ENGEL RELATIONS IN 4-MANIFOLD TOPOLOGY

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Dedicated to Andrew Casson

ABSTRACT. We give two applications of the 2-Engel relation, classically studied in finite and Lie groups, to the 4-dimensional topological surgery conjecture. The A-B slice problem, a reformulation of the surgery conjecture for free groups, is shown to admit a homotopy solution. We also exhibit a new collection of universal surgery problems, defined using ramifications of homotopically trivial links. More generally we show how n-Engel relations arise from higher order double points of surfaces in 4-space.

1. INTRODUCTION

Forty years ago Andrew Casson taught us [3] that singularities of surfaces and the fundamental group of their complements are intimately related. We study a classical group relation, 2-Engel, and the corresponding surface singularities. The results include two surprises (to us) regarding topological surgery. What direction they point is presently unknown. They might later be seen as: a step in proving the full surgery conjecture, or contrariwise as pointing toward a surgery obstruction, or possibly as mere curiosities. The purpose of this paper is to explain these surprises and reconsider fundamental conjectures and constructions in this new light.

Topological surgery is known to work in dimension 4 for a class of "good" fundamental groups. Originally this was established in the simply-connected setting by the first author in [6]. It has since been shown [8] that elementary amenable groups, and more recently [14, 25] the groups of subexponential growth are good in this sense. The validity of surgery for arbitrary fundamental groups remains a central open problem. Surgery may be reduced to a collection of *universal* problems [4, 8] with free fundamental groups, therefore the validity of surgery for (non-abelian) free groups is the key open question. It has been reformulated [9, 10] in terms of the *A-B slice* problem for a family of links, the "generalized Borromean rings".

We give applications of the group-theoretic 2-*Engel relation* both to the A-B slice problem and to construction of model surgery problems. The study of the universal relation, stating that all 3-fold commutators of the form $[[y,x],x]$ are trivial in a group G, dates back to the work of Burnside [1]. It is easily seen to be equivalent to the relation that every element x in G commutes with all of its conjugates x^y.

A restricted version of this relation is familiar in low-dimensional topology: when applied to a set of preferred normal generators x of a group G, it is a defining relation of the *Milnor group* MG, see [28] and section 2 below. The results of imposing the relation in these two settings turn out to be quite different: the free Milnor group on n generators, MF_n, is nilpotent of class n. On the other hand, the free group F_n modulo the universal 2-Engel relation is nilpotent of class 3, independent of n (see section 2). This is the property of the Engel relation that we exploit in our applications.

To formulate our first result, we briefly recall the A-B slice problem (a detailed discussion is given in section 3). Surgery for free groups predicts the existence of topological 4-manifolds M which are homotopy equivalent to a wedge of circles and whose boundary is the zero-framed surgery on a Whitehead double of L, for each L in the collection of generalized Borromean rings. These links (GBR's) are obtained from the Borromean Rings by ramification and Bing doubling. Following [9, 10] consider the resulting free group action on the end-point compactification of the universal cover $\widetilde{M}$, which is homeomorphic to the 4-ball. Choosing a fundamental domain for this action, one is led to the notion of a *decomposition* $D^4 = A \cup B$ of the 4-ball into two codimension zero smooth submanifolds, extending the standard genus one Heegaard decomposition of ∂D^4. Given an n-component GBR L, the existence of the free group action is then equivalent to the existence of n decompositions $D^4 = A_i \cup B_i$ and a disjoint embedding problem for these $2n$ submanifolds into D^4, with the boundary condition given by the link L and its parallel copy. If this embedding problem has a solution, the link L is called A-B slice.

Considering handle decompositions of the submanifolds, one gets a pair of links, which we call a "stabilization", corresponding to the 1- and 2-handles. The embedding question can then be reformulated [12] as a *relative-slice problem* for a certain collection of link pairs corresponding to a GBR L. A key feature of the GBR's is that they are homotopically essential in the sense of Milnor. Therefore it is a natural question whether there is a link-homotopy obstruction in the A-B slice problem, in other words whether the relevant relative-slice problems do not even admit a link-homotopy solution. The evidence thus far has pointed to an affirmative answer: partial obstructions of this type have been found for many families of decompositions, see for example [12], [24]. Surprisingly, here we construct first examples of decompositions giving rise to a homotopy solution to the A-B slice problem.

