defining the maps of Bing cells below. The height of a Bing cell C may be read off from T_C as the maximal number of marked vertices along a geodesic joining a leaf of T_C to its root, where the maximum is taken over the leaves of T_C.

Definition 3.2. A *Bing cell* is a model Bing cell with a finite number of self-plumbings and plumbings among the handles and body surfaces of C, subject to the following disjointness requirement:

• Consider two surfaces A, B (they could be handles or body stages) of C. Let a, b be the corresponding vertices in T_C. (For body surfaces this is the corresponding unmarked cone point, for handles this is the associated leaf.) Consider the geodesic joining a, b in T_C, and look at its vertex c closest to the root of T_C – in other words, c is the first common ancestor of a, b. If c is a marked vertex then A, B are required to be disjoint.

In particular, self-plumbings of any handle and body surface are allowed. In the example shown in figures 1, 2 above, the handle h_1 is required to be disjoint from h_2, h_3 is disjoint from h_4; all other intersections are allowed.

A *Bing cell in a* 4-*manifold* M is an embedding of a Bing cell into M. We say that its image is a realization of C in M, and abusing the notation we denote its image in M also by C.

The main technical result of Ref. 8 shows how Bing cells fit in the context of Milnor's theory of link homotopy. This theorem is used in the analysis of the invariant I_λ below.

Theorem 3.1. *If the components of a link $L \subset S^3 = \partial D^4$ bound disjoint Bing cells in D^4 then L is homotopically trivial.*

Recall[12] that a link L in S^3 is *homotopically trivial* if L is homotopic to the unlink, so that different components stay disjoint during the homotopy. The theorem above builds on a classical result that if the components of L bound disjoint maps of *disks* in D^4 then L is homotopically trivial. The proof of theorem 3.1 is substantially more involved than the argument in the classical case. This is due to the topology of Bing cells which forces additional relations in the fundamental group of the complement. The main new technical ingredients in the proof are the *generalized Milnor group* and an obstruction which is well-defined in the presence of this additional indeterminacy.[8]

The *link groups* $\lambda_n(M)$ are defined as $\{$based loops in a $4-$manifold $M\}$ modulo loops bounding Bing cells of height n. These groups fit in a sequence of surjections

$$\pi_1(M) \longrightarrow \lambda_1(M) \longrightarrow \lambda_2(M) \longrightarrow \dots$$

The groups $\lambda_n(M)$ are topological but not in general homotopy invariants of M. In particular, they are not correlated with the first homology $H_1(M)$, or more generally with the quotients of $\pi_1(M)$ by the terms of its lower central or derived series. Define $\lambda(M)$ to be the direct limit of $\lambda_n(M)$. Given a pair (M, γ) where M is a $4-$manifold and γ is a specified curve in ∂M, consider the invariant $I_\lambda(M, \gamma) \in \{0, 1\}$:

$$I_\lambda(M, \gamma) = 1 \;\text{ if }\; \gamma = 1 \in \lambda(M),$$

set $I_\lambda(M, \gamma) = 0$ otherwise. When the choice of the attaching circle γ of M is clear, we will abbreviate the notation to $I_\lambda(M)$.

Remark 3.2. For the interested reader we point out the "geometric duality" between Bing cells and gropes. Recall the definition:[5] A *grope* is a special pair (2-complex, circle). A grope has a *class* $k = 1, 2, \ldots, \infty$. For $k = 2$ a grope is a compact oriented surface Σ with a single boundary component. For $k > 2$ a *k-grope* is defined inductively as follow: Let $\{\alpha_i, \beta_i, i = 1, \ldots, \text{genus}\}$ be a standard symplectic basis of circles for Σ. For any positive integers p_i, q_i with $p_i + q_i \geq k$ and $p_{i_0} + q_{i_0} = k$ for at least one index i_0, a k-grope is formed by gluing p_i-gropes to each α_i and q_i-gropes to each β_i. A grope has a standard, "untwisted" 4−dimensional thickening, obtained by embedding it into $\mathbb{R}^3$, times I.

Consider a more general collection of 2-complexes, where at each stage one is allowed to attach several parallel copies of surfaces. Then one checks using Kirby calculus that model Bing cells are precisely complements in D^4 of standard embeddings of such generalized gropes. This observation is helpful in the analysis of the $A - B$ slice problem, where gropes play an important role, see section 4.

