GRAPHS, LINKS, AND DUALITY ON SURFACES

VYACHESLAV KRUSHKAL

ABSTRACT. We introduce a polynomial invariant of graphs on surfaces, P_G, generalizing the classical Tutte polynomial. Topological duality on surfaces gives rise to a natural duality result for P_G, analogous to the duality for the Tutte polynomial of planar graphs. This property is important from the perspective of statistical mechanics, where the Tutte polynomial is known as the partition function of the Potts model. For ribbon graphs, P_G specializes to the well-known Bollobás-Riordan polynomial, and in fact the two polynomials carry equivalent information in this context. Duality is also established for a multivariate version of the polynomial P_G. We then consider a 2-variable version of the Jones polynomial for links in thickened surfaces, taking into account homological information on the surface. An analogue of Thistlethwaite's theorem is established for these generalized Jones and Tutte polynomials for virtual links.

1. INTRODUCTION

The Tutte polynomial $T_G(X, Y)$ is a classical invariant in graph theory (see [31, 32, 1]), reflecting many important combinatorial properties of a graph G. For example, the chromatic polynomial, whose values at positive integer values of the parameter Q correspond to the number of colorings of G with Q colors, is a one-variable specialiazation of T_G. The Tutte polynomial is also important in statistical mechanics, where it arises as the partition function of the Potts model, cf [28].

Two properties of the Tutte polynomial are particularly important in these contexts: the contraction-deletion rule, and the duality

$$(1.1) \qquad\qquad T_G(X, Y) \;=\; T_{G^*}(Y, X)$$

where G is a planar graph, and G^* is its dual. (The vertices of G^* correspond to the connected regions in the complement of G in the plane, and two vertices are connected by an edge in G^* whenever the two corresponding regions are adjacent.)

In this paper we introduce a $4-$variable polynomial, $P_{G,\Sigma}(X, Y, A, B)$, which is an invariant of a graph G embedded in a closed orientable surface Σ, which satisfies both the contraction-deletion rule and a duality relation analogous to (1.1). The variables X, Y play the same role as in the definition of the Tutte polynomial, while the additional variables A, B reflect the topological information of G in Σ. It follows

that the Tutte polynomial is a specialization of P_G, where this extra information, reflecting the embedding $G \subset \Sigma$, is disregarded.

The main motivation for this work came from an attempt to understand the combinatorial structure underlying the Potts model on surfaces. As mentioned above, the partition function for the Potts model on the plane is given by the Tutte polynomial, while on surfaces essential loops are weighted differently from trivial loops (for references on the Potts model on surfaces, cf. [6, 8, 15, 26].) This leads to the introduction of additional variables, keeping track of the topological information of graphs on surfaces.

Using topological duality on surfaces, we establish the duality relation

$$(1.2) \qquad\qquad P_G\left(X, Y, A, B\right) \;=\; P_{G^*}\left(Y, X, B, A\right),$$

which may be viewed as a natural analogue of the duality (1.1) of the Tutte polynomial for planar graphs. For the dual graph G^* in (1.2) to be well-defined, it is natural to consider graphs G which are *cellulations* of Σ, that is, graphs such that each component of $\Sigma \smallsetminus G$ is a disk. Equivalently, such graphs may be viewed as orientable *ribbon graphs*, this point of view is presented in more detail in section 4.

For ribbon graphs, there is a well-known 3-variable polynomial defined by B. Bollobás and O. Riordan [2, 3]. We denote this graph polynomial by $BR_G(X, Y, Z)$, its construction is recalled in section 4. We show that this polynomial can be obtained as a specialization of P_G:

$$(1.3) \qquad\qquad BR_G(X, Y, Z) \;=\; Y^g\, P_G(X - 1, Y, YZ^2, Y^{-1}),$$

where g is the genus of the ribbon graph G. In fact the authors prove in [2, 3] that their polynomial is a universal invariant of ribbon graphs with respect to the contraction-deletion rule (we give a precise statement of this result in section 4.) Therefore in principle the two polynomials BR_G, P_G carry equivalent information about the ribbon graph G, although an expression of P_G in terms of BR_G does not seem to be as straightforward as (1.3). We note that the definition of the polynomial P_G could be normalized so the specialization to BR_G is obtained by simply setting one of the variables equal to 1 (see section 4). We chose a normalization making the duality statement (1.2) most natural.

Several authors have established partial results on duality for the Bollobás-Riordan polynomial: B. Bollobás and O. Riordan [2] stated duality for a 1-variable specialization, J. A. Ellis-Monaghan and I. Sarmiento [10] and I. Moffatt [24] (see also [4, 11, 25]) proved duality for a certain 2-variable specialization. These results may be recovered as a consequence of equations (1.2), (1.3), see sections 5.3 and 4.1; our result (1.2) is more general.

A self-contained discussion of the polynomial $P_{G,\Sigma}(X, Y, A, B)$ and its properties may be found in sections 2, 3. In section 4.2 we point out a combinatorial formulation

of this graph polynomial in the context of ribbon graphs, without using homology. The reader interested in a more detailed discussion of topological aspects of graphs on surfaces will find in section 5 the definition of a more general, infinite-variable, polynomial $\widetilde{P}_{G,\Sigma}$. We point out in that section that a general context for the duality of graph polynomials on an oriented closed surface Σ is provided by the intersection pairing and the Poincaré duality, giving rise to a symplectic structure on the first homology group $H_1(\Sigma)$. (The action of the mapping class group of the surface induces a representation of the symplectic group $Sp\,(2g,\mathbb{Z})$ on $H_1(\Sigma,\mathbb{Z})$, where g is the genus of the surface.) Given a subgroup V of $H_1(\Sigma)$, its "orthogonal complement" $V^{\perp}$ with respect to the intersection form may be defined, see (5.2). Using this structure, we define in section 5 a more general version of the polynomial P_G, with coefficients corresponding to subgroups of $H_1(\Sigma)$, its duality property is stated in Lemma 5.1. This more general polynomial may be used to distinguish different embeddings of a graph in Σ. (One may also generalize further and, avoiding the use of homology, consider the *Tutte skein module* of a surface Σ: the vector space spanned by isotopy classes of graphs on Σ, modulo the contraction-deletion relation, see section 5.2. In this case the "polynomial" associated to a graph $G \subset \Sigma$ is the element of the skein module represented by G.) On the other hand, if one considers graphs on Σ up to the action of the diffeomorphism group of Σ (or if one studies ribbon graphs), then the relevant invariant is the finite-variable polynomial P_G, discussed above.

In section 6 a version of the Kauffman bracket and of the Jones polynomial on surfaces is considered, taking into account homological information on the surface. In particular, using the interpretation of a virtual link as an "irreducible" embedding of a link into a surface due to G. Kuperberg [21], this defines a generalization of the Jones polynomial for virtual links. For example, the Jones polynomial $J_L(t, Z)$ acquires a new variable Z which, in the state-sum expression, keeps track of the rank of the subgroup of the first homology group $H_1(\Sigma)$ of the surface represented by a resolution of the link diagram on the surface.

If a link L has an alternating diagram on Σ, the diagram may be checkerboard colored, and there is a graph G (the Tait graph) associated to it. In this context we show (Theorem 6.1) that the generalized Kauffman bracket (and the Jones polynomial $J_L(q, Z)$) is a specialization of the polynomial P_G, generalizing the well-known relation between the Jones polynomial of a link in $3-$space and the Tutte polynomial associated to its planar projection due to Thistlethwaite [29]. The analogue of Thistlethwaite's theorem, relating the Kauffman bracket of virtual links and the Bollobás-Riordan polynomial of ribbon graphs, was established by S. Chmutov and I. Pak in [5]. Theorem 6.1 generalizes these results to the polynomial J_L with the extra homological parameter Z. This relation between the generalized Jones polynomial $J_L(q, Z)$ and the polynomial P_G of the associated graph does not seem to have an immediately obvious analogue in terms of BR_G.