Theorem 1. *The generalized Borromean rings, a collection of links forming model surgery problems, are homotopy A-B slice.*

We present two possible notions of a "homotopy solution", one in the sense of link-homotopy, and a stronger one in terms of disjoint homotopy of 2-handles, see definitions 3.7, 3.9 in section 3. The theorem is true for both notions.

The action of the free group on D^4 by covering transformations is encoded in the requirement that the disjoint embeddings of the A_i, B_i in D^4 are *standard*, in other words isotopic to the original embeddings corresponding to the given decompositions $D^4 = A_i \cup B_i$. It was observed in [23] that there exist solutions to the embedding problem if this requirement is omitted. (However the existence of a solution without the equivariant feature does not have a direct implication for surgery.) Our proof of theorem 1 satisfies the homotopy analogue of the standard embedding requirement, see definition 3.9 and the proof of theorem 1 in section 4.

One way to view theorem 1 is as evidence towards the validity of the surgery conjecture. There is a well established hierarchy of 2-complexes, defined in terms of gropes and capped gropes (cf. [13]), extrapolating between disjoint disks with self-intersections and actual embedded disks. It seems possible that a homotopy solution to the AB slice problem may be further improved using group-theoretic methods. For example n-Engel relations, $n > 2$, are candidates for such an approach, however these higher relations are not as well understood. We refer the reader to [31] for a recent survey of the subject. It is an open question whether a homotopy solution may be improved to a stage that would imply an actual embedded solution to the AB slice problem. To assist the reader who would like to solve this problem we discuss in the Appendix how n-Engel relations relate to higher order self-intersections of a disk.

It has been shown in [15] that Whitehead doubles of (homotopically trivial)$^+$ links (a class of links just slightly smaller than homotopically trivial links) are topologically slice. Therefore (a suitably formulated version of) the AB slice problem has a solution for (homotopically trivial)$^+$ links. Viewing our present work in the context of the relative-slice problem, discussed in section 3, for each GBR we find a stabilization so that the resulting link is homotopically trivial. As remarked above, starting with a (homotopically trivial)$^+$ link a stabilization may be found so that the result is slice. There is gap corresponding to the $+$ assumption, but it is an interesting question whether the two stabilizations may be combined to give a solution.

Overall, the key open problem is to determine whether there still is an obstruction to the AB slice problem in terms of nilpotent invariants of links, specifically Milnor's μ-invariants (with repeating coefficients). Of course such an obstruciton would give a counterexample to surgery for free groups. Conversely, as discussed above the ability to "improve" a homotopy solution could lead to the resolution of the surgery conjecture in the affirmative. An axiomatic framework in terms of *topological arbiters* for an obstruction in the AB slice program has been introduced in [11]. Since our theorem 1 constructs a solution up to homotopy, there is no topological arbiter satisfying an extended "Bing doubling axiom" [11] defined in terms of $\bar{\mu}$-invariants with non-repeating coefficients. Since the method of proof of theorem 1 does not extend to the relevant stabilized link together with parallel copies of its components, μ-invariants with repeating coefficients remain a candidate for a surgery obstruction.

It is interesting to compare the complexity of the homotopy solution to the AB slice problem constructed in theorem 1 with the current state of knowledge about general decompositions $D^4 = A \cup B$. A recent paper [24] gave a thorough analysis of the decompositions of the 4-ball where A has two 2-handles and *one* 1-handle. The answer is quite subtle and the analysis relies on delicate[1] calculations in commutator calculus. In the relevant decomposition $D^4 = A \cup B$ used for the Borromean rings in the proof of theorem 1 (see section 4), the side A has two 2-handles and 36 1-handles. (The B-side has a handle decomposition with the number of 1- and 2-handles reversed.) It seems likely that a novel algebraic structure will be needed to gain further insight into the problem.

In section 5 we describe a slicing problem for a link in a 4-manifold (the "Round Handle Problem") where the existence of a solution depends not just on surgery but also on the 5-dimensional s-cobordism conjecture. At first sight this problem appears similar to the relative-slice formulation of the AB slice problem for GBR's, however the proof of theorem 1 does not extend to this setting. This suggests a subtle distinction between the two problems, with the possibility that a link-homotopy obstruction is still possible to the combination of surgery and s-cobordism conjectures.