4. An obstruction for model decompositions.

In this section we show that the invariant I_λ defined above provides an obstruction for the family of model decompositions. We start the proof of theorem 1.1 by constructing the relevant decompositions of D^4. The simplest decomposition $D^4 = A \cup B$ where A is the 2−handle $D^2 \times D^2$ and B is just the collar on its attaching curve, was discussed in the introduction. Now consider the genus one surface S with a single boundary component α, and set $A_1 = S \times D^2$. Moreover, one has to specify its embedding into D^4 to determine the complementary side, B. Consider the standard embedding (take an embedding of the surface in S^3, push it into the 4−ball and take a regular neighborhood.) Note that given any decomposition, by Alexander duality the attaching curve of exactly one of the two sides vanishes in it homologically, at least rationally. Therefore the decomposition $D^4 = A_1 \cup B_1$ may be viewed as the first level of an "algebraic approximation" to an arbitrary decomposition. The general *model decomposition of height* 1 is analogous to the decomposition $D^4 = A_1 \cup B_1$, except that the surface S may have a higher genus.

Proposition 4.1. *Let $A_1 = S \times D^2$, where S is the genus one surface with a single boundary component α. Consider the standard embedding $(A_1, \alpha \times \{0\}) \subset (D^4, S^3)$. Then the complement B_1 is obtained from the collar on its attaching curve, $S^1 \times D^2 \times I$, by attaching a pair of zero-framed 2−handles to the Bing double of the core of the solid torus $S^1 \times D^2 \times \{1\}$, figures 5, 6.*

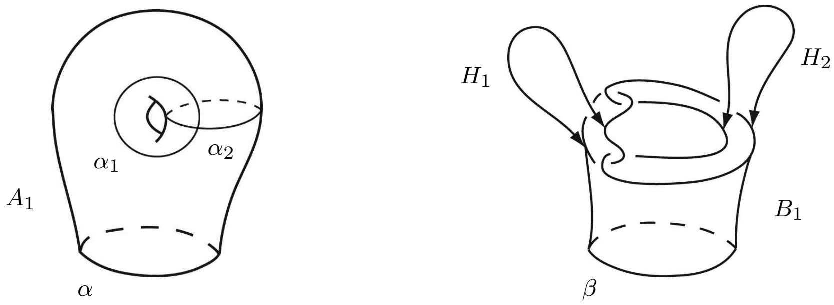

Fig. 5. A model decomposition $D^4 = A_1 \cup B_1$ of height 1: a schematic (spine) picture (figure 5) and a precise description in terms of Kirby diagrams, figure 6.

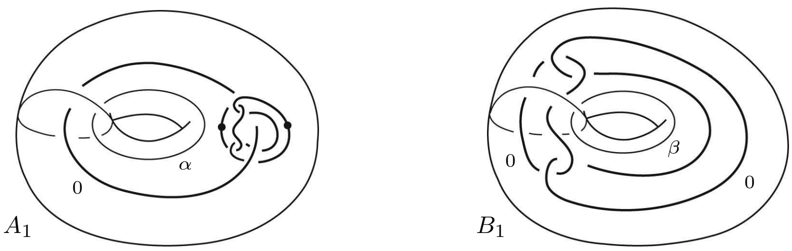

Fig. 6.

The proof is a standard exercise in Kirby calculus, see for example Ref. 4. A precise description of these 4$-$manifolds is given in terms of Kirby diagrams in figure 6. Rather than considering handle diagrams in the 3$-$sphere, it is convenient to draw them in the solid torus, so the 4$-$manifolds are obtained from $S^1 \times D^2 \times I$ by attaching the 1$-$ and 2$-$handles as shown in the diagrams. To make sense of the "zero framing" of curves which are not null-homologous in the solid torus, recall that the solid torus is embedded into $S^3 = \partial D^4$ as the attaching region of a 4$-$manifold, and the 2$-$handle framings are defined using this embedding.

