Filling links in 3-manifolds

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(joint work with Michael Freedman)

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Motivation.

A general theme: links (or knots) can be built in a general 3-manifold which in a sense are as "robust" as an embedded 1-complex can be. For example:

1. Bing's theorem (1958): A closed 3-manifold M is diffeomorphic to S^3 if and only if every knot K in M is contained ("engulfed") in a 3-ball.

2. "Disk busting curves" (Myers, 1982): For any compact 3-manifold M there is a knot K in M so that every essential sphere or disk meets K.

(Suitable versions of these questions are open in higher dimensions.)

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These theorem trivially hold when K is replaced with a 1-complex.

This talk is about another instance of such a problem.

Preliminary definition: a *1-spine* or *spine* of a 3-manifold M:

(a) (The “rank” definition) A spine is a 1-complex in M of least first Betti number, surjecting onto $\pi_1(M)$.

[1-complexes are considered up to $I - H$ moves (a.k.a. Whitehead moves); so they may be thought of as handlebodies.]

Figure: I-H

(b) A spine is a handlebody in M up to isotopy with the property that it is onto on π_1 but no smaller handlebody obtained from compression of a non-separating disk is onto.

We'll work with the (easier) rank definition:

(a) (The "rank" definition) A spine is a 1-complex in M of least first Betti number, surjecting onto $\pi_1(M)$.

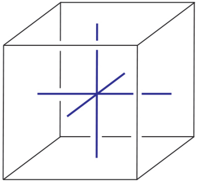

Figure: Example: the standard spine of the 3-torus.

The main notion of this talk:

Given a compact 3-manifold M, is there a link L in M so that whenever G is a spine in M and G is disjoint from L, then $\pi_1(G) \longrightarrow \pi_1(M \setminus L)$ is injective?

If there is such a link L, we call it *filling* in M.

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If there is such a link L, we call it *filling* in M.

It is easy to find a filling 1-complex in any 3-manifold (a spine of a Heegaard handlebody):

Lemma *Let $M = H \cup H^*$ be a Heegaard decomposition. Then given a spine $G \subset M$, any embedding $i\colon G \longrightarrow M \setminus H^*$ induces an injection $\pi_1 G \rightarrowtail \pi_1(M \setminus H^*)$.*

Lemma (filling 1-complex) *Let $M = H \cup H^*$ be a Heegaard decomposition. Then given a spine $G \subset M$, any embedding $i\colon\ G \longrightarrow M \smallsetminus H^*$ induces an injection $\pi_1 G \rightarrowtail \pi_1(M \smallsetminus H^*)$.*

[Recall: A spine is a 1-complex in M of least first Betti number, surjecting onto $\pi_1(M)$.]

Proof. Let π be the image of $\pi_1(G)$ in $\pi_1(M \smallsetminus H^*) \cong \pi_1(H)$.

Being a subgroup of a free group, π is free. Also $\mathrm{rank}(\pi)$ is less than or equal $\mathrm{rank}(\pi_1(G))$ since the map $\pi_1(G) \longrightarrow \pi$ is onto.

If that map has a kernel, by the Hopfian property of free groups $\mathrm{rank}(\pi) < \mathrm{rank}(\pi_1(G))$, contradicting minimal rank. $\square$

$$\pi_1(G) \longrightarrow \pi_1(M \smallsetminus H^*) \longrightarrow \pi_1(M)$$

Elementary observations:

• We can find a *knot* K giving π_1-injectivity for a *fixed* embedding of a spine H (of a Heegaard handlebody):

Consider a minimal genus Heegaard decomposition $M^3 = H \cup H^*$, and let K be a diskbusting curve in the handlebody H^*. If $\pi_1 H \longrightarrow \pi_1(M \setminus K)$ had kernel, by the loop theorem there would be a compressing disk in H^* disjoint from K, a contradiction.

• The problem of finding a filling link L (so an *arbitrary* embedding of a spine is π_1-injective in $M \setminus L$) has a trivial solution for genus one 3-manifolds:

$M^3 = H \cup H^*$ where H, H^* are solid tori. A filling knot is given by the core circle of H^*.

This talk will focus on the case of $M =$ the 3-torus T^3.

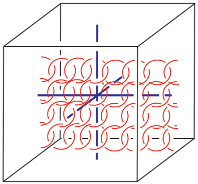
The problem of π_1-injectivity of *any* embedding of a spine G in the complement of a given link L in T^3 is subtle.