A *multivariate* version of the Tutte polynomial, where the edges of a graph are
weighted, is important in the analysis of the Potts model [28]. We define its gen-
eralization, a multivariate version of the polynomial P_G, and establish a duality
analogous to (1.2) in section 7. (A mutlivariate version of the Bollobás-Riordan
polynomial has been considered by I. Moffatt in [24], and F. Vignes-Tourneret [33]
established a partial duality result for a signed version of the multivariate Bollobás-
Riordan polynomial.)

The Tutte polynomial and the definition of the new polynomial P_G, as well as a
discussion of its basic properties, are given in section 2. Its duality relation (1.2) is
proved in section 3. We review the notion of a ribbon graph and the definition of the
Bollobás-Riordan polynomial, and we establish the relation (1.3) in section 4. Section
4.1 shows that our duality result (1.2) implies the previously known partial results
on duality for the Bollobás-Riordan polynomial. Section 5 recalls basic notions of
symplectic linear algebra, allowing one to generalize P to a polynomial $\widetilde{P}_{G,\Sigma}$ with
coefficients taking values in subgroups of the first homology group of the surface.
Section 6 defines the relevant versions of the Jones polynomial and of the Kauffman
bracket and establishes a relationship between them and the polynomial P_G. Finally,
section 7 discusses a multivariate version of the polynomial P_G and the corresponding
duality relation.

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2. The Tutte polynomial and graphs on surfaces

Consider the following normalization of the Tutte polynomial of a graph G:

$$(2.1) \qquad\qquad T_G(X,Y) = \sum_{H \subset G} X^{c(H)-c(G)} Y^{n(H)}.$$

The summation is taken over all spanning subgraphs H of G, that is the subgraphs
H such that the vertex set of H coincides with the vertex set of G. Therefore the
sum contains 2^e terms, where e is the number of edges of G. In (2.1) $c(H)$ denotes
the number of connected components of the graph H, and $n(H)$ is the *nullity* of H,
defined as the rank of the first homology group $H_1(H)$ of H. (Note that the nullity

$n(H)$ may also be computed as $c(H) + e(H) - v(H)$, where e and v denote the number of edges and vertices of H, respectively.)

Now suppose G is a graph embedded in a surface Σ. We need to introduce some preliminary topological notions which will be used in the definition (2.3) of the graph polynomial below. We note that in the context of ribbon graphs, there is a formulation of this graph polynomial in purely combinatorial terms, see section 4.2.

Definition 2.1. For a spanning subgraph H of G, let $s(H)$ be twice the genus of the surface obtained as a regular neighborhood $\mathcal{H}$ of the graph H in Σ. ($\mathcal{H}$ is a surface with boundary, and its genus is defined as the genus of the closed surface obtained from $\mathcal{H}$ by attaching a disk to each boundary circle of $\mathcal{H}$.) Similarly, let $s^{\perp}(H)$ denote twice the genus of the surface obtained by removing a regular neighborhood $\mathcal{H}$ of H from Σ. Denote by i the embedding $G \longrightarrow \Sigma$, and consider the induced map on the first homology groups with real coefficients (we mention [14, 23] as general references on algebraic topology, in particular for the background on the homology groups). Define

$$(2.2) \qquad k(H) := \dim\left(\ker\left(i_* \colon H_1(H; \mathbb{R}) \longrightarrow H_1(\Sigma; \mathbb{R})\right)\right).$$

For example, for the graph H on the surface of genus 3, consisting of a single vertex and 3 edges, shown on the left in figure 2, $s(H) = s^{\perp}(H) = 2$, $k(H) = 0$.

Note that $k(H)$, which enters the definition (2.3) below as the exponent of Y, may be replaced by the nullity $n(H)$ (which is the exponent of Y in the Tutte polynomial (2.1)), the result would be a different normalization of the polynomial P. See formula (4.7) in section 4 below relating $n(H)$ and the parameters used in the definition of P. The choice of the exponent of Y in (2.3) was motivated by the duality relation (3.1) which is most naturally stated with this normalization. We introduce the polynomial $P_{G,\Sigma}$ which is the main object of study in this paper:

$$(2.3) \qquad P_{G,\Sigma}(X, Y, A, B) \;=\; \sum_{H \subset G} X^{c(H) - c(G)} \, Y^{k(H)} \, A^{s(H)/2} \, B^{s^{\perp}(H)/2}$$

The reader interested in a more general topological context for analyzing polynomial invariants of graphs on surfaces and their duality properties should compare $P_{G,\Sigma}$ with the more general version defined in section 5. (The invariants $s(H), s^{\perp}(H)$ fit naturally in that context, and this explains, in part, their normalization as *twice* the genus of the corresponding surface.) Some elementary properties of the polynomial $P_{G,\Sigma}$ are summarized in the following statement. A surface Σ usually will be fixed, and the subscript Σ will be omitted from the notation.

Lemma 2.2.
(1) If e is an edge of G which is neither a loop nor a bridge, then $P_G = P_{G \smallsetminus e} + P_{G/e}$.
(2) If e is a bridge in G, then $P_G = (1 + X) \, P_{G/e}$.
(3) If e is a loop in G which is trivial in $H_1(\Sigma)$, then $P_G = (1 + Y) \, P_{G \smallsetminus e}$.

Proof. The proof of lemma 2.2 is similar to the proof of the corresponding statements for the Tutte polynomial. To prove (1), consider an edge e which is neither a loop nor a bridge. Since e is not a loop, the sum (2.3) splits into two parts $P_G = S_1 + S_2$. S_1 consists of the terms with H containing the edge e, and S_2 consists of the terms with H not containing e. In the first case, the embedding $H \subset \Sigma$ is homotopic to the embedding $H/e \subset \Sigma$, and all of the invariants $c, k, s, s^\perp$ of H coincide with those of H/e. Therefore, $S_1 = P_{G/e}$. The terms in S_2 are in $1 - 1$ correspondence with the terms in $P_{G \smallsetminus e}$. Moreover, since e is not a bridge, $c(G) = c(G \smallsetminus e)$. It follows that $S_2 = P_{G \smallsetminus e}$.

To prove (2), suppose e is a bridge in G. Again the sum (2.3) splits:

$$P_G = \sum_{H \subset (G \smallsetminus e)} X^{c(H)-c(G)} Y^{k(H)} A^{s(H)/2} B^{s^\perp(H)/2} + \sum_{H \subset (G/e)} X^{c(H)-c(G)} Y^{k(H)} A^{s(H)/2} B^{s^\perp(H)/2}$$

More precisely, the subgraphs H parametrizing the second sum are all subgraphs of H containing e. Contracting e leaves each term in the second sum unchanged, and moreover the second sum is precisely the expansion of $P_{G/e}$.

There is a $1 - 1$ correspondence between the subgraphs H (not containing e) of G parametrizing the first sum and the subgraphs parametrizing the second sum. Given $H \subset G$, $e \notin H$, this correspondence associates to it the subgraph $\widetilde{H} \subset G/e$ obtained by identifying the two endpoints of e in H. Since e is a bridge, the homological invariants $k, s, s^\perp$ of H are identical to those of $\widetilde{H}$. However $c(H) - c(G) = c(\widetilde{H}) - c(G/e) + 1$. Therefore each term in the first sum equals the corresponding term in the expansion of $P_{G/e}$ times X. This concludes the proof of (2).

