

HANDLE SLIDES AND LOCALIZATIONS OF CATEGORIES

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ABSTRACT. We propose a means by which some categorifications can be evaluated at a root of unity. This is implemented using a suitable localization in the context of prior work by the authors on categorification of the Jones-Wenzl projectors. Within this construction we define objects, invariant under handle slides, which decategorify to the $SU(2)$ quantum invariants at low levels.

1. INTRODUCTION

Constructions of invariants of 3-manifolds associated to Chern-Simons theory rely on the specialization of a parameter q at a root of unity [23, 26]. In Khovanov’s categorification of the Jones polynomial [13] the variable q is represented by an extra grading. Understanding how to “set” this grading to a root of unity has been a difficult problem in the program to derive invariants of 3-manifolds from categorifications of knot polynomials. In this paper we propose a means by which such categorifications can be evaluated at a root of unity. This proposal is motivated in section 5 by a construction of objects which are invariant under handle slides.

In the Temperley-Lieb algebra the evaluation of q at a root of unity is closely related to taking a suitable quotient by the Jones-Wenzl projector p_N . In [6] the authors constructed chain complexes P_N within the universal categorification of the Jones polynomial (see [14, 2]) that become Jones-Wenzl projectors in the image of the Grothendieck group. These chain complexes are uniquely characterized up to homotopy by the defining properties of the Jones-Wenzl projectors p_N . In this paper we consider a quotient of the homotopy category of chain complexes by the projector P_N . The Verdier quotient classically considered in homological algebra turns out to be unsuitable in this context (see section 4.9). We propose a different notion in section 4 called the coset construction. In this context we define objects which categorify Reshetikhin-Turaev invariants at low levels.

We will now outline a few more details of the construction. The Temperley-Lieb algebra has coefficients in $\mathbb{Z}[q, q^{-1}]$. The evaluation of a polynomial $f(q)$ at a root of unity may be implemented algebraically by passing to the quotient $\pi(f)$ where $\pi : \mathbb{Z}[q] \rightarrow \mathbb{Z}[q]/(\varphi_p(q))$ is the quotient map and $\varphi_p(q)$ is the p th cyclotomic polynomial. For example, when p is a prime number

$$\varphi_p(q) = q^{p-1} + \cdots + q + 1.$$

Given an element $e \in \text{TL}$ such that $\varphi_p(q) \mid \text{Tr}(e)$, the ideal generated by $\varphi_p(q)$ contains the ideal generated by $\text{Tr}(e)$. Therefore the quotient obtained by setting q to be a p th root of unity factors through the quotient of TL by e :

$$\text{TL} \longrightarrow \text{TL} / \langle e \rangle \longrightarrow \text{TL} / \langle \varphi_p(q) \rangle.$$

When $N = p - 1$ the Jones-Wenzl projectors $p_N \in \text{TL}_N$ provide a choice of e . In particular, when p is a prime number the trace of the N th projector $\text{Tr}(p_N)$ is given by the quantum integer $[p]$ and

$$q^{p-1}[p] = q^{2(p-1)} + \cdots + q^2 + 1 = \varphi_p(q)\varphi_p(-q).$$

Since the graded Euler characteristic of the categorified projector agrees with the Jones-Wenzl projector, if a quotient $\text{Ho}(\text{Kom})/(P_N)$ could be defined it would be closely related a desired evaluation of $\text{Ho}(\text{Kom})$ at a root of unity. In this paper we prove that there is an interesting quotient of this form and motivate the claim that this is sufficient for topological considerations by identifying structures which are familiar from low dimensional topology.

Different approaches to categorification of 3-manifold invariants have been proposed in [20, 25]. These authors consider 3-manifold invariants in different contexts, not involving evaluation at a root of unity.

Sections 2 and 3 review constructions of $SU(2)$ TQFTs and the relevant material on categorification from [2] and [6]. We introduce a filtration on coefficients corresponding to the level. The emphasis in both TQFT and categorified settings is on the Jones-Wenzl projectors. In section 4 we define a suitable version of localization. The unusual nature of this construction is motivated in section 4.9 by examining the structure of Verdier localization. Section 5 defines objects Ω which are invariant under handle slides.

2. TEMPERLEY-LIEB SPACES AND A CONSTRUCTION OF $SU(2)$ TQFTS

This section summarizes parts of the constructions of $SU(2)$ quantum invariants which are relevant in this paper, the Turaev-Viro theory associated to a surface and the Reshetikhin-Turaev invariant of 3-manifolds based on a surgery presentation. The general outline presented here follows the ‘‘picture TQFT’’ approach of [9, 27]. Our exposition differs from those available in the literature in the choice of the coefficient ring, and the main point of this section is contained in 2.8 which shows that a version of the TQFT may be defined using the Jones-Wenzl projectors, without a reference to a root of unity.

A compact oriented surface Σ with boundary is *labelled* when paired with a map $\phi : \partial\Sigma \rightarrow \coprod(S^1, \hat{x})$ which identifies each boundary circle with a model circle containing a fixed set of marked points \hat{x} .

Definition 2.1. The *Temperley-Lieb space* $\text{TL}(\Sigma)$ of a labelled surface Σ is the set of all $\mathbb{Z}[q, q^{-1}]$ linear combinations of isotopy classes of 1-manifolds (“multi-curves”) $F \subset \Sigma$ intersecting $\partial\Sigma$ transversely at marked points and subject to the local relation: removing a simple closed curve bounding a disk in Σ from a multi-curve is equivalent to multiplying the resulting element by the quantum integer $[2] = q + q^{-1}$.

$\text{TL}(\Sigma)$ is a 2-dimensional version of the Kauffman skein module of $\Sigma \times I$. The *Temperley-Lieb algebra* TL_n is given by $\text{TL}(D^2, \phi)$ where ϕ identifies ∂D^2 with a circle containing $2n$ marked points. If we view D^2 as a rectangle with n marked points on top and n marked points on the bottom then vertical stacking corresponds to the usual algebra structure on TL_n . The construction of $\text{TL}(\Sigma)$ is compatible with gluing surfaces along common labels. In particular, the restriction to genus zero surfaces gives the structure of a planar algebra.