$\ker[\pi_1 G \longrightarrow \pi_1 T^3]$ is the commutator subgroup of the free group $\pi_1 G$ on 3 generators.

$$\begin{tikzcd}
 & \pi_1 T^3 \\
\pi_1 G \arrow[ur] \arrow[r, "1{-}1\ ?"] & \pi_1(T^3 \smallsetminus L) \arrow[u]
\end{tikzcd}$$

A standard classical tool for showing injectivity of maps of the free group is the Stallings theorem. However it does not directly apply here (as I will explain next).

The existence of a filling link in T^3 is an open question.



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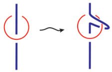

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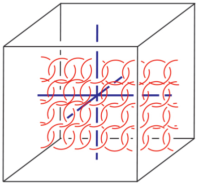

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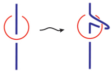


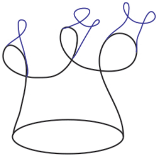
We establish a weaker result:

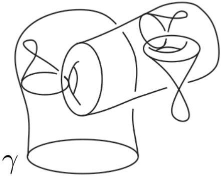
A link $L \subset M$ is *k-filling* if whenever G is a spine in M and G is disjoint from L, the injectivity holds modulo the kth term of the lower central series: $\pi_1 G/(\pi_1 G)_k \rightarrowtail \pi_1(M \smallsetminus L)/\pi_1(M \smallsetminus L)_k$.

Theorem (Freedman-K., 2020)
For any $k \geq 2$ there exists a k-filling link in T^3.

It is interesting to note the similarity of the problem with the current state of knowledge of the topological 4-dimensional surgery theorem for free non-abelian groups.

The underlying technical statement, the *π_1-null disk lemma*, also has a variable homotopy, and the question is whether the map on π_1 can be made *trivial*. One can solve the problem modulo any term of the lower central series, but the question itself is open.





Does γ bound a π_1-null disk in a 4D thickening of the capped grope?

[Digression: the Stallings theorem]

Given a group A, its lower central series is defined inductively by $A_1 = A$, $A_k = [A_{k-1}, A]$; $A_\omega = \cap_{k=1}^{\infty} A_k$.

Stallings' theorem (1965) *Let $f\colon A \longrightarrow B$ be a group homomorphism inducing an isomorphism on H_1 and an epimorphism on H_2. Then f induces an isomorphism $A/A_k \longrightarrow B/B_k$ for all finite k, and an injective map $A/A_\omega \longrightarrow B/B_\omega$.*

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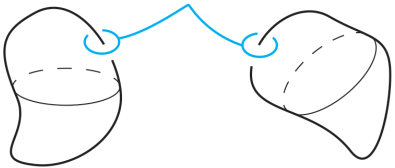
Example: Consider $S^2 \coprod S^2 \hookrightarrow \mathbb{R}^4$.



Take $A = \pi_1(S^1 \vee S^1), B = \pi_1(\mathbb{R}^4 \smallsetminus (S^2 \sqcup S^2))$. It follows that $\pi_1(\mathbb{R}^4 \smallsetminus (S^2 \sqcup S^2))$ is isomorphic to the free group modulo any finite term of the l.c.s. Moreover, $\mathrm{Free}_2 \hookrightarrow \pi_1(\mathbb{R}^4 \smallsetminus (S^2 \sqcup S^2))$.

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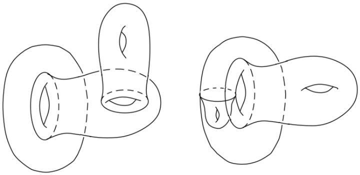
Dwyer extended the theorem, relaxing the surjectivity to be onto H_2 modulo the k-th term of the *Dwyer filtration*:

$$\phi_n(A) \;=\; \ker[H_2(A) \longrightarrow H_2(A/A_n)]$$



(Geometrically $\phi_n(A)$ is represented by maps of *gropes* of height $n-1$.)

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Dwyer's theorem (1975) *Assuming that $f\colon A \longrightarrow B$ is an isomorphism on H_1, f induces an isomorphism $A/A_{k+1} \longrightarrow B/B_{k+1}$ if and only if it induces an epimorphism $H_2(A)/\phi_k(A) \longrightarrow H_2(B)/\phi_k(B)$.*

The Stallings theorem does *not* work in general for the *derived series*.

For a group B, its derived series is defined by
$B^{(0)} = B$, $B^{(n+1)} = [B^{(n)}, B^{(n)}]$.