The proof of (3) is analogous, noting that removing a loop e which is homologically trivial on the surface from a subgraph H decreases $k(H)$ by 1 and leaves other exponents in the expansion (2.3) unchanged. The proof that the exponent of B does not change relies on the fact that deleting a homologically trivial loop a disconnects the surface. This fact conceptually is a consequence of the Poincaré-Lefschetz duality [14]. It may also be observed using a more elementary argument as follows. Supposing the opposite is true, one immediately finds a loop b in Σ which intersects a in a single point. This is a contradiction with the basic and fundamental fact in homology theory that a boundary has trivial intersection number with any cycle. (See [23] for a general discussion of curves on surfaces as well as for applications of homology in this context.)   □

Remark. Note that if G_1, G_2 are disjoint graphs in Σ, it is *not* true in general that $P_{G_1 \cup G_2} = P_{G_1} P_{G_2}$, see for example figure 1. (A similar comment applies to the case when G_1, G_2 in Σ are disjoint except for a single vertex v.) This is quite different form the case of the classical Tutte polynomial. However, the polynomial P_G is multiplicative with respect to disjoint unions in the context of ribbon graphs,

see lemma 4.3. The proof of that lemma also shows that if G_1, G_2 are graphs in disjoint surfaces, $G_1 \subset \Sigma_1$, $G_2 \subset \Sigma_2$, then $P_{G_1 \sqcup G_2, \Sigma_1 \sqcup \Sigma_2} = P_{G_1, \Sigma_1} P_{G_2, \Sigma_2}$.
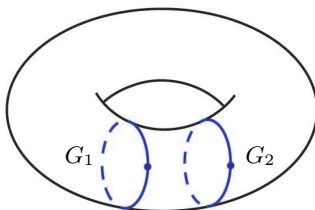

FIGURE 1. In general the polynomial P_G is not multiplicative with respect to disjoint unions: in this example,

$$P_{G_1} = P_{G_2} = 1 + B, \ P_{G_1 \cup G_2} = 2 + B + Y.$$

Note that P_G is multiplicative for *ribbon* graphs, see section 4.

Lemma 2.3. *The Tutte polynomial (2.1) is a specialization of P_G:*

$$T_G(X, Y) = Y^g P_{G, \Sigma}(X, Y, Y, Y^{-1}),$$

where g is the genus of the surface Σ.

Proof. Substituting $A = Y$, $B = Y^{-1}$ into the expansion (2.3), one gets terms of the form

(2.4) $$X^{c(H) - c(G)} Y^{k(H) + s(H)/2 - s^\perp(H)/2}.$$

We claim that

(2.5) $$n(H) = k(H) + g + s(H)/2 - s^\perp(H)/2.$$

This formula shows that each term of the form (2.4) above, multiplied by Y^g, gives the corresponding term in the expansion (2.1) of the Tutte polynomial.

The conceptual framework for the formula (2.5) is provided by the structure of the first homology group $H_1(\Sigma)$ of the surface given by the intersection numbers of curves in Σ. This point of view is discussed in more detail in section 5 of the paper, specifically see the identities (5.5). At outline of the argument may be seen as follows. Following definition 2.1, given a spanning subgraph H of G let $\mathcal{H}$ be its regular neighborhood in Σ. Denote by $\mathcal{H}^*$ its complement: $\mathcal{H}^* = \Sigma \smallsetminus \mathcal{H}$, so Σ is represented as the union of two compact surfaces $\mathcal{H}, \mathcal{H}^*$ along their boundary. The dimension $2g$ of $H_1(\Sigma)$ equals $s(H) + s^\perp(H) + 2l$ for some integer $l \geq 0$. Observe that the dimension of the image of $H_1(\mathcal{H})$ in $H_1(\Sigma)$ is precisely $s(H) + l$. Indeed, if the dimension of the image were greater than $s(H) + l$ then in fact the genus of $\mathcal{H}$ must be greater than $s(H)/2$, a contradiction. Similarly, if the dimension of the image were less than $s(H) + l$ then the genus of $\mathcal{H}^*$ must be greater than $s^\perp(H)/2$. This shows that $2g = s(H) + s^\perp(H) + 2l$, and together with $n(H) = k(H) + s(H) + l$ this establishes

(2.5), concluding the proof of lemma 2.3. See section 5 for a more detailed discussion of the underlying homological structure. We note that a less direct combinatorial proof of the formula (2.5) may be given using the combinatorial interpretation of the invariants $s(H), s^\perp(H)$. (In particular, $s(H) = c(H) - bc(H) + n(H)$, see section 4.2.) $\square$

Remark. The polynomial P_G can be normalized to make the relation with the Tutte polynomial easier to state. For example, if one chose the exponent of Y in (2.3) to be $n(H) = \text{rank}\, H_1(H)$ rather than $k(H)$, $T_G(X, Y)$ would be the specialization of the resulting polynomial obtained simply by setting $A = B = 1$. We chose the convention (2.3) to have a natural expression of duality (1.2), proved in theorem 3.1 below.

3. DUALITY.

In this section we prove a duality result for the polynomial P_G defined in (2.3), which is analogous to the duality $T_G(X, Y) = T_{G^*}(Y, X)$ satisfied by the Tutte polynomial of planar graphs. The following result applies to *cellulations* of surfaces: graphs $G \subset \Sigma$ such that each connected component of $\Sigma \smallsetminus G$ is a disk. This is a natural condition guaranteeing that the dual G^* is well-defined. Equivalently, one may view G as a *ribbon graph*, see section 4.

Theorem 3.1. *Suppose G is a cellulation of a closed orientable surface Σ (equivalently, let G be an oriented ribbon graph.) Then the polynomial invariants of G and its dual G^* are related by*

$$(3.1) \qquad\qquad P_G(X, Y, A, B) \;=\; P_{G^*}(Y, X, B, A)$$

Proof. Consider the expansions (2.3) of both sides in the statement of the theorem:

$$(3.2) \qquad P_G(X, Y, A, B) \;=\; \sum_{H \subset G} X^{c(H)-c(G)}\, Y^{k(H)}\, A^{s(H)/2}\, B^{s^\perp(H)/2}$$

$$(3.3) \qquad P_{G^*}(Y, X, B, A) \;=\; \sum_{H^* \subset G^*} Y^{c(H^*)-c(G^*)}\, X^{k(H^*)}\, B^{s(H^*)/2}\, A^{s^\perp(H^*)/2}.$$

Recall that the vertices of G^* correspond to the components of the complement $\Sigma \smallsetminus G$ (which are disks since G is a cellulation), and two vertices are connected by an edge in G^* if and only if the two corresponding components share an edge. Therefore the edges of G and G^* are in $1-1$ correspondence, with each edge e of G intersecting the corresponding edge e^* of G^* in a single point, and e is disjoint from all other edges of G^*. For each spanning subgraph $H \subset G$, consider the spanning subgraph $H^* \subset G^*$ whose edges are precisely all those edges of G^* which are disjoint from all edges of H. The theorem follows from the claim that the term corresponding to H
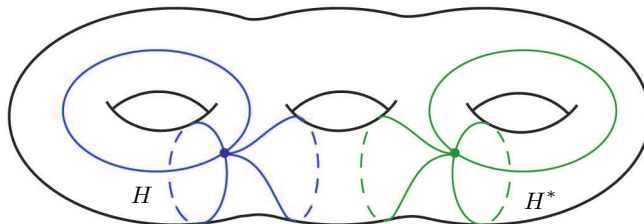
FIGURE 2. A subgraph H of a cellulation G of the genus 3 surface Σ, and the corresponding subgraph H^* of the dual cellulation G^*.

in the expansion of P_G equals the term corresponding to H^* in the expansion of P_{G^*} above.

The simplest example of a cellulation G of a surface Σ of genus g is a graph consisting of a single vertex and $2g$ edges which are loops representing a symplectic basis of $H_1(\Sigma)$. Then its dual is the graph G^* also with a single vertex and $2g$ loops. Figure 2 shows the surface of genus 3 and a subgraph $H \subset G$ formed by 3 edges on the left in the figure. In this case H^* also consists of 3 edges as illustrated on the right in the figure.