This example illustrates the general principle that (in all examples considered in this paper) the 1$-$handles of each side are in one-to-one correspondence with the 2$-$handles of the complement. This is true since the embeddings in D^4 considered here are all standard, and in particular each 2$-$handle is unknotted in D^4. The statement follows from the fact that 1$-$handles may be viewed as standard 2$-$handles removed from a collar, a standard technique in Kirby calculus (see Chapter 1 in Ref. 7.) Moreover, in each of our examples the attaching curve α on the $A-$side bounds a surface in A, so it has a zero framed 2−handle attached to the core of the solid torus. On the 3−manifold level, the zero surgery on this core transforms the solid torus corresponding to A into the solid torus corresponding to B. The Kirby diagram for B is obtained by taking the diagram for A, performing the surgery as above, and replacing all zeroes with dots, and conversely all dots with zeroes. (Note that the 2−handles in all our examples are zero-framed.)

Note that a distinguished pair of curves α_1, α_2, forming a symplectic basis in the surface S, is determined as the meridians (linking circles) to the cores of the 2−handles H_1, H_2 of B_1 in D^4. In other words, α_1, α_2 are fibers of the circle normal bundles over the cores of H_1, H_2 in D^4.

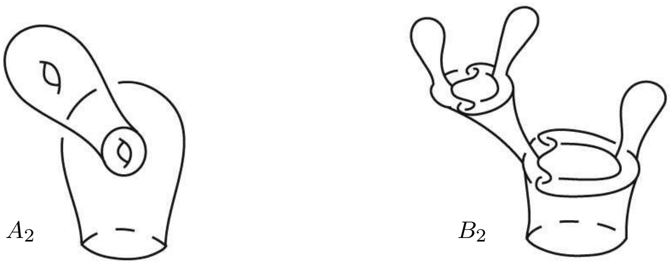

Fig. 7. A model decomposition $D^4 = A_2 \cup B_2$ of height 2.

An important observation[4] is that this construction may be iterated: consider the 2−handle H_1 in place of the original 4−ball. The pair of curves (α_1, the attaching circle β_1 of H_1) form the Hopf link in the boundary of H_1. As discussed in the beginning of this section, it is natural to consider two possibilities: either α_1 or β_1 bounds a surface in H_1. For simplicity of exposition, we again assume at this point that this is a surface of genus one. The first possibility (α_1 bounds) is shown in figure 7: note that in this decomposition one side, A_2, is a grope of height 3 (discussed in remark 3.2) and its complement B_2 is an example of a Bing cell.

Consider the second possibility: β_1 bounds a surface in H_1. As discussed above, its complement in H_1 is given by two zero-framed 2−handles attached to the Bing double of α_1. Assembling this data, consider the new decomposition $D^4 = A'_2 \cup B'_2$, figures 8, 9. As above, the diagrams are drawn in solid tori (complements in S^3 of unknotted circles drawn dashed in the figures.) The decompositions $D^4 = A_2 \cup B_2$, $D^4 = A'_2 \cup B'_2$ are examples of model decompositions of height 2. To get a general decomposition of this type, one also considers the alternative as above for the pair of curves α_2, β_2 in the 4−ball H_2. For simplicity of illustration, in the examples shown in figures 7 - 9 the curve β_2 bounds a surface of genus zero. One gets models of an arbitrary height by an iterated application of the construction above, and in general one considers (orientable) surfaces of an arbitrary genus at each stage. See figure 10 for examples of model decompositions of height 3.

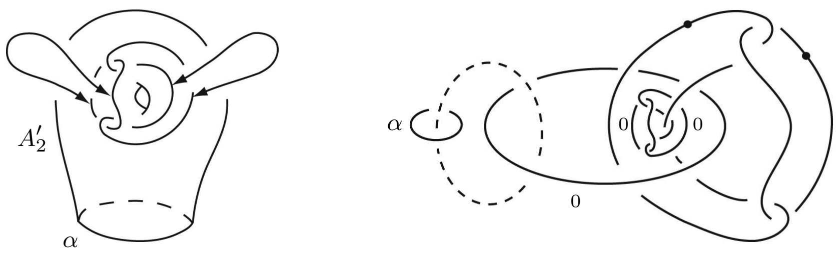

Fig. 8.