Another application of the 2-Engel relation yields a new set of universal surgery problems. The "usual" model surgery kernels [13] are given by $S^2 \vee S^2$-like capped gropes. They are universal in the sense that if solvable they imply solutions to all 4-dimensional surgery problems with the vanishing Wall obstruction, see section 6. There is a corresponding collection of slicing problems for links $\{Wh(Bing(Hopf))\}$ (where the slice complement in the 4-ball is required to have free fundamental group, generated by the meridians). The links in question are Whitehead doubles of the generalized Borromean rings mentioned above and discussed further in section 6. We introduce a new collection of universal slicing problems:

Theorem 2. *There is a family of links $\{K\}$ for which the problem of constructing free slices constitutes a universal problem, where each $K \in \{K\}$ is of the form:*

$$D(Ram(h\text{-}triv)),$$

a general double of a ramified homotopically trivial link.

The "double" in this statement is a generalization of the notion of a Whitehead double of a link, introduced in section 5.1. The key (and surprising) feature of this new collection of links is that they are defined starting from *homotopically trivial* links, see remark 6.1.

The organization of the paper is as follows. Section 2 discusses the 2-Engel relation and introduces a geometric analogue, *weak homotopy* of links. A proof is given that

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[1]It is important that a certain system of cubic equations has no integral solutions whereas it manifestly has a solution over $\mathbb{Z}[1/4]$.

2-Engel groups are 3-nilpotent. In section 3 we recall the formulation of the A-B slice problem with a particular focus on the notions of a homotopy solution and a standard embedding, important for theorem 1. The proof of theorem 1 is given in section 4. Section 5 formulates the Round Handle Problem, providing a comparison of our results with the setting of the s-cobordism theorem. New model surgery problems are constructed in section 6. In the Appendix we show how n-Engel relations correspond to higher order intersections of disks.

2. The 2-Engel relation

The Milnor group provides a convenient setting for the analysis of the 2-Engel relation and for the main results of the paper. We start by briefly reviewing the Milnor group and link homotopy in section 2.1, the reader is referred to [28] for a detailed introduction. Section 2.2 presents the 2-Engel relation and shows that 2-Engel groups are 3-nilpotent. A geometric realization of this relation, weak homotopy of links, is discussed in section 2.3.

2.1. **The Milnor group.**

Definition 2.1. Let G be a group normally generated by a fixed finite collection of elements $g_1, \ldots, g_n$. The *Milnor group* of G, defined with respect to the given normal generating set $\{g_i\}$, is given by

$$(2.1) \qquad MG := G \, / \, \langle\!\langle \, [g_i, g_i^y] \ \ i = 1, \ldots, n, \ \ y \in G \rangle\!\rangle.$$

The Milnor group MG is generated by $g_1, \ldots, g_n$. Moreover, it is a finitely presented nilpotent group of class $\leq n$, see [28].

Given an n-component link L in S^3, let G denote $\pi_1(S^3 \smallsetminus L)$. Consider meridians g_i to the components l_i of L: g_i is an element of G obtained by following a path α_i in $S^3 \smallsetminus L$ from the basepoint to the boundary of a regular neighborhood of L, followed by a small circle (a fiber of the circle normal bundle) linking l_i, then followed by α_i^{-1}. G is normally generated by the elements $g_1, \ldots, g_n$. Then MG, defined with respect to the meridians, is called the Milnor group ML of the link L.

Denoting by $F_{g_1, \ldots, g_n}$ the free group generated by the $\{g_i\}$, $i = 1, \ldots, n$, consider the Magnus expansion

$$(2.2) \qquad M \colon F_{g_1, \ldots, g_n} \longrightarrow \mathbb{Z}[\![x_1, \ldots, x_n]\!]$$

into the ring of formal power series in non-commuting variables $\{x_i\}$, defined by

$$M(g_i) = 1 + x_i, \ \ M(g_i^{-1}) = 1 - x_i + x_i^2 - x_i^3 \pm \ldots$$

The Magnus expansion induces a homomorphism

$$(2.3) \qquad MF_{g_1, \ldots, g_n} \longrightarrow R_{x_1, \ldots, x_n},$$

into the quotient $R_{x_1,...,x_n}$ of $\mathbb{Z}[\![x_1,...,x_n]\!]$ by the ideal generated by all monomials $x_{i_1} \cdots x_{i_k}$ with some index occuring at least twice. The homomorphism (2.3) is well-defined and injective [28]. Using the Magnus expansion it is not difficult to see that the Milnor group MF_n of the free group F_n on n generators is nilpotent of class precisely equal to n.