But there is a version of the theorem for the *torsion-free derived series* (a notion due to Harvey). A corollary when the domain is the free group:

Theorem (Cochran - Harvey, 2008) *Suppose F is a free group, B is a finitely-related group, $\phi\colon F \longrightarrow B$ induces a monomorphism on $H_1(-;\mathbb{Q})$, and $H_2(B;\mathbb{Q})$ is spanned by $B^{(n)}$-surfaces. Then ϕ induces a monomorphism $F/F^{(n+1)} \subset B/B^{(n+1)}$.*

The notion of $B^{(n)}$-surfaces (maps of surfaces into $K(B,1)$ where the image on π_1 is in $B^{(n)}$ gives an analogue of the Dwyer filtration in the derived setting.

[Back to the existence of fillings links]

Stallings' theorem *Let $f\colon\ A \longrightarrow B$ be a group homomorphism inducing an isomorphism on H_1 and an epimorphism on H_2. Then f induces an isomorphism $A/A_k \longrightarrow B/B_k$ for all finite k.*

The Stallings theorem does not apply to the map $\pi_1 G \longrightarrow \pi_1(T^3 \smallsetminus L)$ because it is not surjective on second homology, and it is injective, rather than an isomorphism, on H_1.

The complexity of the problem reflects the fact that the image of the map on π_1 depends on the embedding of the spine G.

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The complexity of the problem reflects the fact that the image of the map on π_1 depends on the embedding of the spine G.

One may attempt to apply the Stallings theorem to the map $[\pi_1 G, \pi_1 G] \longrightarrow K$, where K is the kernel $\pi_1(T^3 \smallsetminus L) \longrightarrow \pi_1(T^3)$. But injectivity of the infinitely generated first homology of the commutator subgroup is hard to establish when the embedding $G \longrightarrow T^3 \smallsetminus L$ changes by an arbitrary homotopy.

[Back to the main theorem]

A link $L \subset M$ is *k-filling* if whenever G is a spine in M and G is disjoint from L, the injectivity holds modulo the kth term of the lower central series: $\pi_1 G/(\pi_1 G)_k \rightarrowtail \pi_1(M \smallsetminus L)/\pi_1(M \smallsetminus L)_k$.

Theorem (Freedman-K., 2020)
For any $k \geq 2$ there exists a k-filling link in T^3.

To prove the theorem we give an extension of the Stallings theorem using powers of the augmentation ideal $\mathbb{Z}[\mathbb{Z}^3]$, which applies uniformly to all embeddings $G \longrightarrow T^3 \smallsetminus L$, where the conclusion holds modulo a given term of the lower central series.

Main steps of the proof:

- An equivariant homological framework for analyzing the effect of homotopies of a spine in terms of powers of the augmentation ideal.
- Construction of links satisfying the homological conditions.
- An extension of the Stallings theorem, relating powers of the augmentation ideal to the lower central series.

Consider the *relative case* where the construction of k-filling links is easier to describe, $M = T^2 \times I$.

Fix the standard "relative spine" $G = (\{*\} \times I) \cup (T^2 \times \partial I)$, and the dual spine $G^* = S^1 \vee S^1 \subset T^2 \times \{1/2\}$. Their preimages in the universal cover:

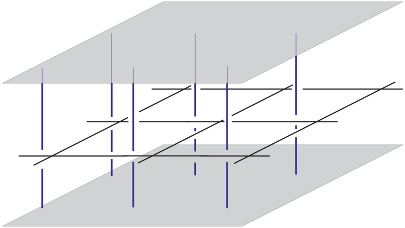


Figure: The preimage $\widetilde{G}$ in the universal cover $\mathbb{R}^2 \times I$ of the standard relative spine $G = (\{*\} \times I) \cup (T^2 \times \partial I)$ consists of the top and bottom shaded panels union the vertical line segments. The mid-level horizontal grid is the preimage of the dual spine $S^1 \vee S^1 \subset T^2 \times \{1/2\}$.

The goal is to analyze the map on π_1 induced by inclusion when G^* is replaced by a link. In this figure the link L is obtained by "resolving" the dual spine G^* into two disjoint essential circles.

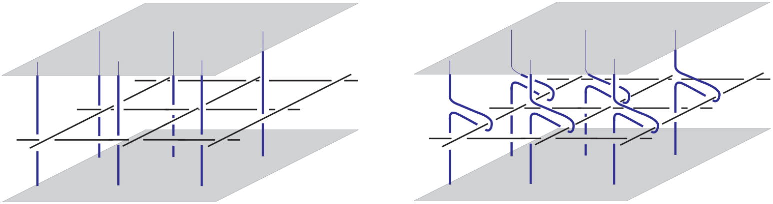


Figure: An example of a link $L \subset T^2 \times (0, 1)$, and a finger move.