2.2. The Jones-Wenzl Projectors. For each $n \geq 1$ the Temperley-Lieb algebra TL_n contains special idempotent elements called the *Jones-Wenzl projectors*. If we depict elements of TL_n graphically with n incoming and n outgoing chords then

$$(2.1) \quad \begin{array}{c} | \\ | \\ \boxed{n} \\ | \\ | \end{array} = \begin{array}{c} | \\ | \\ \boxed{n-1} \\ | \\ | \end{array} \Bigg| - \frac{[n-1]}{[n]} \begin{array}{c} | \\ | \\ \boxed{n-1} \\ | \\ | \end{array} \Bigg|$$

where the first element $p_1 \in \text{TL}_1$ is given by a single chord. Jones-Wenzl projectors will be central to the discussion in section 2.8 and sections 4, 5. For more detail, see [6, 12].

2.3. Restricted Coefficients. Recall the quantum integers,

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \dots + q^{-n+3} + q^{-n+1}.$$

Definition 2.4. Consider the power series

$$s_k = \sum_{i=0}^{\infty} (q^{(2i+1)k-1} - q^{(2i+1)k+1})$$

determined by the expansion of $1/[k]$ into positive powers of q . For each $N > 0$ the *restricted ring*

$$R^N = \mathbb{Z}[q, q^{-1}][s_1, \dots, s_N] \subset \mathbb{Z}[q^{-1}][[q]]$$

is obtained by adjoining s_1, \dots, s_N to the ring of Laurent polynomials $\mathbb{Z}[q, q^{-1}]$.

We have $R^1 = \mathbb{Z}[q, q^{-1}]$ and inclusions

$$R^1 \subset \dots \subset R^N \subset R^{N+1} \subset \dots \subset \mathbb{Z}[q^{-1}][[q]].$$

Definition 2.5. Let Σ be a labeled surface (see definition 2.1). Recall that $\text{TL}(\Sigma)$ is the $\mathbb{Z}[q, q^{-1}]$ module given by the Temperley-Lieb space. Define the N th restricted Temperley-Lieb space by extension of coefficients,

$$\text{TL}^N(\Sigma) = \text{TL}(\Sigma) \otimes_{\mathbb{Z}[q, q^{-1}]} R^N.$$

The restricted space TL^N is precisely the extension of TL necessary to define the first N Jones-Wenzl projectors: $p_1, p_2, \dots, p_N \in \text{TL}^N$ (after $1/[k]$ is replaced by $s_k(q)$ in equation (2.1)). Each restricted space TL^N will have a categorified analogue Kom^N (see section 3.2) and these special coefficients will allow us to control the quotient later.

We would like to consider the quotient of TL^N by the ideal $\langle p_N \rangle$. Note that $1/\text{Tr}(p_N) = 1/[N+1] \notin R^N$, this motivates the definition of the restricted ring R^N above.

Definition 2.6. Let $Q \in \text{TL}^N(D^2, \hat{x})$. The ideal $\langle Q \rangle$ generated by Q in $\text{TL}^N(\Sigma)$ is the smallest submodule containing all elements obtained by gluing Q to B where $\Sigma = (D^2, \hat{x}) \cup (\Sigma \setminus D^2, \hat{x})$ with $D^2 \subset \text{int}(\Sigma)$, and B is any element of $\text{TL}^N(\Sigma \setminus D^2, \hat{x})$.

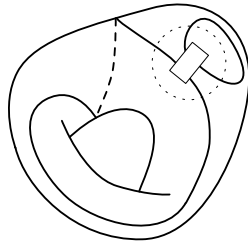


FIGURE 1. An element of the ideal $\langle Q \rangle \subset \text{TL}^N(\Sigma)$ where Q is represented by a box, and the surface Σ is a torus.

The notion of an ideal $\langle Q \rangle$ allows one to consider an element Q of the Temperley-Lieb algebra as a local relation “ $Q = 0$ ” among multi-curves on a surface Σ , see [27, 9] for more details. The term “ideal” is motivated by the notion of an ideal in the planar algebra setting (which corresponds to restricting our discussion to surfaces of genus zero).

2.7. Construction of $\text{SU}(2)$ quantum invariants. We start by briefly recalling a construction of the Turaev-Viro theory [26] associated to a surface Σ , following [9]. We will then present a slight modification of the definition where the evaluation at a root of unity is replaced by taking a suitable quotient by a Jones-Wenzl projector.

Let $N = p - 1$ and consider the Temperley-Lieb space $\overline{\text{TL}}^N(\Sigma)$ with coefficients in the ring $K^p = \mathbb{Z}[q, q^{-1}]/\langle \varphi_p(q^2) \rangle$, where $\varphi_p(q)$ denotes the p th cyclotomic polynomial. Then the Turaev-Viro $\text{SU}(2)$ theory associated to Σ , $\text{TV}^N(\Sigma)$, is the quotient $\text{TL}(\Sigma)$ by the ideal $\langle p_N \rangle$, where p_N denotes the N th Jones-Wenzl projector in the Temperley-Lieb algebra. Note that our definition is somewhat different from that in [9], specifically rather than setting q to be a root of unity in the ground ring (which may be algebraically implemented by taking a quotient by the cyclotomic polynomial $\varphi_p(q)$) we define the coefficient ring K^p as the quotient by $\varphi_p(q^2)$. The relation between the two is $\varphi_p(q^2) = \varphi_p(q)\varphi_p(-q)$. The reason for our choice of the coefficient ring is explained in section 2.8.

The spaces TV^N suffice for the study of mapping class group representations. Such spaces are determined in a canonical way by a 3-dimensional TQFT [4]. On the other hand in order to define a 3-dimensional TQFT using such spaces one must choose maps associated to the addition of 3-dimensional handles which transform in accordance with handle slides and cancellations. When such a choice exists it is unique up to multiplication by a scalar [8, 17]. In our context this information is determined by the special element ω_N (see section 2.12).

For reasons explained in section 3, it will be convenient for us to consider *cosets* of the submodule $\langle P_N \rangle$ generated by P_N in $\text{TL}(\Sigma)$. Of course, the quotient $\text{TL}(\Sigma)/\langle P_N \rangle$ may be equivalently considered as the set of cosets of $\langle P_N \rangle$ in $\text{TL}(\Sigma)$. We stress this minor detail so it is easier to follow the categorical construction in section 3.