The two cellulations G, G^* give rise to dual handle decompositions of the surface Σ. In the handle decomposition corresponding to G, the $0-$handles are disk neighborhoods of the vertices of G, the $1-$handles are regular neighborhoods of the edges of G, the $2-$handles correspond to the $2-$cells $\Sigma \smallsetminus G$. Let $\mathcal{H}$ (respectively $\mathcal{H}^*$) denote the union of the $0-$ and $1-$handles corresponding to H (respectively H^*). Note that $\mathcal{H}$ is a regular neighborhood of the graph H, and similarly $\mathcal{H}^*$ is a regular neighborhood of the graph H^*. If H is the entire graph G, H^* consists of all vertices of G^* and no edges. Removing one edge from H at a time, observe that the effect on $\mathcal{H}$ is the removal of a $1-$handle, while the effect on the dual handle decomposition $\mathcal{H}^*$ is the addition of the co-core of the removed $1-$handle. To summarize, Σ is the union of two surfaces $\mathcal{H}, \mathcal{H}^*$ along their boundary, where $\mathcal{H}$ is a regular neighborhood of H, and $\mathcal{H}^*$ is a regular neighborhood of H^*.

It follows from definition of $s, s^\perp$ that $s(H) = s^\perp(H^*)$ and $s^\perp(H) = s(H^*)$. One also checks that $c(H^*) - c(G^*) = k(H)$ and $c(H) - c(G) = k(H^*)$. A geometric argument may be given for this fact, where one considers the induction on the number of edges in H and observes that adding an edge to H (and therefore removing an edge from H^*) either decreases both $c(H) - c(G)$ and $k(H^*)$ by one, or leaves both quantities unchanged. We give a more direct, algebraic-topological proof: by Poincaré-Lefschetz duality (cf [14]), since $\Sigma = \mathcal{H} \cup \mathcal{H}^*$, one has an isomorphism of the relative second homology group $H_2(\Sigma, \mathcal{H})$ and the 0th cohomology $H^0(\mathcal{H}^*)$. The dimension of $H^0(\mathcal{H}^*) \cong H_0(\mathcal{H}^*)$ equals $c(H^*)$, the number of connected components of H^*. The

group $H_2(\Sigma, \mathcal{H})$ fits in the long exact sequence (cf [14]) of the pair $(\Sigma, \mathcal{H})$:

$$0 \longrightarrow H_2(\Sigma) \longrightarrow H_2(\Sigma, \mathcal{H}) \longrightarrow H_1(\mathcal{H}) \longrightarrow H_1(\Sigma) \longrightarrow \ldots,$$

therefore

$$dim(H_2(\Sigma, \mathcal{H})) = dim(ker[H_1(\mathcal{H}) \longrightarrow H_1(\Sigma)]) + dim(H_2(\Sigma)).$$

The first term in this sum is the definition of $k(H)$, and each connected component of Σ contributes 1 to the second term. Since G^* is a cellulation, the number of connected components of G^* equals the number of connected components of Σ. Combining these equalities, one has $c(H^*) = k(H) + c(G^*)$. This proves $c(H^*) - c(G^*) = k(H)$, and analogously one has $c(H) - c(G) = k(H^*)$.

This shows that the terms corresponding to H, H^* in (3.2, 3.3) are equal, concluding the proof of theorem 3.1.   □

Note that duality results for certain specializations of the Bollobás-Riordan polynomial have been previously obtained by several authors. We discuss these results and show that they may be derived as a consequence of our theorem 3.1 in section 4.1 below.

4. Ribbon graphs and the Bollobás-Riordan polynomial

A *ribbon graph* is a pair (G, S) where G is a graph embedded in a surface (with boundary) S such that the embedding $G \hookrightarrow S$ is a homotopy equivalence. It is convenient to consider the surface S with a handle decomposition corresponding to the graph G: the 0-handles are disk neighborhoods of the vertices of G, and the 1-handles correspond to regular neighborhoods of the edges. (Other terms: cyclic graphs, fat graphs are also sometimes used in the literature to describe ribbon graphs.) G is an *orientable* ribbon graph if S is an orientable surface. Given a ribbon graph (G, S), one obtains a closed surface Σ by attaching a disk to S along each boundary component. Therefore a ribbon graph may be viewed as a *cellulation* of a closed surface Σ, i.e. a graph G embedded in Σ such that each component of the complement $\Sigma \smallsetminus G$ is a disk. Conversely, given a cellulation G of Σ, one has a ribbon graph structure (G, S) where S is a regular neighborhood of G in Σ. We will use the notions of a ribbon graph and of a cellulation interchangeably.

Consider the Bollobás-Riordan polynomial of ribbon graphs [2, 3] (in this paper we only consider *orientable* ribbon graphs, therefore there are three, rather than four, variables): given a ribbon graph (G, S),

$$(4.1) \qquad BR_{G,S}(X, Y, Z) = \sum_{H \subset G} (X - 1)^{r(G) - r(H)} Y^{n(H)} Z^{c(H) - bc(H) + n(H)}.$$

The summation is taken over all spanning subgraphs H of G, and moreover each H inherits the ribbon structure from that of G: the relevant surface is obtained as the union of all 0-handles and just those 1-handles which correspond to the edges of H. To explain the notation in this definition, let $v(H), e(H)$ denote the number of vertices, respectively edges, of H, and let $c(H)$ be the number of connected components. ($v(H) = v(G)$ since H is a spanning subgraph of G.) Then $r(H) = v(G) - c(H)$, $n(H) = e(H) - r(H)$, and $bc(H)$ is the number of boundary components of the surface S. Note that $n(H)$ equals the rank of the first homology group $H_1(H)$, and the exponent of Z, $c(H) - bc(H) + n(H)$, equals $2g(H) = s(H)$, twice the genus of the surface underlying the ribbon graph H. To simplify the notation, we will often omit the reference to the surface S and denote the polynomial by BR_G.

Lemma 4.1. *The Bollobás-Riordan polynomial of a ribbon graph may be obtained as a specialization of the polynomial P_G:*

(4.2) $$BR_{G,S}(X, Y, Z) = Y^g P_{G,\Sigma}(X - 1, Y, YZ^2, Y^{-1}),$$

where Σ is the closed surface obtained by attaching a disk to S along each boundary component, and g is the genus of Σ.

The proof consists of showing that the corresponding terms in the expansions (2.3), (4.1) are equal. Indeed, substituting $A = YZ^2$, $B = Y^{-1}$ in (2.3) gives summands of the form

$$(X - 1)^{c(H) - c(G)} Y^{k(H) + s(H)/2 - s^{\perp}(H)/2} Z^{s(H)}.$$

Using the formulas (5.5), established in the following section, observe that

$$n(H) = k(H) + g + s(H)/2 - s^{\perp}(H)/2,$$

therefore these summands are equal to Y^{-g} times the corresponding terms in (4.1). $\square$

To discuss the relation between the polynomial P_G and the Bollobás-Riordan polynomial further, recall that the polynomial BR_G satisfies the following universality property. Let $\mathcal{G}$ denote the set of isomorphism classes [3] of connected ribbon graphs. Define the maps C_{ij} from $\mathcal{G}$ to $\mathbb{Z}[X]$ by $BR = \sum_{i,j} C_{ij} Y^i Z^j$. Further, given a commutative ring R and an element $x \in R$, $C_{ij}(x)$ will denote the map from $\mathcal{G}$ to R obtained by composing C_{ij} with the ring homomorphism $\mathbb{Z}[X] \longrightarrow R$ mapping X to x.