Different embeddings of the vertical interval are related by homotopies that may pass through link components and may be thought of as finger moves.

G: the standard embedding of the relative spine into $T^2 \times I$; $\widetilde{G}$: its preimage in the universal cover. The notation G', $\widetilde{G}'$ will be used for an arbitrary embedding, i.e. related to the standard embedding by finger moves. The fundamental group of $\widetilde{G}$, and also of $\widetilde{G}'$ is

$$K := \ker[\mathbb{Z}^2 * \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2].$$

Let L be any link in $T^2 \times (0,1)$ whose components are all essential in $\pi_1(T^2)$, and let $\widetilde{L}$ denote its preimage: a $\mathbb{Z}^2$-equivariant collection of lines in the universal cover.

Note: $\pi_1(\mathbb{R}^2 \times I \smallsetminus \widetilde{L})$ is free.

The starting point is to analyze the injectivity of the map α in the commutative triangle

$$\begin{tikzcd}
\pi_1(G) \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \pi_1(T^2 \times I \smallsetminus L) \arrow[d, "\gamma"] \\
& \pi_1(T^2 \times I)
\end{tikzcd}$$

The focus is on the map

$$K \longrightarrow \pi_1(\mathbb{R}^2 \times I \smallsetminus \widetilde{L}). \tag{1}$$

Denote by J the first homology of $\widetilde{G}$, $J = K/[K, K]$, and let H denote $H_1(\mathbb{R}^2 \times I \smallsetminus \widetilde{L})$. Since $\pi_1(\mathbb{R}^2 \times I \smallsetminus \widetilde{L})$ is a free group, if $J \longrightarrow H$ were injective, the Stallings theorem would imply that the map

$$K \longrightarrow \pi_1(\mathbb{R}^2 \times I \smallsetminus \widetilde{L}). \tag{2}$$

is injective (for the standard spine G).

Denote the map $J \longrightarrow H$ by Lk. The group H is generated by meridians $m(l)$, one for each line l in $\widetilde{L}$. The map Lk is given by $\mathbb{Z}^2$-equivariant linking, sending a 1-cycle c in $\widetilde{G}$ to a linear combination of meridians $\sum_i a_i\, m(l_i)$, where the coefficient $a_i \in \mathbb{Z}[\mathbb{Z}^2]$ is the linking "number" of c and l_i. Since there is a single generator $m(l)$ for each line l, when there is no risk of confusion we will write

$$Lk(c) = \sum_i a_i\, l_i.$$

As a module over $\mathbb{Z}[\mathbb{Z}^2]$, J is generated by the boundaries of two vertical "plaquettes", denoted P_x and P_y. Think of elements of J as linear combinations of these plaquettes, with coefficients in $\mathbb{Z}[\mathbb{Z}^2]$. The translations in the directions perpendicular to P_x, P_y are denoted respectively by x, y. Note that the relation

$$(1-x)P_x + (1-y)P_y = 0 \qquad (3)$$

holds in J.

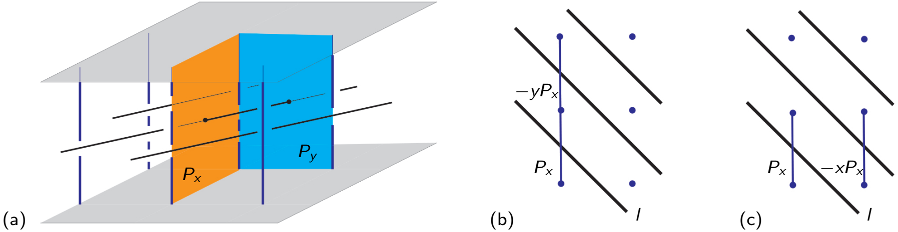
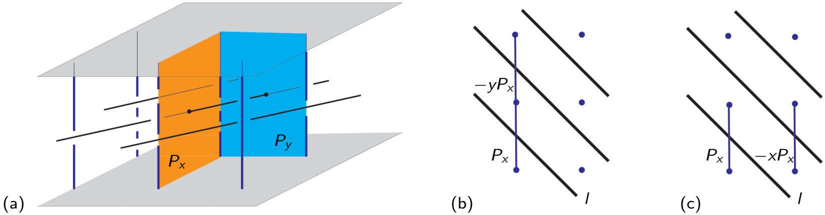


Figure:
(a): Plaquettes P_x, P_y generating J over $\mathbb{Z}[\mathbb{Z}^2]$.
(b), (c): Projection onto $\mathbb{R}^2$; dots represent the preimage of the edge $\{*\} \times I$ of the relative spine.