2.8. Evaluating at a root of unity by killing a projector. Set $N = p - 1$ and recall the sequence of steps used in the definition of $\text{TV}^N(\Sigma)$ above: first the coefficients of $\text{TL}(\Sigma)$ are set to be K^p (this is similar to setting the parameter q to be a root of unity), and then one takes the quotient of $\text{TL}^N(\Sigma)$ by the submodule $\langle p_N \rangle$. We will next show that in fact the first step may be omitted.

Consider the ring $K^p = \mathbb{Z}[q, q^{-1}]/\langle \varphi_p(q^2) \rangle$ as above. Recall the restricted ring R^N (definition 2.4) and consider the ideal $I = \langle p_N \rangle \subset \text{TL}^N(S^2) = R^N$.

Lemma 2.9. *Given a prime p , let $N = p - 1$. Then*

$$(R^N/I) \otimes \mathbb{Q} \cong K^p \otimes \mathbb{Q}.$$

Proof. Since $[k]$ is a unit in $K^p \otimes \mathbb{Q}$ the quotient map $\mathbb{Z}[q, q^{-1}] \otimes \mathbb{Q} \rightarrow K^p \otimes \mathbb{Q}$ extends to a map $R^N \otimes \mathbb{Q} \rightarrow K^p \otimes \mathbb{Q}$. Moreover, since any element of I evaluates to zero in $K^p \otimes \mathbb{Q}$, this map descends to a map

$$\phi : (R^N/I) \otimes \mathbb{Q} \rightarrow K^p \otimes \mathbb{Q}.$$

On the other hand, observe that the cyclotomic polynomial $\varphi_p(q^2)$ is in the ideal I , since $\varphi_p(q^2) = q^N \text{Tr}(P_N)$. This gives an inverse map

$$\psi : K^p \otimes \mathbb{Q} \rightarrow (R^N/I) \otimes \mathbb{Q}.$$

□

Setting the variable q to a root of unity corresponds to quotienting the ring of coefficients by the ideal $\langle \varphi_p(q) \rangle$. Since $\langle \varphi_p(q^2) \rangle \subset \langle \varphi_p(q) \rangle$ the lemma implies

$$\text{TL} \longrightarrow \text{TL} / \langle p_N \rangle \longrightarrow \text{TL} / \langle \varphi_p(q) \rangle.$$

Remark. The proof of this lemma generalizes to give a “picture TQFT” construction of $\text{SU}(2)_N$ Turaev-Viro theory for a surface Σ by killing P_N in the $\text{SU}(2)$ Kauffman skein module of $\Sigma \times I$ over R^N .

2.10. Kirby Calculus and Handle Slides. It is a classical theorem of Lickorish and Wallace that any 3-manifold M can be obtained by surgery on a framed link $L \subset S^3$. Kirby calculus is a means of describing when surgery on two different framed links produce homeomorphic 3-manifolds:

Theorem 2.11. [19] *Two 3-manifolds M and N obtained by surgery on S^3 along framed links L and L' are diffeomorphic if and only if the associated link diagrams $D(L)$ and $D(L')$ are related by a sequence of Kirby moves:*



FIGURE 2

The first move describes the addition of a disjoint ± 1 -framed unknot to the diagram. It is the second, *handle slide*, move that will be the focus of this paper. The invariance under the first Kirby move is achieved by a suitable normalization of the quantum invariant (cf. [4, 18]). This is not addressed in the present paper.

2.12. The magic element. In [23] (see also [12]) a special element ω_N is defined which can be represented by a linear combination of projectors:

$$(2.2) \quad \omega_N = \sum_{k=0}^N \text{Tr}(p_k) \phi_k.$$

Here ϕ_k is the element of the annulus category obtained by placing the projector p_k in the annulus. This ω_N , sometimes referred to as the magic linear combination of Temperley-Lieb elements, is the element which is invariant under handle slides used in most constructions of quantum 3-manifold invariants.

Lemma 2.13. *The element ω_N is invariant under handle slides in the module $\mathrm{TL}^N(S^1 \times I)/\langle p_N \rangle$.*

A beautiful proof of this lemma was given by Lickorish (see [18]) when q equals a root of unity. We observe that his proof remains true in the annulus module above, where the evaluation at a root of unity is replaced by taking a quotient by the ideal generated by the Jones-Wenzl projector.

The Reshetikhin-Turaev invariant of a 3-manifold M is defined by presenting M as a surgery on a framed link L in S^3 , labeling each link component by the element ω_N and evaluating the Jones polynomial of the resulting labeled link at the corresponding root of unity, cf. [12, 18]. To relate this to the context of the Temperley-Lieb spaces and the Turaev-Viro theory $\mathrm{TV}^N(\Sigma)$ associated to surfaces, note that the Jones polynomial of a link may be viewed as the evaluation of the link in the skein module associated to S^2 . The Turaev-Viro theory discussed above is the quantum double of the Reshetikhin-Turaev TQFT, however in the case of the 2-sphere both theories are 1-dimensional. Therefore it makes sense to analyze the handle-slide property in the module $\mathrm{TL}^N(S^1 \times I)/\langle p_N \rangle$ which was defined above in the Turaev-Viro setting.

The reader should note that the omegas introduced in section 5 are not precisely the same as the special element ω_N defined above. The two definitions are compared in section 5.9.

2.14. Spin 3-manifolds. A refined version of $\mathrm{SU}(2)$ quantum invariants may be defined for a 3-manifold with a spin structure [3]. One considers a decomposition $\omega = \omega_0 + \omega_1$, corresponding to the $\mathbb{Z}/2$ grading (parity) of the summation indexing set in (2.2). Given a presentation of a 3-manifold M with a spin structure as the surgery on a framed link $L \subset S^3$, one labels the components of the characteristic sublink by ω_1 and the rest of the components by ω_0 . A spin analogue of the Kirby calculus is then used [3] to prove that an appropriate normalization is an invariant of M^3 with a given spin structure.

3. CATEGORIFICATION OF TEMPERLEY-LIEB SPACES

In this section we recall a version of Dror Bar-Natan's graphical formulation [2] of the Khovanov categorification [14]. It will be used throughout the remainder of this paper. The Bar-Natan formulation extends to planar algebras and surfaces in a transparent way and is essentially universal (see [15]) and so has the advantage of allowing our constructions to apply to a number of variant categorifications which exist in the literature.