Theorem 4.2. [2, 3] *Let R be a commutative ring and $x \in R$ and $\phi \colon \mathcal{G} \longrightarrow R$ a map satisfying*
(1) $\phi(G) = \phi(G/e) + \phi(G \smallsetminus e)$ if e is neither a loop nor a bridge, and
(2) $\phi(G) = x \, \phi(G/e)$ if e is a bridge.
Then there are elements $\lambda_{ij} \in R$, $0 \le j \le i$, such that

$$\phi = \sum_{i,j} \lambda_{ij} C_{ij}(x).$$

The polynomial P_G satisfies the properties (1), (2) in this theorem, therefore it follows that the coefficients of P_G may be expressed as linear combinations of the coefficients of BR_G. The main difference in the definitions of the two polynomials is that each term in the expansion (4.1) of BR_G is defined in terms of the invariants of a ribbon subgraph H, while the terms in the expansion (2.3) involve the invariants associated to the embedding of H into the original fixed surface Σ. Indeed, note that the four parameters $c(h), k(H), s(H), s^\perp(H)$ in the definition (2.3) of P_G are independent invariants of H, in the sense that there are examples of graphs showing that no three of the parameters determine the other one. Therefore it does not seem likely that there is a straightforward expression for P_G in terms of BR_G similar to that in lemma 4.1, however it would be interesting to find an explicit expression.

Returning to the properties of the polynomial P_G, observe that the multiplicativity for disjoint unions and for one-point unions holds in the context of ribbon graphs (compare with the remark after lemma 2.2):

Lemma 4.3. *Properties (1)–(3) in lemma 2.2 hold for ribbon graphs G. In addition, for disjoint ribbon graphs G_1, G_2,*

(4) $P_{G_1 \sqcup G_2} = P_{G_1 \vee G_2} = P_G \cdot P_{G'}$.

Here by the polynomial P_G of a ribbon graph (G, S) we mean $P_{G,\Sigma}$ where as above Σ is the closed surface associated to S. For example, the closed surface associated to the graphs G_1, G_2 with the ribbon structure inherited from their embedding into the torus in figure 1 is the 2-sphere (and the surface associated to $G_1 \cup G_2$ is the disjoint union of two spheres), and not the torus. This illustrates the difference between the validity of the property (4) for ribbon graphs, but not in general for graphs on surfaces as in figure 1.

Proof. The proof of (1)–(3) is identical to that in lemma 2.2. To prove (4) for the disjoint union $G_1 \sqcup G_2$, consider subgraphs $H_1 \subset G_1$, $H_2 \subset G_2$ and let V_i denote the image of $H_1(H_i)$ in $H_1(\Sigma)$, $i = 1, 2$. Since the surface associated to $G_1 \sqcup G_2$ is the disjoint union of surfaces associated to G_1 and G_2, one has $k(H_1 \cup H_2) = k(H_1) + k(H_2), s(H_1 \cup H_2) = s(H_1) + s(H_2)$, and $s^\perp(H_1 \cup H_2) = s^\perp(H_1) + s^\perp(H_2)$. The proof for the one-vertex union $G_1 \vee G_2$ is directly analogous. $\qquad\square$

4.1. **Prior results on duality of the Bollobás-Riordan polynomial.** Several authors have established duality for certain specializations of the Bollobás-Riordan polynomial. For example, [3] notes that

$$(4.3) \qquad\qquad BR_G(1 + t, t, t^{-1}) = BR_{G^*}(1 + t, t, t^{-1}).$$

By lemma 4.1, $BR_G(1 + t, t, t^{-1}) = Y^g P_G(t, t, t^{-1}, t^{-1})$, therefore (4.3) is a consequence of theorem 3.1. More generally, it is shown in [10, 24] (see also [4, 11, 25]) that there is duality for a $2-$variable specialization:

$$(4.4) \qquad BR_G(1+X, Y, (XY)^{-1/2}) \; = \; (X^{-1}Y)^g \, BR_{G^*}(1+Y, X, (XY)^{-1/2})$$

Observe that according to lemma 4.1,

$$
\begin{aligned}
BR_G(1+X, Y, (XY)^{-1/2}) &\; = \; Y^g \, P_G(X, Y, X^{-1}, Y^{-1}), \\
BR_{G^*}(1+Y, X, (XY)^{-1/2}) &\; = \; X^g \, P_{G^*}(Y, X, Y^{-1}, X^{-1}).
\end{aligned}
$$

Therefore (4.4) may also be viewed as a consequence of theorem 3.1. It would be interesting to understand the full duality relation (3.1) in terms of the Bollobás-Riordan polynomial, since as discussed above, the polynomials P_G, BR_G carry equivalent information about a ribbon graph G.

4.2. A reformulation of the polynomial P_G for ribbon graphs.

We conclude this section by noting that one may give a combinatorial formulation of the polynomial P_G (defined by (2.3)) in the context of ribbon graphs.

Given a ribbon graph (G, S), as above consider G as a cellulation of a closed surface Σ, so each component of $\Sigma \smallsetminus G$ is a disk. The dual cellulation $G^* \subset \Sigma$ is then well-defined: the vertices of G^* correspond to the components of the complement $\Sigma \smallsetminus G$, and two vertices are connected by an edge in G^* if and only if the corresponding components of $\Sigma \smallsetminus G$ share an edge. Therefore the edges of G and G^* are in $1-1$ correspondence, with each edge e of G intersecting the corresponding edge e^* of G^* in a single point, and e is disjoint from all other edges of G^*. For each spanning subgraph $H \subset G$, consider the spanning subgraph $H^* \subset G^*$ whose edges are precisely all those edges of G^* which are disjoint from all edges of H.

Given a ribbon graph (G, S), consider

$$(4.5) \quad P'_{G,S}(X, Y, A, B) \; = \; \sum_{H \subset G} X^{c(H)-c(G)} \, Y^{n(H)} \, A^{c(H)-bc(H)+n(H)} B^{c(H^*)-bc(H^*)+n(H^*)}.$$

The summation is taken over all spanning ribbon subgraphs H of G. Note that the exponent of A, $c(H) - bc(H) + n(H)$, equals $2g(H)$, twice the genus of the surface underlying the ribbon graph H. Similarly, the exponent of B equals twice the genus of the dual ribbon graph H^*. Since these quantities correspond to the invariants $s(H), s^{\perp}(H)$ (see definition 2.1), the polynomial P' may be rewritten as

$$(4.6) \qquad P'_{G,S}(X, Y, A, B) \; = \; \sum_{H \subset G} X^{c(H)-c(G)} \, Y^{n(H)} \, A^{s(H)} B^{s^{\perp}(H)}.$$

The invariants $n(H), k(H), s(H), s^{\perp}(H)$ may be related using the formulas (5.5), established in the following section:

$$(4.7) \qquad\qquad n(H) = k(H) + g + s(H)/2 - s^{\perp}(H)/2,$$

where g is the genus of the surface underlying the ribbon graph G. Then it is straight-forward to see that the polynomial P_G, defined by (2.3), and P'_G, combinatorially defined above, are equivalent (may be obtained from each other by substitutions of variables). For example, P_G may be expressed in terms of P'_G as follows:

$$P_G(X, Y, A, B) = Y^{-g} P'_G(X, Y, AY^{-1/2}, BY^{1/2}).$$

5. Symplectic linear algebra and a more general graph polynomial

In this section we show that the polynomial $P_{G,\Sigma}$, defined in section 2, fits in a more general topological framework. (The material in this section is not directly used in sections 6, 7, and therefore the reader who is interested in applications to knot theory, or in the multivariate version of the graph polynomial P_G, may choose to proceed directly to the subsequent sections of the paper.) First we recall a number of basic facts and introduce certain notation in the symplectic linear algebra setting which will be useful for the definition of the more general graph polynomial $\widetilde{P}_{G,\Sigma}$. Let Σ be a (not necessarily connected) closed oriented surface, and consider the intersection pairing

$$w \colon H_1(\Sigma, \mathbb{R}) \times H_1(\Sigma, \mathbb{R}) \longrightarrow \mathbb{R}.$$

The intersection pairing may be viewed geometrically, as the intersection number (where the intersection points are counted with signs) of oriented cycles representing homology classes in $H_1(\Sigma)$, or dually as the cup product on first cohomology $H_1(\Sigma)$, see [14]. The invariants considered below do not depend on the orientation. Poincaré duality [14] implies that the bilinear form w is non-degenerate, in other words it is a symplectic form on the vector space $H_1(\Sigma, \mathbb{R})$. A note on the homology coefficients: the invariants below may be defined using either $\mathbb{Z}$ or $\mathbb{R}$, and these coefficients will be used interchangeably.