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The figure shows the case when the link L has a single component, the $(1,1)$-curve in the torus $T^2 \times \{1/2\}$. In this case the two translations act the same way on $\tilde{L}$: for any line l, $xl = yl$. (b, c) show the projection onto $\mathbb{R}^2$ of two elements of J: $(1-y)P_x$, $(1-x)P_x$. Denoting the line intersecting the plaquette P_x by l_0, we have

$$Lk((1-y)P_x) = (1-y)\, l_0, \quad Lk((1-x)P_x) = (1-x)\, l_0.$$

Since $(1-y)\, l_0 = (1-x)\, l_0$ in this example, the map $J \longrightarrow H$ is not injective.

Consider elements of $\mathbb{Z}[\mathbb{Z}^2]$ as Laurent polynomials in two commuting variables x, y. Let I denote the augmentation ideal of $\mathbb{Z}[\mathbb{Z}^2]$. The following lemma provides a convenient tool for analyzing the injectivity of the linking map for an arbitrary spine, modulo powers of the augmentation ideal.

Lemma

If

$$i_k\colon\ I^k J/I^{k+1}J \longrightarrow I^k H/I^{k+1}H$$

is injective for some k, then for any relative spine $G' \subset T^3 \smallsetminus L$,

$$i'_k\colon\ I^k J'/I^{k+1}J' \ \longrightarrow\ I^k H/I^{k+1}H$$

is injective. Here i_k, i'_k are the map induced by the inclusions of G, G' into $T^3 \smallsetminus L$.

Lemma

For any k there exists a link $L_k \subset T^2 \times I$ such that $i_j\colon\ I^j J/I^{j+1}J \longrightarrow I^j H/I^{j+1}H$ is injective for all $1 \le j \le k$.

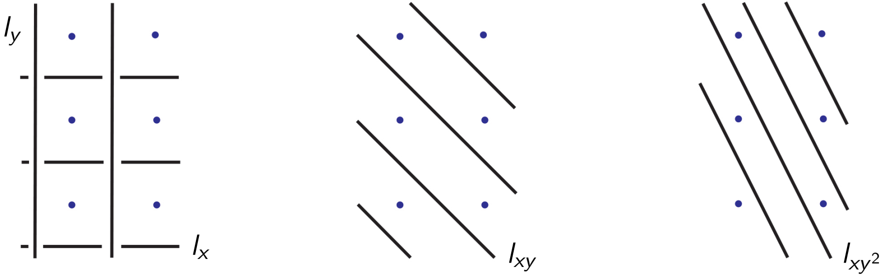

Figure: The preimage of the curves in $T^2 \times I$ in the universal cover: projection of $\mathbb{R}^2 \times I$ onto $\mathbb{R}^2$ is shown; the dots represent the preimage of the edge $\{*\} \times I$ of the relative spine.

Powers of the augmentation ideal and the lower central series

Let $i'\colon\ G' \longrightarrow T^3 \smallsetminus L$ denote any spine homotopic to the standard spine G, where L is a link whose components are all essential in $\pi_1 T^3$.

Recall the notation:
F denotes $\pi_1 G'$, the free group on three generators, and

$$K := \ker\left[\,\pi_1(T^3 \smallsetminus L) \longrightarrow \pi_1 T^3\,\right]$$

is isomorphic to $\pi_1(\mathbb{R}^3 \smallsetminus \widetilde{L})$, a free group. J' denotes the first homology of the preimage $\widetilde{G}'$ of G' in $\mathbb{R}^3$ and H denotes $H_1(\mathbb{R}^3 \smallsetminus \widetilde{L})$,

$$J' \cong F_2/[F_2, F_2],\ \ H \cong K/[K, K].$$

J' and H are considered as modules over $\mathbb{Z}[\mathbb{Z}^3]$, and I^k denotes the k-th power of the augmentation ideal I of $\mathbb{Z}[\mathbb{Z}^3]$.

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J' and H are considered as modules over $\mathbb{Z}[\mathbb{Z}^3]$, and I^k denotes the k-th power of the augmentation ideal I of $\mathbb{Z}[\mathbb{Z}^3]$.