The defining relations of the Milnor group (2.1) are well suited for studying links L in S^3 up to link homotopy. Recall that two links are *link-homotopic* if they are connected by a 1-parameter family of link maps where different components stay disjoint for all values of the parameter, see figure 2.1. If L, L' are link-homotopic then their Milnor groups ML, ML' are isomorphic, and moreover an n-component link L is homotopically trivial (link-homotopic to the n-component unlink) if and only if ML is isomorphic to the free Milnor group $MF_{m_1,...,m_n}$.
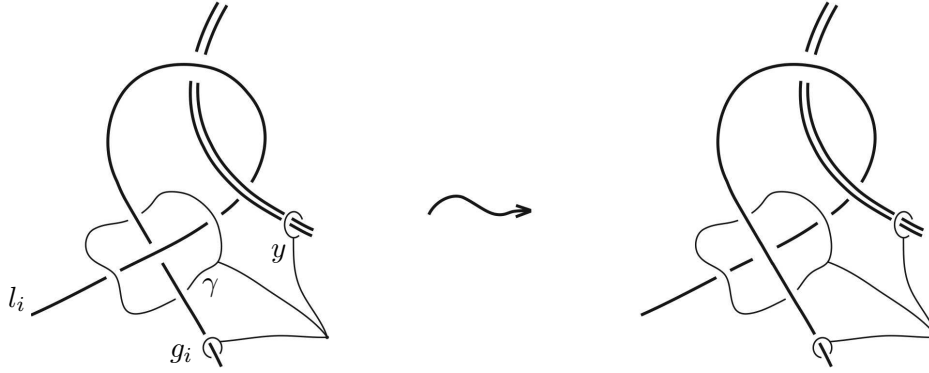

FIGURE 2.1. An illustration of a non-generic "crossing-time" during a link homotopy. The based curve γ in the link complement, corresponding to the defining relation $[g_i, g_i^y]$ of the Milnor group, becomes trivial after a self-intersection of the component l_i.

The Milnor group is also useful for studying surfaces Σ in the 4-ball where the components are disjoint but may have self-intersections. In this context the Clifford tori linking the double points in D^4 give rise to the relations (2.1) in $M\pi_1(D^4 \smallsetminus \Sigma)$. Link homotopy theory may be interpreted as the study of links up to singular concordance (links $L \subset S^3 \times \{0\}$, $L' \subset S^3 \times \{1\}$ bounding disjoint maps of annuli into $S^3 \times [0,1]$). In particular, a link L is homotopically trivial if and only if its components bound disjoint immersed disks Δ in D^4, and in this case $M\pi_1(S^3 \smallsetminus L) \cong M\pi_1(D^4 \smallsetminus \Delta) \cong MF_n$.

2.2. **2-Engel groups.** We start off by fixing the notation. The lower central series of a group G is defined inductively by $G^1 = G, G^n = [G^{n-1}, G]$. Given $g_1,...,g_n \in G$, the commutator $[[... [g_1, g_2],..., g_{n-1}], g_n]$ will be concisely denoted $[g_1, g_2,..., g_n]$.

The main focus of this section is on 2-Engel groups, that is groups satisfying the universal relation $[y, x, x] = 1$, or equivalently $[x, x^y] = 1$. Unlike the setting of the Milnor group (2.1), this relation holds for *all* elements x, y of a 2-Engel group.

In reference [1] (which is at the foundation of the subject of Engel groups) W. Burnside showed any elements of a 2-Engel group G satisfy that the identities

$$[x, y, z] = [y, z, x], \quad [x, y, z]^3 = 1.$$

In a later paper [20] C. Hopkins showed that 2-Engel groups G have nilpotency class ≤ 3, that is $G^4 = \{1\}$. (Also see [26].) We give a proof of this result below in the context of the Milnor group, to establish a reference point for geometric applications in later sections. Corollary 2.3 summarizes the relevant facts.

It is interesting to note that the 2-Engel relation is functorial, that is any group homomorphism $G \longrightarrow H$ induces a homomorphism of groups modulo the relation. This contrasts the Milnor group setting (2.1): only homomorphisms taking chosen generators of G to the chosen generators of H are guaranteed to induce a homomorphism of the Milnor groups, $MG \longrightarrow MH$, defined with respect to these generators. It follows that the quotient needed for making the theory functorial necessarily kills most of non-abelian information:

Lemma 2.1. *Any 2-Engel group is nilpotent of class ≤ 3.*

We will use the following basic result about Milnor groups. Given a group G normally generated by $g_1, \ldots, g_n$, consider "basic commutators" $[g_{i_1}, \ldots, g_{i_m}]$.