Let H be a graph embedded in the surface Σ, and let $i \colon H \hookrightarrow \Sigma$ denote the embedding. Denote

$$(5.1) \qquad\qquad V = V(H) = \mathrm{image}\,(i_* \colon H_1(H; \mathbb{R}) \longrightarrow H_1(\Sigma; \mathbb{R})).$$

In other words, V is the subgroup of the first homology group of the surface, generated by the cycles in the graph G. The "symplectic orthogonal complement" of V may be defined by

$$(5.2) \qquad V^\perp = V^\perp(H) \;=\; \{u \in H_1(\Sigma, \mathbb{R}) \,|\, \forall v \in V(H),\, w(u, v) = 0\}.$$

The invariants $s(H)$, $s^\perp(H)$ of a graph H on Σ, introduced in definition 2.1, may be defined in this framework as follows:

$$(5.3) \qquad s(H) = \dim(V/(V \cap V^\perp)), \quad s^\perp(H) = \dim(V^\perp/(V \cap V^\perp)).$$

Said differently, $s(H)$ is the dimension of a maximal symplectic subspace of V (with respect to the symplectic form w on $H_1(\Sigma, \mathbb{R})$), and similarly $s^\perp(H)$ is the dimension of a maximal symplectic subspace in $V^\perp$. (The fact that (5.3) gives the same invariants as definition 2.1 may be observed by considering a regular neighborhood $\mathcal{H}$ of H in Σ and noting that the homology classes corresponding to the boundary curves of $\mathcal{H}$ in $H_1(\Sigma)$ are in the intersection $V \cap V^\perp$.) Also it will be useful to consider

$$(5.4) \quad l(H) := \dim(V \cap V^\perp), \quad k(H) := \dim(\ker(i_* \colon H_1(H; \mathbb{R}) \longrightarrow H_1(\Sigma; \mathbb{R}))).$$

Note the identities relating these invariants for any graph $H \subset \Sigma$:

$$(5.5) \quad s(H) + s^\perp(H) + 2l(H) = 2g, \quad k(H) + l(H) + s(H) = \dim(H_1(H)),$$

where g denotes the genus of Σ.

5.1. A more general graph polynomial.

Now suppose G is a graph embedded in a surface Σ, let $i \colon G \longrightarrow \Sigma$ denote the embedding. Consider a collection of formal variables corresponding to the subgroups of $H_1(\Sigma)$. Given a subgroup $V < H_1(\Sigma)$, let $[V]$ denote the corresponding variable associated to it. Define

$$(5.6) \qquad \widetilde{P}_{G,\Sigma}(X, Y) = \sum_{H \subset G} [i_*(H_1(H))] \, X^{c(H)-c(G)} \, Y^{k(H)}.$$

Here $[i_*(H_1(H))]$ is the formal variable associated to the subgroup equal to the image of $H_1(H)$ in $H_1(\Sigma)$ under the homomorphism i_* induced by inclusion; $k(H)$ is defined in (2.2). Therefore $\widetilde{P}_{G,\Sigma}$ may be viewed as a polynomial in X, Y with coefficients corresponding to the subgroups of $H_1(\Sigma)$. This polynomial may be used to distinguish different embeddings of a graph G into Σ.

However if two graphs G, G' in Σ are considered equivalent whenever there is a diffeomorphism taking G to G', one needs to consider a polynomial invariant in terms of quantities which are invariant under the action of the mapping class group. This is the context in which the polynomial $P_{G,\Sigma}$ (defined in section 2) is useful, indeed it may be viewed as a specialization of $\widetilde{P}_{G,\Sigma}$ where $[i_*(H_1(H))]$ is specialized to $A^{s(H)/2}B^{s^\perp(H)/2}$. In section 5.3 below we point out that the polynomial $\widetilde{P}_{G,\Sigma}$ satisfies a natural duality relation, generalizing theorem 3.1.

5.2. The Tutte skein module.

One may generalize the polynomial P_G further and, avoiding the use of homology, consider the *Tutte skein module* of a surface Σ: the vector space spanned by isotopy classes of graphs on Σ, modulo relations (1)-(3) in lemma 2.2. For example, the contraction-deletion relation states that $G = G \smallsetminus e + G/e$, where the three graphs $G, G \smallsetminus e, G/e$ are viewed as vectors in the skein module. In this case the "polynomial" associated to a graph $G \subset \Sigma$ is the element

of the skein module represented by G. There is an expansion, analogous to (5.6), where each term in the expansion is an element of the skein module, and to get the polynomial $\widetilde{P}_{G,\Sigma}$ one applies homology to that expansion.

Note that a relative version of this skein module, specialized to $Y = 0$, in the rectangle – the *chromatic algebra* – was considered in [12, 13]. See also remark 6 following the statement of theorem 6.1 below concerning the relation between the Tutte skein module of Σ and the Kauffman skein module of $\Sigma \times I$.

5.3. **Duality.** In the remaining part of this section we show that the polynomial $\widetilde{P}_{G,\Sigma}$ satisfies a natural duality relation, generalizing theorem 3.1:

Lemma 5.1. *Suppose G is a cellulation of a closed orientable surface Σ (equivalently, let G be an oriented ribbon graph.) Then $\widetilde{P}_{G^*}(Y, X)$ is obtained from $\widetilde{P}_G(X, Y)$ by replacing each coefficient $[V]$ (formally corresponding to a subgroup of $H_1(\Sigma)$) with its symplectic orthogonal complement $[V^\perp]$.*

The proof of this lemma follows along the lines of the proof of theorem 3.1, one shows that each term $[i_*(H_1(H))] \, X^{c(H)-c(G)} \, Y^{k(H)}$ in the expansion of $\widetilde{P}_G(X,Y)$ equals the corresponding term $[i_*(H_1(H^*))] \, X^{k(H^*)} \, Y^{c(H^*)-c(G^*))}$ in the expansion of $\widetilde{P}_{G^*}(Y, X)$, and moreover that $i_*(H_1(H^*)) \cong (i_*(H_1(H)))^\perp$. Here for each spanning subgraph $H \subset G$, H^* is the "dual" subgraph of G^*.

The proof of theorem 3.1 established that $c(H^*) - c(G^*) = k(H)$ and $c(H) - c(G) = k(H^*)$. The remaining step is to show that, in the notation of (5.1), (5.2),

(5.7) $$V(H^*) \cong V(H)^\perp.$$

Consider the regular neighborhoods $\mathcal{H}, \mathcal{H}^*$ of H, H^* in Σ. Then Σ is represented as the union of two surfaces $\mathcal{H}, \mathcal{H}^*$ along their boundary. Since the intersection of any 1-cycle in $\mathcal{H}$ with any 1-cycle in $\mathcal{H}^*$ is zero, it is clear that $V(H^*) \subset V(H)^\perp$. To prove the opposite inclusion, consider the Mayer-Vietoris sequence [14]:

$$\ldots \longrightarrow H_1(\mathcal{H}) \oplus H_1(\mathcal{H}^*) \overset{\alpha}{\longrightarrow} H_1(\Sigma) \overset{\beta}{\longrightarrow} H_0(\partial) \longrightarrow \ldots$$

where ∂ denotes $\partial\mathcal{H} = \partial\mathcal{H}^* = \mathcal{H} \cap \mathcal{H}^*$. It follows from the geometric decomposition $\Sigma = \mathcal{H} \cup \mathcal{H}^*$ that if a non-trivial element $h \in H_1(\Sigma)$ is not in the image of α then it intersects non-trivially with $H_1(\partial\mathcal{H})$, so in this case $h \notin V(H)^\perp$. this implies that $V(H)^\perp \subset \text{image}(\alpha)$, so $V(H)^\perp \subset (V(H) \cap V(H)^\perp) \cup V(H^*) = V(H^*)$. Therefore $V(H)^\perp = V(H^*)$, and this completes the proof of lemma 5.1.