Let Σ be a labelled surface. There is an additive category $\mathrm{Pre-Cob}(\Sigma)$ whose objects are isotopy classes of formally q -graded 1-manifolds $F \subset \Sigma$ intersecting all boundary components transversely at marked points (compare to definition 2.1).

The morphisms are given by $\mathbb{Z}[\alpha]$ linear combinations of isotopy classes of orientable cobordisms bounded in $\Sigma \times [0, 1]$ between two surfaces Σ containing such diagrams. Each cobordism is constant along the boundary. The *degree* of a cobordism $C : q^i A \rightarrow q^j B$ is given by $\deg(C) = \deg_t(C) + \deg_q(C)$ where the topological degree $\deg_t(C) = \chi(C) - n$ is given by the Euler characteristic of C and the q -degree $\deg_q(C) = j - i$ is given by the relative difference in q -gradings. Homological degree is not part of the definition $\deg(C)$.

It has become a common notational shorthand to represent a handle by a dot and a saddle by a flattened diagram containing a dark line.



FIGURE 3

(The topological degrees of the cobordisms above are $-2, -1$ respectively.) When $(\Sigma, \phi) = (D^2, 2n)$ we would like a category \mathcal{C} such that $K_0(\mathcal{C}) \cong \text{TL}_n$ so we require that the object represented by a closed circle be isomorphic to sum of two empty objects in degrees ± 1 respectively. If such maps are to be degree preserving then the most natural choice for these maps is given below.

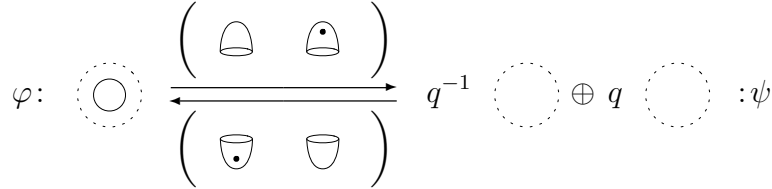


FIGURE 4

In order to obtain $\varphi \circ \psi = 1$ and $\psi \circ \varphi = 1$ we form a new category $\text{Cob}(\Sigma)$ obtained as a quotient of the category $\text{Pre-Cob}(\Sigma)$ by the local relations given below.

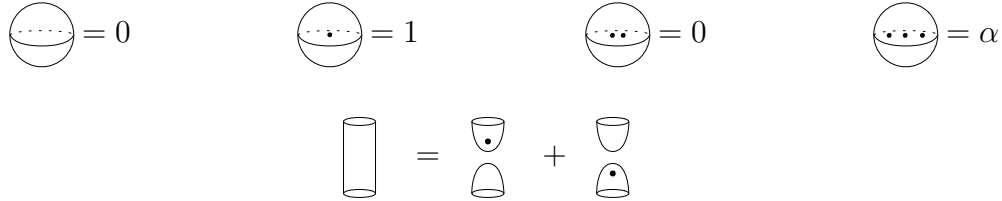


FIGURE 5

The neck cutting relation can be applied to any compressing disk. In particular, that this means the isomorphisms ϕ and ψ cannot be applied to remove a non-trivial circle from the annulus. The cylinder or neck cutting relation implies that closed surfaces Σ_g of genus $g > 3$ evaluates to 0. The relations above imply that a sheet with two dots is equal to α times a sheet with no dots. In what follows we will let α be a free variable and absorb it into our base ring ($\Sigma_3 = 8\alpha$). One can think of α as a deformation parameter, see [2] for further details.

Proposition 3.1. *For any labelled surface Σ there is an isomorphism of $\mathbb{Z}[q, q^{-1}]$ -modules,*

$$K_0(\text{Cob}(\Sigma)) \cong \text{TL}(\Sigma).$$

The proof follows directly from the construction above. Note that when $\Sigma = D^2$ this is precisely the same setting as [6] and [7]. For an extended discussion of Grothendieck groups in this context see [6].

Remark. Surfaces with common boundary labels can be glued and the construction of $\text{Cob}(\Sigma)$ is compatible with this gluing.

3.2. Filtered Categories and Universal Projectors.

Definition 3.3. If Σ is a labelled surface let $\text{Kom}^\infty(\Sigma) = \text{Kom}(\text{Mat}(\text{Cob}(\Sigma)))$ be the category of chain complexes of finite direct sums of objects in $\text{Cob}(\Sigma)$. In Kom^∞ we allow chain complexes K_* of unbounded positive homological degree and bounded negative homological degree. Let $\text{Kom}^\infty(n) = \text{Kom}^\infty(D^2, 2n)$ denote the category associated to the disk. Let $\text{Kom}^1(\Sigma)$ be the category of bounded complexes.

In previous work the authors showed that the category $\text{Kom}^\infty(n)$ contains special objects P_n which categorify the Jones-Wenzl projectors $p_n \in \text{TL}_n$.

Theorem 3.4. [6] *There exists a chain complex $P_n \in \text{Kom}^\infty(n)$ called the universal projector which satisfies*

- (1) P_n is positively graded with degree zero differential.
- (2) The identity diagram appears only in homological degree zero and only once.
- (3) The chain complex P_* is contractible under turnbacks.

These three properties guarantee that P_n is unique up to homotopy, $P_n \otimes P_n \simeq P_n$ and $K_0(P_n) = p_n \in \text{TL}_n$.

Other authors [10, 24] constructed projectors using different techniques.

Definition 3.5. Given $N \in \mathbb{Z}_+$ and a labeled surface Σ , let $\text{Kom}^N(\Sigma)$ be the full subcategory of $\text{Kom}^\infty(\Sigma)$ consisting of those chain complexes whose image under K_0 is contained in $\text{TL}^N(\Sigma)$.

By construction,

$$\mathrm{Kom}^1(\Sigma) \subset \cdots \subset \mathrm{Kom}^N(\Sigma) \subset \mathrm{Kom}^{N+1}(\Sigma) \subset \cdots \subset \mathrm{Kom}^\infty(\Sigma)$$

where $\mathrm{Kom}^1(\Sigma)$ is the category of bounded complexes and $\mathrm{Kom}^\infty(\Sigma)$ is as in definition 3.3.