Proposition 2.2. *Any basic commutator $[g_{i_1}, \ldots, g_{i_m}]$ where at least two of the indices coincide, $i_j = i_k$ for some $j \neq k$, is trivial in the Milnor group MG.*

One way to prove this fact is to use the Magnus expansion. Every monomial (other than 1) in the expansion of $[g_{i_1}, \ldots, g_{i_m}]$ has a variable $x_{i_j} = x_{i_k}$ occurring at least twice. Since the Magnus expansion (2.3) is injective it follows that $[g_{i_1}, \ldots, g_{i_m}] = 1 \in MF_{g_1, \ldots, g_n}$.

One can also use the commutator identities (2.5) to directly show that $[g_{i_1}, \ldots, g_{i_m}]$ with repeating labels is a product of the defining relations (2.1) of the Milnor group.

Proof of lemma 2.1. We will use the Hall-Witt identity (2.4) and basic commutator identities (2.5), cf. [27, Theorem 5.1].

(2.4) $$[x, y, z^x] \cdot [z, x, y^z] \cdot [y, z, x^y] = 1,$$

(2.5) $$[x, yz] = [x, z] [x, y]^z, \quad [xz, y] = [x, y]^z [z, y], \quad [x^{-1}, y] = [y, x]^{x^{-1}}.$$

It suffices to show that the free group $F_n = F_{g_1, \ldots, g_n}$ modulo the 2-Engel relation is nilpotent of class 3. This quotient factors through the Milnor group MF_n. Using the identities (2.5) and proposition 2.2, $(MF_n)^4$ is seen to be normally generated by

commutators $[g_{i_1}, \ldots, g_{i_4}]$ with non-repeating indices. It suffices to show that F_4, the free group on 4 generators is nilpotent of class 3, after dividing out $<2\text{-Engel}>$.

Denote the generators of F_4 by x, y, z, w. First we focus on 3-fold commutators. Denoting by $\equiv$ the equivalence up to the 2-Engel relation, in the free Milnor group MF_4 one has, by definition:

$$(2.6) \qquad\qquad\qquad\qquad 1 \equiv [z, xy, xy].$$

Expanding this commutator according to (2.5) yields a product of four terms (where the conjugations are omitted, for a reason discussed below) in (2.7). The second equality follows from proposition 2.2:

$$(2.7) \qquad [z, xy, xy] \;=\; [z, x, x] \cdot [z, x, y] \cdot [z, y, x] \cdot [z, y, y] \;=\; [z, x, y] \cdot [z, y, x].$$

It is a basic fact that conjugation as in (2.4), (2.5) may be disregarded in Milnor group calculations of this type. One way to see this is to consider the Magnus expansion (2.3). The effect of conjugation is an introduction of higher order terms. Each higher order monomial that comes up in applications of the identities (2.5) to (2.7) has repeated indices, so is trivial in the target ring R of the Magnus expansion. Since the expansion (2.3) is injective, it follows that conjugation resulting from the commutator identities (2.5) does not change the terms appearing in (2.7).

It follows from (2.6), (2.7) that

$$(2.8) \qquad\qquad\qquad\qquad [z, x, y] \;\equiv\; [z, y, x]^{-1} \;=\; [y, z, x].$$

Similarly,

$$(2.9) \qquad 1 \equiv [x, yz, yz] = [x, y, z] \cdot [x, z, y], \text{ so } [x, y, z] \;\equiv\; [x, z, y]^{-1} \;=\; [z, x, y].$$

Then the Hall-Witt identity (where conjugation is again irrelevant) implies:

$$(2.10) \qquad\qquad 1 \;=\; [x, y, z] \cdot [z, x, y] \cdot [y, z, x] \;\equiv\; [x, y, z]^3.$$

Using (2.5) and disregarding conjugation in the Milnor group as above, it follows that 4-fold commutators are also of order 3:

$$(2.11) \qquad\qquad\qquad [x, y, z, w]^3 \;=\; [[x, y, z]^3, w] \;\equiv\; 1.$$