$\square$

6. The generalized Kauffman bracket and Jones polynomial of links on surfaces

Various relations between the Tutte polynomial and link polynomials are well known, for example see [29, 16]. More recently [5] such relations have been established for link polynomials and the Bollobás-Riordan polynomial of associated graphs on surfaces. (See also [7, 24] for other related results.) In this section we consider a 2-variable generalization of the Jones polynomial of links in (surfaces) $\times I$, and more generally a $4-$variable Kauffman bracket of link diagrams on a surface, and we establish an analogue of Thistlethwaite's theorem [29] relating these polynomials for alternating links in $\Sigma \times I$ and the polynomial P_G of the associated Tait graph on the surface Σ. Using the interpretation of virtual links as "irreducible" embeddings of classical links into surfaces [21], these results apply to virtual links.

Let L be a link embedded in $\Sigma \times I$, where Σ is a closed orientable surface. Consider a projection D of L onto the surface. By general position D is a diagram with a finite number of crossings. Each crossing may be resolved as shown in figure 3. Given a diagram D with n crossings, consider the set $\mathcal{S}$ of its 2^n resolutions. Each resolution $S \in \mathcal{S}$ is a disjoint collection of closed curves embedded in Σ. Denote by $\alpha(S)$ the number of resolutions of type (1) that were used to create it, by $\beta(S)$ the number of resolutions of type (2), and let $c(S)$ be the number of components of S. Consider the inclusion map $i\colon S \subset \Sigma$, and denote

$$k(S) \;=\; \mathrm{rank}\,(\,\mathrm{ker}\,\{i_*\colon\; H_1(S) \longrightarrow H_1(\Sigma)\}).$$
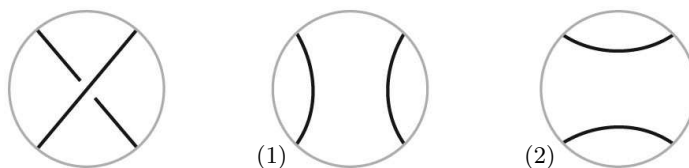

FIGURE 3. Resolutions of a crossing.

Generalizing the classical definition of the Kauffman bracket (cf [20, 1]), consider

$$(6.1) \qquad \widetilde{K}_L(A, B, d) \;=\; \sum_{S \in \mathcal{S}} [i_*(H_1(S))]\, A^{\alpha(S)}\, B^{\beta(S)}\, d^{k(S)}$$

Here $[i_*(H_1(S))]$ denotes a formal variable associated to the subgroup $i_*(H_1(S))$ of $H_1(\Sigma)$. This definition is closely related to (more precisely, it may be viewed as a specialization of) the *surface bracket polynomial*, defined in the context of virtual links in [9, 22]. Two diagrams in Σ, representing isotopic embeddings of a link L in $\Sigma \times I$, are related by the usual Reidemeister moves, and the usual specialization

(6.2) $\widetilde{J}_L(t) \; = \; (-1)^{w(L)} \, t^{3w(L)/4} \, \widetilde{K}_D(t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2}),$

where $w(L)$ is the writhe of L, is an invariant of an embedded oriented link $L \subset \Sigma \times I$. The polynomial $\widetilde{J}_L(t)$ with coefficients corresponding to subgroups of $H_1(\Sigma)$ may be used to distinguish non-isotopic links in $\Sigma \times I$ (also see remark 6 following theorem 6.1 below.) However if one is interested in studying links up to to the action of the diffeomorphisms of Σ, or in studying virtual links, a relevant invariant is the following finite-variable specialization. Denoting the rank of $i_*(H_1(S))$ by $r(S)$, define

(6.3) $K_D(A, B, d, Z) \; = \; \sum_{S \in \mathcal{S}} A^{\alpha(S)} \, B^{\beta(S)} \, d^{k(S)} \, Z^{r(S)},$

and the corresponding version of the Jones polynomial:

(6.4) $J_L(t, Z) \; = \; (-1)^{w(L)} \, t^{3w(L)/4} \, K_D(t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2}, Z).$

Note that all of the polynomials considered here may be defined for *virtual* links [19], using their "irreducible" embeddings into surfaces [21]. Since $k(S) + r(S)$ equals the number $c(S)$ of components of S, it follows that for a virtual link L, the invariant K_D defined above specializes to the usual Kauffman bracket by setting $Z = d$:

$$[L](A, B, d) \; = \; d^{-1} K_L(A, B, d, d).$$
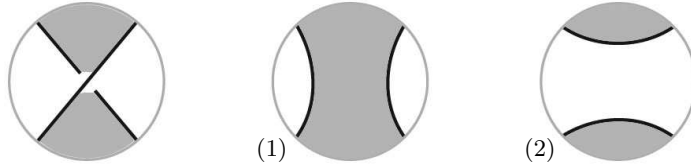

(1)   (2)

FIGURE 4. Checkerboard coloring near a crossing of an alternating diagram.

We now turn to the analogue for links on surfaces of Thistlethwaite's theorem [29] relating the Jones polynomial $J_L(t)$ of an alternating link L in S^3 to the specialization $T_G(-t, -t^{-1})$ of the Tutte polynomial of an associated Tait graph. Suppose L is a link in $\Sigma \times I$ which has an alternating diagram D on Σ. Then this diagram may be checkerboard-colored, as shown near each crossing on the left in figure 4. The associated Tait graph is the graph $G_D \subset \Sigma$ whose vertices correspond to the shaded regions of the diagram, and two vertices are connected by an edge whenever the corresponding shaded regions meet at a crossing (an example of an alternating link on the torus and the corresponding Tait graph are shown in figure 5 - compare with the example in [5].) The Tait graph is a well-defined graph $G \subset \Sigma$ if each
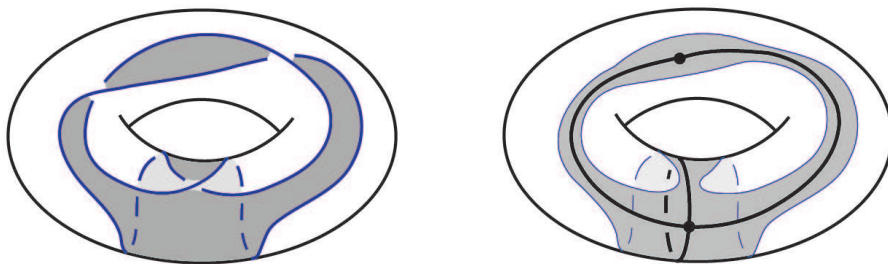
FIGURE 5. An alternating link diagram (left) and its Tait graph (right) on the torus.

component in the complement of a link diagram D is a disk; this condition holds for virtual links due to the irreducibility of their embedding into $\Sigma \times I$.

Theorem 6.1. *The generalized Kauffman bracket (6.3) of an alternating link diagram D on a surface Σ may be obtained from the polynomial P_G, defined by (2.3), of the associated Tait graph G as follows:*

$$(6.5) \quad K_D(A, B, d, Z) \; = \; A^{g+v(G)-c(G)} \, B^{-g+n(G)} \, d^{c(G)} \, Z^g \, P_G \left(\frac{Bd}{A}, \frac{Ad}{B}, \frac{A}{BZ}, \frac{B}{AZ} \right).$$

In particular, substituting $A = t^{-1/4}, B = t^{1/4}, d = -t^{1/2} - t^{-1/2}$ as in (6.4) yields an expression for the 2-variable Jones polynomials $J_L(t, Z)$ in terms of the polynomial P_G of the associated Tait graph.