4. LOCALIZATION

The idea of localization is central to this paper. We seek to carry out a categorified analogue of the ‘‘picture TQFT’’ construction in sections 2.7, 2.8, and a central step in this program is to kill P_N in $\mathrm{Kom}^N(\Sigma)$ for all Σ . In order to accomplish this we find a subcategory $\langle\langle P_N \rangle\rangle$ of $\mathrm{Kom}^N(\Sigma)$ the image of which corresponds to the ideal $\langle P_N \rangle$ in the Grothendieck group $K_0(\mathrm{Kom}^N(\Sigma)) \cong \mathrm{TL}^N(\Sigma)$. A natural family of morphisms is then inverted (definition 4.4) in which we identify any object X in $\mathrm{Kom}^N(\Sigma)$ with any cone of X on any object $P' \in \langle\langle P_N \rangle\rangle$. This localization can be understood concretely (definition 4.7) in terms of iterated cones over the ideal $\langle\langle P_N \rangle\rangle$. In section 4.9 we explain why the usual notion of Verdier localization is not suitable in our context, leading us to a different version of localization described next.

In what follows, we fix $N > 0$ and use P to denote P_N and Kom to denote $\mathrm{Kom}^N(\Sigma)$ for some Σ . Also $\mathrm{Ho}(\mathrm{Kom})$ is the homotopy category of Kom .

Definition 4.1. Given Q in $\mathrm{Ho}(\mathrm{Kom}(D^2, \hat{x}))$, the *ideal* $\langle\langle Q \rangle\rangle \subset \mathrm{Ho}(\mathrm{Kom}(\Sigma))$ generated by Q is the smallest full subcategory of $\mathrm{Ho}(\mathrm{Kom}(\Sigma))$ which contains all objects obtained by gluing Q to B where $\Sigma = (D^2, \hat{x}) \cup (\Sigma \setminus D^2, \hat{x})$ with $D^2 \subset \mathrm{int}(\Sigma)$, B is any object of $\mathrm{Ho}(\mathrm{Kom}(\Sigma \setminus D^2, \hat{x}))$, and which is closed under cones and grading shifts.

The reader should compare this to definition 2.6.

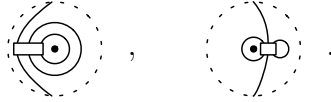


FIGURE 6. Examples of objects in the ideal $\langle\langle P_3 \rangle\rangle$ in $\mathrm{Kom}(S^1 \times I)$, where the annulus $S^1 \times I$ has two marked points on the boundary.

Notice that $A \oplus B \in \langle\langle P \rangle\rangle \not\cong A \in \langle\langle P \rangle\rangle$; the subcategory $\langle\langle P \rangle\rangle$ is not thick, compare with section 4.9. The definition is chosen so that the following proposition holds.

Proposition 4.2. *If $\mathrm{Ho}(\mathrm{Kom})$ and $P \in \mathrm{Ho}(\mathrm{Kom})$ are as above, then*

$$K_0(\langle\langle P \rangle\rangle) = \langle P \rangle$$

where $\langle P \rangle \subset K_0(\mathrm{Ho}(\mathrm{Kom})) \cong \mathrm{TL}^N$ is the ideal generated by P (definition 2.6).

The proof follows from definition 4.1. For more on K_0 of infinite complexes see [6]. For a discussion of K_0 of triangulated categories see [28].

Proposition 4.3. (*Localization of a category*) *Given a collection S of morphisms of a category \mathcal{C} , there exists a category $\mathcal{C}[S^{-1}]$ and a functor $Q : \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$ satisfying the universal property:*

- *If $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor and $F(s)$ is an isomorphism for all $s \in S$ then there exists a functor $G : \mathcal{C}[S^{-1}] \rightarrow \mathcal{D}$ such that $F = G \circ Q$.*

The category $\mathcal{C}[S^{-1}]$ is constructed by realizing \mathcal{C} as a directed graph, adding edges corresponding to inverses of elements of S and taking morphisms to be paths in this graph subject to the most obvious relations. For details see [11].

Definition 4.4. (“brusque quotient”) Given $P' \in \langle\langle P \rangle\rangle \subset \text{Ho}(\text{Kom})$, if $f : X \rightarrow P'$ or $g : P' \rightarrow X$ there are maps $\pi_f : \text{Cone}(f) \rightarrow X$ and $i_g : X \rightarrow \text{Cone}(g)[-1]$. We wish to invert all such maps. Set

$$S = \{\pi_f | f : X \rightarrow P', P' \in \langle\langle P \rangle\rangle\} \cup \{i_g | g : P' \rightarrow X, P' \in \langle\langle P \rangle\rangle\},$$

and define

$$\text{Ho}(\text{Kom})/\langle\langle P \rangle\rangle = \text{Ho}(\text{Kom})[S^{-1}].$$

Note that this version of localization has some unusual properties. For instance, $\text{Id}_{A \oplus B} \cong 0 \not\cong \text{Id}_A \cong 0$ in the quotient, and the familiar triangulated structure is now gone. Section 4.9 explains how this definition is different from the usual Verdier localization.

We will now show that the above construction can be reformulated in a more concrete way. (The reader may note that the theme of characterizing localization in terms of iterated extensions appears in algebraic topology literature, cf. [5]). We begin with an equivalence relation on objects in Kom .

Definition 4.5. ($\langle\langle P \rangle\rangle$ -equivalence) Let $\langle\langle P \rangle\rangle \subset \text{Ho}(\text{Kom})$ be as above then for any $A, B \in \text{Ho}(\text{Kom})$ we say that A is $\langle\langle P \rangle\rangle$ -equivalent to B if there exists a sequence of maps $\{f_i\}_{i=1}^m$ and objects $Q_i \in \langle\langle P \rangle\rangle$ such that

- (1) $f_1 : B \rightarrow Q_1$ or $f_1 : Q_1 \rightarrow B$ for some $Q_1 \in \langle\langle P \rangle\rangle$
- (2) $f_i : \text{Cone}(f_{i-1}) \rightarrow Q_i$ or $f_i : Q_i \rightarrow \text{Cone}(f_{i-1})$

and $\text{Cone}(f_m) \cong A$

This equivalence relation amounts to isomorphism in the category $\text{Ho}(\text{Kom})/\langle\langle P \rangle\rangle$ defined above:

Proposition 4.6. *$A \cong B$ in $\text{Ho}(\text{Kom})/\langle\langle P \rangle\rangle$ if and only if A is $\langle\langle P \rangle\rangle$ -equivalent to B in $\text{Ho}(\text{Kom})$.*

Proof. The first direction is obvious. Suppose $A \cong B$ in $\text{Ho}(\text{Kom})/\langle\langle P \rangle\rangle$ then by definition an isomorphism $\phi : A \rightarrow B$ is a map of the form $\phi = h_n \phi_n h_{n-1} \phi_{n-1} \cdots h_1 \phi_1$ where ϕ_k is an isomorphism in $\text{Ho}(\text{Kom})$ and h_k is an isomorphism of the form $\pi_{f_k}^{\pm 1}$ or $i_{g_k}^{\pm 1}$ as in definition 4.4 above. A map $h_k : X_k \rightarrow X_{k+1}$ tells us that either X_k is equivalent to X_{k+1} coned on $P' \in \langle\langle P \rangle\rangle$ or vice versa. \square

Given an object $A \in \text{Ho}(\text{Kom})$ we can combine all of the objects which are $\langle\langle P \rangle\rangle$ -equivalent to it to form a category which is a natural analogue of the coset $A + \langle P \rangle$ in the image of the Grothendieck group.

Definition 4.7. (Coset categories) Given $A \in \text{Ho}(\text{Kom})$, the *coset* associated to A is the full subcategory, denoted $A + \langle\langle P \rangle\rangle \subset \text{Ho}(\text{Kom})$, consisting of objects Y which are $\langle\langle P \rangle\rangle$ -equivalent to A in $\text{Ho}(\text{Kom})$.

Isomorphism classes of coset subcategories are in bijection with $\langle\langle P \rangle\rangle$ -equivalence classes:

Proposition 4.8. *A is $\langle\langle P \rangle\rangle$ -equivalent to B if and only if $A + \langle\langle P \rangle\rangle \cong B + \langle\langle P \rangle\rangle$*

Notice that the Grothendieck group of a coset category $\text{K}_0(A + \langle\langle P \rangle\rangle)$ can be identified with the coset $\text{K}_0(A) + \langle P \rangle$. In particular, the non-triviality of $\text{TL}^N / \langle P_N \rangle$ implies that the quotient $\text{Ho}(\text{Kom})/\langle\langle P \rangle\rangle$ is highly non-trivial. More precisely, in light of remark at the end of section 2.8, the machinery developed in this section may be viewed as a categorification of the 2-dimensional part of the Turaev-Viro theory.

Remark. In [16] Khovanov considered additive monoidal categories $(\mathcal{C}, \otimes, 1)$ of graded objects in which $1 \oplus q1 \oplus \cdots \oplus q^{n-1}1 \cong 0$. Such an isomorphism implies that the polynomial $1 + q + \cdots + q^{n-1}$ is equal to 0 in the Grothendieck group $\text{K}_0(\mathcal{C})$.

In this paper, $\text{Tr}(P_N)$ is isomorphic to a chain complex consisting only of empty diagrams [6] and the relation $\text{Tr}(P_N) = 0$ implies that the polynomial $1 + q^2 + \cdots + q^{2N}$ is zero in the Grothendieck group.

There are two other more notable differences between these ideas. In our context, the quotient must be homotopy invariant because P_N is only determined up to homotopy. Also inasmuch as P_N is not determined by its graded Euler characteristic, the effect of the quotient on the category has less to do with the polynomial $\text{K}_0(\text{Tr}(P_N))$ and more to do with the structure of P_N itself.

4.9. On Verdier Quotients. In section 4 we quotiented $\text{Kom}^N(\Sigma)$ by P_N in an unusual way. In this section we recall elements of the most common quotient construction for triangulated categories. In particular, we explain why it is not useful in the context of this paper. Readers interested in a detailed discussion of the Verdier quotient should consult [11, 22].

Given a triangulated category \mathcal{C} we would like to invert some family of maps T so that the objects of a full triangulated subcategory $\mathcal{D} \subset \mathcal{C}$ become isomorphic to zero in a triangulated quotient category $\mathcal{C}[T^{-1}] = \mathcal{C}/\mathcal{D}$.

In order to produce a triangulated quotient category $\mathcal{C}[T^{-1}]$ the maps in T are required to admit a calculus of left and right fractions. The collection T is additionally required to be closed under homological grading shifts and if (f, g, h) is a map between triangles with $f, g \in T$ then there exists an $h' \in T$ so that (f, g, h') is a map between triangles. Most of these conditions are not imposed by the brusque quotient of section 4.

If \mathcal{D} is a full triangulated subcategory of a triangulated category \mathcal{C} , then T is defined to be the collection of maps $f : X \rightarrow Y$ which fit into an exact triangle:

$$X \rightarrow Y \rightarrow Z \rightarrow X[1]$$

where $Z \in \mathcal{D}$. If the collection T is chosen in this manner then it satisfies all of the properties described above. The quotient functor $Q : \mathcal{C} \rightarrow \mathcal{C}[T^{-1}]$ is a triangulated functor and universal in the category of small triangulated categories. The quotient functor F in proposition 4.3 does not satisfy these properties.

Definition 4.10. A *thick* (or *épaisse*) subcategory $\mathcal{D} \subset \mathcal{C}$ is a subcategory which is closed under retracts: $A \oplus B \in \mathcal{D} \Rightarrow A \in \mathcal{D}$.

The Verdier quotient only sees the smallest thick subcategory containing the subcategory under consideration. More specifically, if $\mathcal{D} \subset \mathcal{C}$ is a subcategory and $\bar{\mathcal{D}}$ is the smallest thick subcategory containing \mathcal{D} then $\mathcal{C}/\mathcal{D} \cong \mathcal{C}/\bar{\mathcal{D}}$.

The following proposition explains our problem.

Proposition 4.11. *The smallest thick subcategory containing the ideal $\langle\langle P_2 \rangle\rangle$ is $\text{Ho}(\text{Kom}(\Sigma))$.*

Proof. In [6] it was shown that $\text{Tr}(P_2) \simeq q^{-2}\mathbb{Z} \oplus \mathbb{Z} \oplus \dots$ □

It follows that $\text{Ho}(\text{Kom}(\Sigma))/\langle\langle P_2 \rangle\rangle \cong 0$. The following general statement also holds.