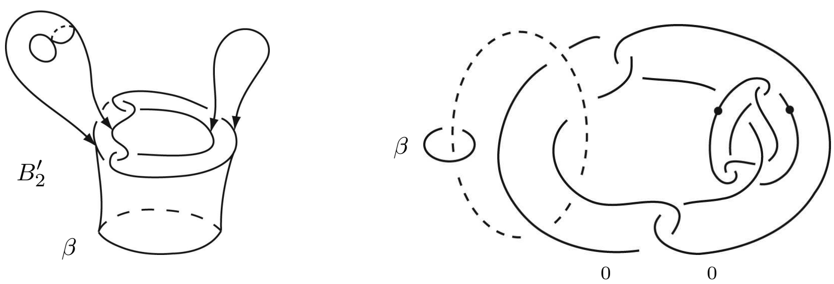

Fig. 9. Another example of a model decomposition $D^4 = A'_2 \cup B'_2$ of height 2.

It follows from theorem 3.1 that the following lemma implies our main result, theorem 1.1:

Lemma 4.2. *Let $D^4 = A \cup B$ be a model decomposition. Then*

$$I_\lambda(A, \alpha) + I_\lambda(B, \beta) = 1.$$

Indeed, suppose a link $L = (l_1, \dots, l_n)$ is $A - B$ slice where each decomposition $D^4 = A_i \cup B_i$, $i = 1, \dots, n$ is a model decomposition. According to lemma 4.2, the invariant I_λ of precisely one part of the decomposition equals 1. For each i, denote $C_i = A_i$ if $I_\lambda(A_i) = 1$ and $C_i = B_i$ otherwise.

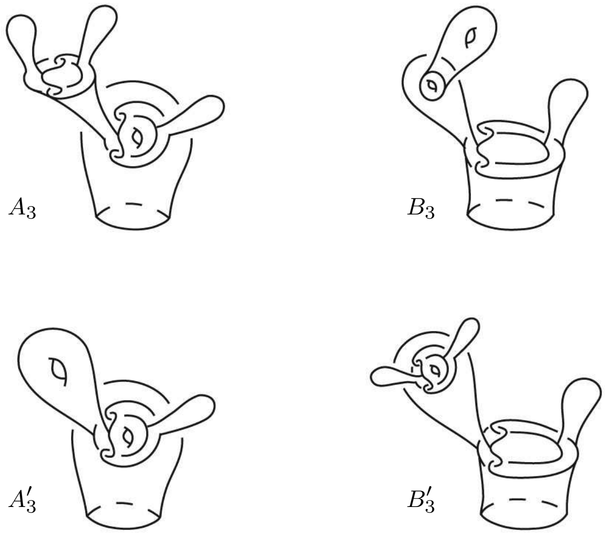

Fig. 10. Examples of model decompositions $D^4 = A_3 \cup B_3$, $D^4 = A_3' \cup B_3'$ of height 3.

Let γ_i denote the attaching curve of C_i. It follows from the definition of I_λ that γ_i bounds a Bing cell in C_i. Since the collections $\{\alpha_i\}, \{\beta_i\}$ form the link L and its parallel copy, the collection of curves $(\gamma_1, \ldots, \gamma_n)$ is isotopic to L. This contradicts theorem 3.1 since L is homotopically essential. This concludes the proof of theorem 1.1, assuming lemma 4.2.

Proof of lemma 4.2. It suffices to prove that given a model decomposition $D^4 = A \cup B$, either $\alpha = 1 \in \lambda(A)$ or $\beta = 1 \in \lambda(B)$. Then theorem 3.1 implies that precisely one of these two possibilities holds. The proof of the statement above is inductive. Given a model decomposition of height 1 (figure 5), observe that one of the two parts of the decomposition - the handlebody B_1 in the example in figure 5 - is a model Bing cell of height 1. (In this case the planar surface C in definition 3.1 is the annulus.) Therefore $\beta = 1 \in \lambda(B_1)$. In the case that A_1 is a surface of genus $g > 1$, the handlebody description of B_1 consists of first taking g parallel copies of the core curve of the solid torus, Bing doubling them and then attaching zero-framed $2-$handles to the resulting link. One observes that the attaching curve β still bounds a model Bing cell of height 1 in this handlebody, indeed there are g choices of Bing cells bounded by β.