Next we show that 4-fold commutators are also of order 4. The Hall-Witt identity (2.4) (applied to $[x, y], z, w$) implies in the Milnor group MF_4:

$$[x, y, z, w] \cdot [w, [x, y], z] \cdot [z, w, [x, y]] = 1.$$

Now interchange the order of the terms $w, [x, y]$ in the second commutator above (this inverts the term) and expand the last commutator using the Hall-Witt identity (applied to x, y and $[z, w]$):

$$(2.12) \qquad\qquad [x, y, z, w] \cdot [x, y, w, z]^{-1} \cdot [z, w, x, y] \cdot [z, w, y, x]^{-1} = 1.$$

Since $[x, y, zw, zw] \equiv 1$ is a 2-Engel relation, the first two terms above are equal. Similarly, it follows from $[z, w, xy, xy] \equiv 1$ that the last two terms are equal. Denoting $A := [x, y, z, w]$, $B := [z, w, x, y]$, the equation (2.12) then asserts: $A^2 B^2 = 1$.

It follows from the 2-Engel relation $[y, xz, xz]$ that $[xz, y, xz] \equiv 1$ and $[xz, y, xz, w] \equiv 1$. Therefore $[x, y, z, w] \equiv [z, y, x, w]^{-1}$. Similarly, y and w can also be interchanged at the expense of inverting the term. This implies $A \equiv B$, so $A^4 = [x, y, z, w]^4 \equiv 1$.

Therefore, $[x, y, z, w]$ is both of order 3 and 4, so is trivial. It follows that all 4-fold commutators in $F_{x,y,z,w}$ are trivial mod the 2-Engel relation. This concludes the proof of lemma 2.1. $\qquad\square$

The following corollary of the proof of lemma 2.1 will be used in later sections.

Corollary 2.3. *Suppose G is a group normally generated by $g_1, \ldots, g_n$. Let $g \in G^k$ be an element of the k-th term of the lower central series, $4 \leq k \leq n$. Then g may be represented in the Milnor group MG as a product of (conjugates of) k-fold commutators of the form $[h_1, \ldots, h_k]$ where two of the elements h_i are equal to each other and to a product of two generators, $h_j = h_m = g_{i_1} g_{i_2}$ for some $j \neq m$, and each other element h_i is one of the generators $g_1, \ldots, g_n$.*

The proof for $k = 4$ follows from a direct inspection of the instances where the 2-Engel relation is used in the proof of lemma 2.1: equations (2.6), (2.9) and two paragraphs following (2.12). For any $k \geq 4$, g (considered as an element of $(MG)^k$) equals a product of conjugates of k-fold basic commutators $[g_{i_1}, \ldots, g_{i_k}]$ with distinct indices. The argument for $k = 4$ shows that "the initial segment" $[g_{i_1}, \ldots, g_{i_4}]$ of each factor is a product $\prod[h_{j_1}, \ldots, h_{j_4}]$. Using the identities (2.5), with the conjugation disregarded in the Milnor group, $[g_{i_1}, \ldots, g_{i_k}] = [\prod[h_{j_1}, \ldots, h_{j_4}], g_{i_5}, \ldots, g_{i_k}] = \prod[h_{j_1}, \ldots, h_{j_4}, g_{i_5}, \ldots, g_{i_k}]$. $\qquad\square$

The following observations are useful for estimating the number of commutators $[h_1, \ldots, h_k]$ needed for a given element $g \in G^k$. Since the statement takes places in the Milnor group, it may be assumed that all generators g_i that appear in each commutator $h_1, \ldots, h_k$ are distinct. Also, it suffices to consider commutators where (in the notation of the corollary) $1 \leq j, m \leq 4$.