Proposition 4.12. *If L is any link or $(1, 1)$ -tangle then the smallest thick subcategory containing the ideal $\langle\langle L \rangle\rangle$ is $\text{Ho}(\text{Kom}(\Sigma))$.*

Proof. The chain complex associated to L is homotopy equivalent to a chain complex in which the Rasmussen invariant can be computed from the grading of a trivial direct summand (see [21]). □

A conceptual explanation could be formulated by instead considering the category $\text{D}^{\text{perf}}(H^*(\mathbb{C}P^1)\text{-mod})$ (compare [14]). Here thick subcategories are in correspondence with closed subschemes [1] and $\text{Spec}(\mathbb{Z}[x]/(x^2))$ has no interesting subschemes.

Other quotients such as those of Bousfield and Drinfeld also respect thickness in the manner described above. It seems that the intricacies of these homotopy theories are not seen by the usual quotients.

$$\begin{array}{|} \hline \square \\ \hline \end{array} = \text{Cone} \left(\begin{array}{|} \hline | \\ \hline \end{array} \rightarrow \left[\begin{array}{c} \cup \\ \cap \end{array} \right] \right) \quad \text{and} \quad \begin{array}{|} \hline | \\ \hline \end{array} \simeq \text{Cone} \left(\left[\begin{array}{c} \cup \\ \cap \end{array} \right] \rightarrow \begin{array}{|} \hline \square \\ \hline \end{array} \right)$$

where the map defining the first cone is the saddle, d_0 , appearing in the definition of P_2 and the map in the second cone is the inclusion of the tail into P_2 .

Definition 5.3. (Ω_2) In the category $\text{Kom}^2(S^1 \times [0, 1])$ let Ω_+ be the object consisting of a single curve about the origin and Ω_- be the object consisting of a single nullhomotopic simple closed curve. Let $\Omega = \Omega_+ \oplus \Omega_-$, we represent Ω by a curve decorated with a dot. Graphically,

$$\Omega_+ = \begin{array}{c} \text{---} \\ \circ \\ \text{---} \end{array}, \quad \Omega_- = \begin{array}{c} \text{---} \\ \circ \\ \text{---} \end{array}, \quad \begin{array}{c} \text{---} \\ \circ \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circ \\ \text{---} \end{array} \oplus \begin{array}{c} \text{---} \\ \circ \\ \text{---} \end{array}.$$

Theorem 5.4. The coset $\bar{\Omega} = \Omega + \langle\langle P_2 \rangle\rangle$ associated to the object Ω defined above is invariant under handle slides.

Proof. Consider first the partial trace of P_2 in $\text{Kom}^2(S^1 \times [0, 1])$,

$$\begin{array}{c} \text{---} \\ \circ \\ \text{---} \end{array} = \text{Cone} \left(\begin{array}{c} \text{---} \\ \circ \\ \text{---} \end{array} \rightarrow \left[\begin{array}{c} \text{---} \\ \circ \\ \text{---} \end{array} \right] \right).$$

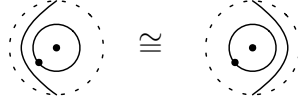
As above, the square brackets denote the tail of the second projector. Consider also the following diagram:

$$\begin{array}{c} \text{---} \\ \circ \\ \text{---} \end{array} = \text{Cone} \left(\begin{array}{c} \text{---} \\ \circ \\ \text{---} \end{array} \rightarrow \left[\begin{array}{c} \text{---} \\ \circ \\ \text{---} \end{array} \right] \right).$$

Note that the tail of the first equation and the tail of the second equation above are equal chain complexes. Also the left-hand sides of both equations are in the ideal $\langle\langle P_2 \rangle\rangle$ (definition 4.1), since they contain the second projector. These two observations taken together imply that the equation

$$\begin{array}{c} \text{---} \\ \circ \\ \text{---} \end{array} \simeq \left[\begin{array}{c} \text{---} \\ \circ \\ \text{---} \end{array} \right] = \left[\begin{array}{c} \text{---} \\ \circ \\ \text{---} \end{array} \right] \simeq \begin{array}{c} \text{---} \\ \circ \\ \text{---} \end{array}$$

holds among $\langle\langle P_2 \rangle\rangle$ cosets. For example, the first isomorphism holds since the diagram on the left is a cone of the inclusion of the tail into an element of $\langle\langle P_2 \rangle\rangle$. This says that a single strand placed on the lefthand side of Ω_+ is isomorphic to a single strand placed on the righthand side of Ω_- as $\langle\langle P_2 \rangle\rangle$ cosets and vice versa. This equation together with its vertical reflection imply that



as $\langle\langle P_2 \rangle\rangle$ cosets, establishing the handle-slide property of Ω . \square

5.5. Handle Slides and The Third Projector. We start this section by recalling the minimal (in the sense that it is not homotopic to a chain complex with fewer Temperley-Lieb diagrams) chain complex for P_3 , introduced in [6]. Then we show how it can be used to define an element Ω in the annular category $\text{Kom}^3(S^1 \times [0, 1])$ which is invariant under handle slides in the associated coset category.

Recall that the third projector is given by

$$\begin{array}{ccccccc}
 ||| & \xrightarrow{A} & q^1 \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \oplus \begin{array}{c} | \\ | \end{array} \right) & \xrightarrow{B} & q^2 \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \oplus \begin{array}{c} \text{---} \\ \text{---} \end{array} \right) & \xrightarrow{C} & q^4 \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \oplus \begin{array}{c} \text{---} \\ \text{---} \end{array} \right) \\
 & & & & & & \downarrow D \\
 \dots & \longleftarrow & q^8 \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \oplus \begin{array}{c} \text{---} \\ \text{---} \end{array} \right) & \xleftarrow{B} & q^7 \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \oplus \begin{array}{c} | \\ | \end{array} \right) & \xleftarrow{E} & q^5 \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \oplus \begin{array}{c} | \\ | \end{array} \right)
 \end{array}$$

Where $A = \begin{pmatrix} - & \text{---} & | \\ & \text{---} & \text{---} \end{pmatrix}^{\text{tr}}$ and

$$\begin{array}{l}
 B = \begin{pmatrix} \text{---} & - & \text{---} \\ - & \text{---} & \text{---} \end{pmatrix} \quad C = \begin{pmatrix} \text{---} + \text{---} & \text{---} \\ \text{---} & \text{---} + \text{---} \end{pmatrix} \\
 D = \begin{pmatrix} \text{---} & - & \text{---} \\ - & \text{---} & \text{---} \end{pmatrix} \quad E = \begin{pmatrix} | & + & | \\ \text{---} & & \text{---} \\ \text{---} & + & \text{---} \end{pmatrix}
 \end{array}$$

After the initial identity term the complex becomes 4 periodic.