Remarks.
1. Recall that $\widetilde{K}_L(A, B, d)$, defined in (6.1), is a polynomial in A, B, d with coefficients corresponding to subgroups $V < H_1(\Sigma)$. The equation (6.5) follows from a more general relation, which can be deduced from the proof of theorem 6.1, between the polynomials $\widetilde{K}_L$ and $\widetilde{P}_G(X, Y)$ (defined in (5.6)). In particular, each coefficient $[V], V < H_1(\Sigma)$, of $\widetilde{P}_G$ is replaced with $V \cap V^\perp$ to get the corresponding coefficient of $\widetilde{K}_L$.

2. It would be interesting to establish a relation, analogous to (6.5) between these generalized versions of the Kauffman bracket, the Jones polynomial, and the Bollobás-Riordan polynomial. In principle, such a relationship follows from theorem 6.1 (see the discussion following theorem 4.2), but an explicit formula does not seem to be as straightforward as (6.5).

3. Theorem 6.1 asserts that the generalized Kauffman bracket $K_D(A, B, d, Z)$ may be obtained as a specialization of the polynomial $P_G(D)$. It would be interesting to find out whether K_D and P_G (or K_D and the Bollobás-Riordan polynomial BR_G) in fact determine each other.

4. Suppose D is an alternating link diagram (associated to a link $L \subset \Sigma \times I$) on an orientable surface Σ. Switching each crossing, one gets an alternating link diagram D' whose checkerboard coloring is precisely that of D with the colors switched on each face. Therefore the associated graphs G, G' are duals of each other. (To make this statement precise, it is convenient to consider virtual links, so the embedding $L \subset \Sigma \times I$ is "irreducible" [21], and then the Tait graphs G, G' are cellulations.) In this context theorem 6.1 gives a different perspective on the duality relation (3.1) for P_G.

5. Adapting the proof in [5], one may establish a generalization of theorem 6.1 from alternating diagrams to checkerboard-colored diagrams, using a *signed* version of the polynomial P_G. The proof uses the observation [18] that any such link diagram on a surface can be made alternating by switching some of the crossings, and then one labels by -1 each edge of the Tait graph where a switch has been made.

6. One may generalize the correspondence between the Jones polynomial and P_G to *skein modules*. (This is a further generalization from the polynomials $\widetilde{K}_L$ and $\widetilde{P}_G$ whose coefficients are subgroups of $H_1(\Sigma)$.) Specifically, one may consider the isotopy classes of graphs on Σ modulo the contraction-deletion relation, see section 5.2, and the skein module of links modulo the Kauffman skein relation in figure 3, cf. [27, 30]. The author would like to thank Józef Przytycki for pointing out this perspective on the problem.

Proof of theorem 6.1. The terms in the expansions (2.3), (6.3) are in $1-1$ correspondence. Specifically, for each spanning subgraph $H \subset G(D)$ parametrizing the sum (2.3), consider the corresponding resolution $S(H)$: each crossing of the diagram D is resolved as in figure 4, where the resolution (1) is used if the corresponding edge is included in H, and the resolution (2) is used otherwise. Observing the effect of the resolutions on the shaded regions in figure 4, note that the collection of embedded curves $S(H) \subset \Sigma$ is the boundary of a regular neighborhood of H in Σ. Moreover, the number $\alpha(S)$ of resolutions of type (1) is precisely the number $e(H)$ of edges of H, and $\beta(S)$ equals $e(G) - e(H)$.

Observe

$$\alpha(S(H)) = e(H) = v(H) - c(H) + n(H),$$

$$\beta(S(H)) = e(G) - e(H) = n(G) - n(H) + c(H) - c(G).$$

Also note that since S is the boundary of a regular neighborhood of H, $r(S) = l(H)$, and $k(S) = c(H) + k(H)$. Therefore the summands $A^{\alpha(S)} B^{\beta(S)} d^{k(S)} Z^{r(S)}$ in (6.3) may be rewritten as

$$A^{v(H)-c(H)+n(H)} B^{n(G)-n(H)+c(H)-c(G)} d^{c(H)+k(H)} Z^{l(H)}.$$

Substituting the required variables, the summands in the expansion (2.3) of P_G are of the form

$$\left(\frac{Bd}{A}\right)^{c(H)-c(G)} \left(\frac{Ad}{B}\right)^{k(H)} \left(\frac{A}{BZ}\right)^{s(H)/2} \left(\frac{B}{AZ}\right)^{s^{\perp}(H)/2}.$$

The proof is completed by using the relations (5.5) to identify the exponents of A, B, d, Z on the two sides of (6.5). $\qquad\square$

7. A MULTIVARIATE GRAPH POLYNOMIAL

We conclude the paper by pointing out a multivariate version of the polynomial P_G, and observing the corresponding duality relation. (Note that a multivariate version of the Bollobás-Riordan polynomial has been considered in [24]. A duality result for a certain specialization of the signed multivariate Bollobás-Riordan polynomial has been established in [33].) Let G be a graph on a surface Σ, and let

$$\mathbf{v} = \{v_e\}_{e \in E(G)}$$

be a collection of commuting indeterminates associated to the edges of G. Following the notation used in (2.3), consider

$$(7.1) \qquad \overline{P}_G(q, \mathbf{v}, A, B) = \sum_{H \subset G} q^{c(H)} A^{s(H)/2} B^{s^{\perp}(H)/2} \prod_{e \in E(H)} v_e$$

Clearly, the "usual" multivariate Tutte polynomial Z_G [28] is a specialization of $\overline{P}_G$:

$$Z_G(q, \mathbf{v}) = \overline{P}_{G,\Sigma}(q, \mathbf{v}, 1, 1),$$

The relation to the polynomial $P_G(X, Y, A, B)$ defined in (2.3) is given by

$$P_G(X, Y, A, B) = X^{-c(G)} Y^{-g-v(G)} \overline{P}_G(XY, Y, A/Y, BY),$$

where as usual $c(G)$ denotes the number of connected components of the graph G, $v(G)$ is the number of vertices of G, and g is the genus of the surface Σ. That is, to get the polynomial P_G, one sets in the multivariate version $\overline{P}_G$ all edge weights v_e equal to Y, and $q = XY$. The analogue of the duality (3.1) for the multivariate polynomial $\overline{P}_G$ is as follows.

Lemma 7.1. *Let G be a cellulation of a surface Σ (or equivalently a ribbon graph), and let G^* denote its dual. Then*

$$(7.2) \qquad \overline{P}_{G^*}(q, \mathbf{v}, A, B) = q^{-g+c(G^*)-v(G)} \left(\prod_{e \in E(G)} v_e \right) \overline{P}_G(q, q/\mathbf{v}, B/q, Aq).$$

As the notation indicates, the edge weights of G^* in the formula on the right-hand side are given by $\{q/v_e\}_{e\in E}$. Using the relation $c(H) = v(H) - e(H) + n(H)$, note that the expansion of the polynomial $\overline{P}_G$ may be rewritten as

$$(7.3) \qquad \overline{P}_G(q, \mathbf{v}, A, B) \ = \ q^{v(G)} \sum_{H \subset G} q^{n(H)} \, A^{s(H)/2} \, B^{s^\perp(H)/2} \prod_{e \in E(H)} \frac{v_e}{q}$$

The proof of lemma 7.1 consists of identifying the terms in the expansions of the two sides, following the lines of the proof of theorem 3.1.

Note that the usual duality relation for *planar* graphs (cf. [28]):

$$Z_{G^*}(q, \mathbf{v}) \ = \ q^{1-v(G)} \left(\prod_{e \in E(G)} v_e \right) Z_G(q, q/\mathbf{v})$$

may be obtained as a specialization of the relation (7.2).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VA 22904

E-mail address: krushkal@virginia.edu