The main result relating the filtrations of J', H in terms of powers of the augmentation ideal and the lower central series:

Lemma. *Suppose $J'/I^k J' \longrightarrow H/I^k H$ is injective for some k. Then the map*

$$F/F_{k+1} \longrightarrow \pi_1(M \smallsetminus L)/\pi_1(M \smallsetminus L)_{k+1}$$

is injective.

Lemma. *Suppose* $J'/I^kJ' \longrightarrow H/I^kH$ *is injective for some* k. *Then the map*

$$F/F_{k+1} \longrightarrow \pi_1(M \smallsetminus L)/\pi_1(M \smallsetminus L)_{k+1}$$

is injective.

Consider

$$\pi := \mathrm{image}\,[\,F \xrightarrow{i'_*} \pi_1(T^3 \smallsetminus L)\,]. \tag{4}$$

The proof is by induction on k; the inductive assumption is that $F/F_k \longrightarrow \pi/\pi_k$ is an isomorphism. The strategy is motivated by the proof of the Stallings theorem.

Note that both F_2 and π_2 are free groups but the map $F_2 \longrightarrow \pi_2$ is not an isomorphism on H_1. Being an isomorphism on H_1 is equivalent to $J'/I^kJ' \cong \overline{H}/I^k\overline{H}$ for *all* k. Rather the lemma has a weaker assumption, $J'/I^kJ' \cong \overline{H}/I^k\overline{H}$ for *some* fixed k.

Let ϕ_k denote the inclusion $F_k \subset F_2$ composed with the quotient map $F_2 \longrightarrow F_2/[F_2, F_2]$, and consider its kernel:

$$1 \longrightarrow F_k \cap [F_2, F_2] \longrightarrow F_k \xrightarrow{\phi_k} F_2/[F_2, F_2] \qquad (5)$$

Denote the generators of F by x, y, z; the same letters will denote the covering translations of $\mathbb{R}^3$.

A basic example, the triple commutator $[[x, y], z] \in F_3$. The map ϕ_k is implemented by first expanding $[[x, y], z] = [x, y] \cdot ([x, y]^{-1})^z$. The first factor is mapped to the boundary of the plaquette P_z. The second factor is mapped to the boundary of this plaquette with the opposite orientation and shifted one unit up, $-zP_z$. So $\phi_3([[x, y], z]) = (1 - z)P_z$.

The image of ϕ_k is $I^{k-2}J \subset J$.

The main ingredient in the proof of the inductive step:

$$\begin{tikzcd}
1 \arrow[r] & \frac{F_k \cap [F_2,F_2]}{F_{k+1} \cap [F_2,F_2]} \arrow[r] \arrow[d, "\alpha"] & \frac{F_k}{F_{k+1}} \arrow[r] \arrow[d, "\beta"] & \frac{I^{k-2}J}{I^{k-1}J} \arrow[r] \arrow[d, "\gamma"] & 1 \\
1 \arrow[r] & \frac{\pi_k \cap [\pi_2,\pi_2]}{\pi_{k+1} \cap [\pi_2,\pi_2]} \arrow[r] & \frac{\pi_k}{\pi_{k+1}} \arrow[r] & \frac{I^{k-2}\overline{H}}{I^{k-1}\overline{H}} \arrow[r] & 1
\end{tikzcd} \tag{6}$$

Compare with the Stallings' proof:

$$\begin{tikzcd}
H_2(A) \arrow[r] \arrow[d] & H_2(A/A_\alpha) \arrow[r] \arrow[d] & A_\alpha/A_{\alpha+1} \arrow[r] \arrow[d] & H_1(A) \arrow[r] \arrow[d] & H_1(A/A_\alpha) \arrow[d] \\
H_2(B) \arrow[r] & H_2(B/B_\alpha) \arrow[r] & B_\alpha/B_{\alpha+1} & H_1(B) \arrow[l] \arrow[r] & H_1(B/B_\alpha)
\end{tikzcd}$$

More recently Christopher Leininger and Alan Reid proved:

Theorem

Let M be a closed orientable 3-manifold such that $\pi_1(M)$ has rank 2. Then M contains a filling hyperbolic link.

The proof relies on work of Jaco-Shalen:

Let $L \subset M$ be a hyperbolic link with at least 3 components.

The possibilities for the image H of $\pi_1(G)$ in $\pi_1(M \setminus L)$ are:

1. H is free of rank 2, or
2. H is free abelian of rank ≤ 2, or
3. H has finite index in $\pi_1(M \setminus L)$.