Theorem 5.6. [6] *The definition of P_3 given above is a chain complex that satisfies the axioms of the universal projector.*

We define the *tail* of P_3 to be the chain complex

$$\left[\begin{array}{c} \text{X} | \oplus | \text{X} \\ \rightarrow \\ \text{X} \oplus \text{X} \end{array} \right] = \text{Cone} \left(\begin{array}{c} \text{X} \\ \rightarrow \\ \text{X} \end{array} \right)$$

so that

$$\left[\begin{array}{c} \text{X} | \oplus | \text{X} \\ \rightarrow \\ \text{X} \oplus \text{X} \end{array} \right] \simeq q^1 \left(\begin{array}{c} \text{X} | \oplus | \text{X} \\ \rightarrow \\ \text{X} \oplus \text{X} \end{array} \right) \rightarrow q^2 \left(\begin{array}{c} \text{X} \oplus \text{X} \\ \rightarrow \\ \text{X} \oplus \text{X} \end{array} \right) \rightarrow \dots$$

i.e. the “tail” of the chain complex for P_3 given at the beginning of this section.

Definition 5.7. (Ω_3) In the category $\text{Kom}^3(S^1 \times [0, 1])$ let Ω_+ be the object consisting of two curves about the origin and Ω_- be the object consisting of a single curve about the origin and a single nullhomotopic simple closed curve. Let $\Omega = \Omega_+ \oplus \Omega_-$, we represent Ω by a curve decorated with a dot then in terms of pictures. Graphically,

$$\Omega_+ = \begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array} \text{ , } \Omega_- = \begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array} \text{ , } \begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array} \oplus \begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array} \text{ .}$$

Theorem 5.8. *The coset $\bar{\Omega} = \Omega + \langle\langle P_3 \rangle\rangle$ associated to the object Ω defined above is invariant under handle slides.*

Proof. The proof consists of analyzing the partial traces of P_3 in the annulus.

$$\begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array} \rightarrow \left[\begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array} \oplus \begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array} \rightarrow \begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array} \oplus \begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array} \right]$$

where we omit the word Cone above to save horizontal space. Consider the partial trace,

$$\begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array} \rightarrow \left[\begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array} \oplus \begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array} \rightarrow \begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array} \oplus \begin{array}{c} \text{---} \\ \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array} \right] \text{ .}$$

Notice that the two tails above are the same up to changing the order of the summands,

$$\left[\begin{array}{c} \text{Diagram 1} \\ \oplus \\ \text{Diagram 2} \end{array} \rightarrow \begin{array}{c} \text{Diagram 3} \\ \oplus \\ \text{Diagram 4} \end{array} \right] \cong \left[\begin{array}{c} \text{Diagram 5} \\ \oplus \\ \text{Diagram 6} \end{array} \rightarrow \begin{array}{c} \text{Diagram 7} \\ \oplus \\ \text{Diagram 8} \end{array} \right]$$

Since the left hand side of both equations above include the third projector they are both elements of the ideal $\langle\langle P_3 \rangle\rangle$. It follows that

$$\text{Diagram 9} \cong \text{Diagram 10},$$

but this equation says that a free strand can be slid over Ω_+ at the cost of turning the Ω_+ into an Ω_- . This equation together with its vertical reflection imply that the $\langle\langle P_3 \rangle\rangle$ coset associated to the Ω defined above is invariant under handle slides. \square

5.9. Relation to the magic element. Here we compare the ω elements categorified in 5.1 and 5.5 to the special ω_N of section 2.12. We use X to represent a single essential circle in the annulus and the relation $X\phi_k = \phi_{k+1} + \phi_{k-1}$ below.

In section 5.1 when $p_2 = 0$ we set $\omega = X + [2]$. It follows that $X\omega = X^2 + [2]X = \phi_2 + \phi_0 + [2]\phi_1 = \omega_2$. So, $\omega_2 = X\omega$. The handle slide property implies that $\omega_2 = [2]\omega$.

In section 5.5 when $p_3 = 0$ we set $\omega = X^2 + [2]X$. If $\omega_3 = 1 + [2]\phi_1 + [3]\phi_2$ then $X\omega_3 = ([3] + 1)X + [2]X^2$ since $X = X\phi_2$ when $p_3 = 0$ and $[3] + 1 = [2]^2 \Rightarrow X\omega_3 = [2]\omega$. Since ω_3 satisfies the handle slide property, $[2]\omega_3 = X\omega_3 = [2]\omega \Rightarrow \omega_3 = \omega$ since $[2]$ is a unit in TL^3 by construction.

5.10. Spin manifolds. One may use definitions 5.3, 5.7 in the context of 3-manifolds with a spin structure. Specifically, one labels the components of the characteristic sublink by Ω_- and the rest of the components by Ω_+ , compare with section 2.14. Theorems 5.4, 5.8 then state that sliding a strand interchanges the cosets associated to Ω_- and Ω_+ , as one expects in the context of spin Kirby calculus.

5.11. Summary. This section introduced a categorification of the elements ω_2, ω_3 , where ω_N is defined in (2.2). It is shown that these objects are invariant under handle slides, understood in the context of the coset construction of section 4. The authors conjecture that there is also a categorification of ω_N for all N , however it is likely that a construction would require a further non-trivial extension of ideas presented here.

One can construct an object in the brusque quotient category (section 4) by labelling each component of a framed link $L \subset S^3$ with Ω . Theorems 5.4 and 5.8 imply that

the isomorphism type of this object is unchanged by handle slides. This reflects information about the 3-manifold obtained by surgery on L (see section 2.10). We expect that there are maps between such objects associated to 4-dimensional cobordisms.

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