Suppose lemma is proved for model decompositions of height $\leq n$, and let $D^4 = A \cup B$ be a model decomposition of height $n + 1$. The attaching curve of either A or B is trivial in its first homology group. To be specific, assume $\alpha = 0 \in H_1(A;\mathbb{Z})$. First assume the surface Σ bounded by α has genus 1. Then A is obtained by attaching models A', A'' of height $\leq n$ to a symplectic basis of curves α_1, α_2 of Σ, figure 11. Similarly, using the notation of figure 5, B is obtained from the model B_1 of height 1 by replacing its $2-$handles H_1, H_2 by two models B', B'' of height $\leq n$. Here $D^4 = A' \cup B', D^4 = A'' \cup B''$ are two decompositions for which lemma holds according to the inductive assumption. Therefore $I_\lambda(A')+I_\lambda(B') = I_\lambda(A'')+I_\lambda(B'') = 1$. Consider two cases:

Case 1: $I_\lambda(B') = I_\lambda(B'') = 1$

Case 2: At least one of $I_\lambda(A')$, $I_\lambda(A'')$ equals 1.

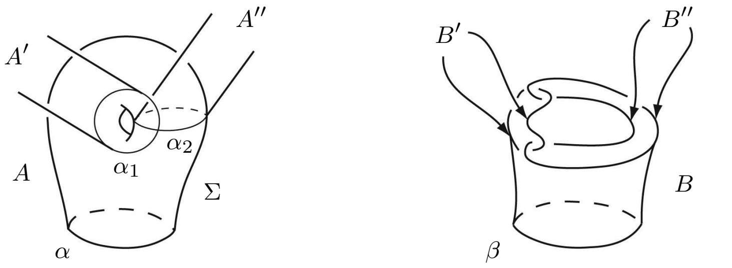

Fig. 11. Proof of lemma 4.2: the inductive step.

We claim that in the first case $I_\lambda(B) = 1$ and in the second case $I_\lambda(A) = 1$. Consider case 1. By assumption, the attaching curve β' of B' bounds a Bing cell C' in B', and similarly the attaching curve β'' bounds a Bing cell C'' in B''. Consider the handlebody C obtained from $S^1 \times D^2 \times I$ by attaching C', C'' to the Bing double of the core of the solid torus. The associated tree T_C is illustrated on the left in figure 12. (Note that the trees $T_{C'}$, $T_{C''}$ join in a *marked* vertex.) Since B' and B'' are disjoint, there are no $C' - C''$ intersections. (Note that such intersections are not allowed in the definition 3.2 of a Bing cell.) Therefore the attaching curve β bounds a Bing cell in B, and $I_\lambda(B,\beta) = 1$.

Consider the second case. Without loss of generality assume $I_\lambda(A') = 1$, so α_1 bounds a Bing cell C' in A'. Surger the first stage surface Σ along α_1, the result is a pair of pants whose boundary consists of α and two copies of α_1. Consider two copies of C' (denote them by C' and $\overline{C}'$) and perturb them so there are only finitely many intersections between surfaces in C' and surfaces in $\overline{C}'$. Consider the handlebody C assembled from the (pair

Fig. 12.

of pants)$\times D^2$ with $C', \overline{C}'$ attached to it. The tree T_C associated to C is shown on the right in figure 12; observe that the trees $T_{C'}$, $T_{\overline{C}'}$ join in an *unmarked* vertex. Note that all $C' - \overline{C}'$ intersections are of the type allowed in definition 3.2, therefore α bounds a Bing cell in A, and $I_\lambda(A, \alpha) = 1$.

In the case when the surface Σ has genus $g > 1$ the proof is analogous to the genus one case discussed above. Specifically, A is obtained by attaching models A_i', A_i'', $i = 1, \ldots, g$ to a symplectic basis of curves in Σ. The complements are denoted B_i', B_i''. One observes that if there exists $1 \leq i \leq g$ such that $I_\lambda(B_i') = I_\lambda(B_i'') = 1$, then $I_\lambda(B) = 1$. On the other hand, if for each i either $I_\lambda(A_i')$ or $I_\lambda(A_i'')$ equals 1, then $I_\lambda(A) = 1$. This concludes the proof of lemma 4.2 and of theorem 1.1. $\square$

Remark 4.1. In the example of the decomposition $D^4 = A_2' \cup B_2'$ in figures 8, 9 the proof above shows that $I_\lambda(A_2', \alpha) = 1$. One may find an explicit construction of a Bing cell bounded by α in A_2' in the proof of [9, Lemma 7.3].

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