2.3. **Weak homotopy of links.** This section introduces a geometric analogue of the 2-Engel relation, and also applications to link theory of lemma 2.1 and corollary 2.3. We say that two n-component links in S^3 are *weakly homotopic* if they are connected by a 1-parameter family of maps which is a smooth isotopy except for a finite collection of times when a "packet" of k parallel strands (for any $k \geq 1$) undergo a simultaneous intersection as shown in figure 2.2. Any number of segments from any of the link components may temporarily be grouped into such a packet. Later packets have no memory of earlier packets. Weak homotopy may be thought of as a
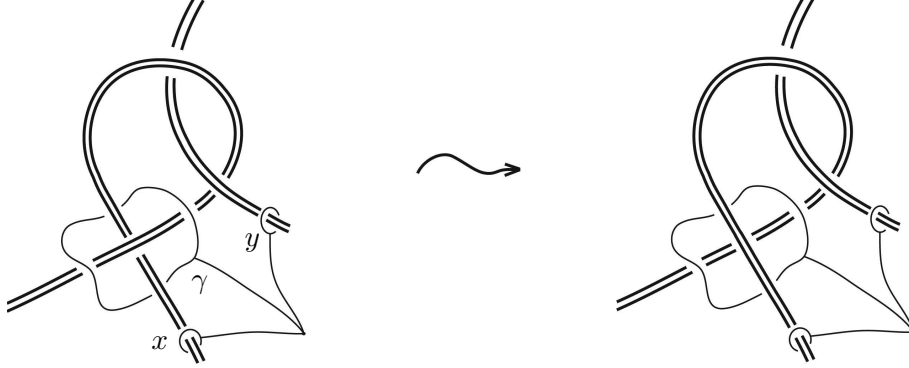
FIGURE 2.2. An elementary weak homotopy. The based curve γ in the link complement, corresponding to the 2-Engel relation $[x, x, y] = [x, x^y]$, becomes trivial after the indicated move. (Compare with figure 2.1 illustrating link homotopy.)

generalization of link-homotopy (figure 2.1) where the self-intersecting component is replaced by an arbitrary number of parallel strands.

Links with non-trivial linking numbers and also links with non-trivial $\bar{\mu}_{123}$ (for example the Borromean rings) are not weakly null-homotopic. There are also non-trivial four component links, but in this case the obstruction is of order 3, see (2.10). The following result, showing that these are the only non-trivial cases, is a consequence of lemma 2.1.

Lemma 2.4. *Let L be an n-component link in S^3, and suppose that $ML/(ML)^5 \cong MF_n/(MF_n)^5$, or equivalently that all $\bar{\mu}$-invariants of L with non-repeating coefficients of length ≤ 4 vanish. Then L is weakly null-homotopic.*

Proof. If $n \leq 4$ then the assumption on the $\bar{\mu}$ invariants implies that L is link-homotopic to the unlink, so the statement trivially follows. If $n \geq 5$ consider the component l_n as an element (defined up to conjugation) of $G := \pi_1(S^3 \smallsetminus (l_1 \cup \ldots, l_{n-1}))$. According to the assumption of the lemma, $l_n \in (MG)^4$. It follows from lemma 2.1 that l_n equals a product of (conjugates of) 2-Engel relations. These relations may be realized by weak homotopies of $l_1, \ldots, l_{n-1}$, after which l_n bounds a disk in the complement of the other components. With l_n gone, run the sequence of weak homotopies on $l_1, \ldots, l_{n-1}$ backwards to arrive at the original link L with the last component missing. If $n - 1 \geq 5$ then repeat the argument. $\square$

It is worth noting (compare with the remark preceding lemma 2.1) that weak homotopy is an equivalence relation on links which arises in an attempt to combine link homotopy and handle slides. Lemma 2.4 shows that a theory making link homotopy invariant under handle slides necessarily makes most links trivial. Three-manifolds are isotopy classes of framed links up to handle slides (and blow-up (down)). If one
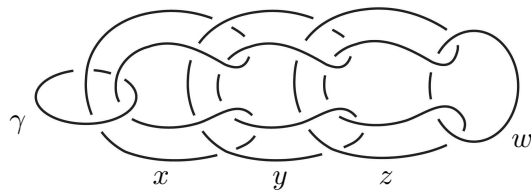
FIGURE 2.3. $\gamma = [x, y, z, w]$.

studied the quotient of 3-manifolds induced by homotopy of framed links, very little would remain: homology, cup and some Massey triple products, certain 4-fold Massey products whose values are 3-torsion.

Our work in later sections does not directly use lemma 2.4, rather we will rely on the more detailed information given by corollary 2.3. To this end, it is useful to illustrate the links which are a geometric analogue of the commutators appearing in the statement of the corollary. Figure 2.3 shows a 5-component link (obtained by iterated Bing doubling the Hopf link) where the left-most component is denoted γ and the meridians to the other 4 components are labeled x, y, z, w. Then γ reads off the commutator $[x, y, z, w]$ in the complement of the other 4 components. Lemmas 2.1, 2.4 imply that this link is weakly null-homotopic.
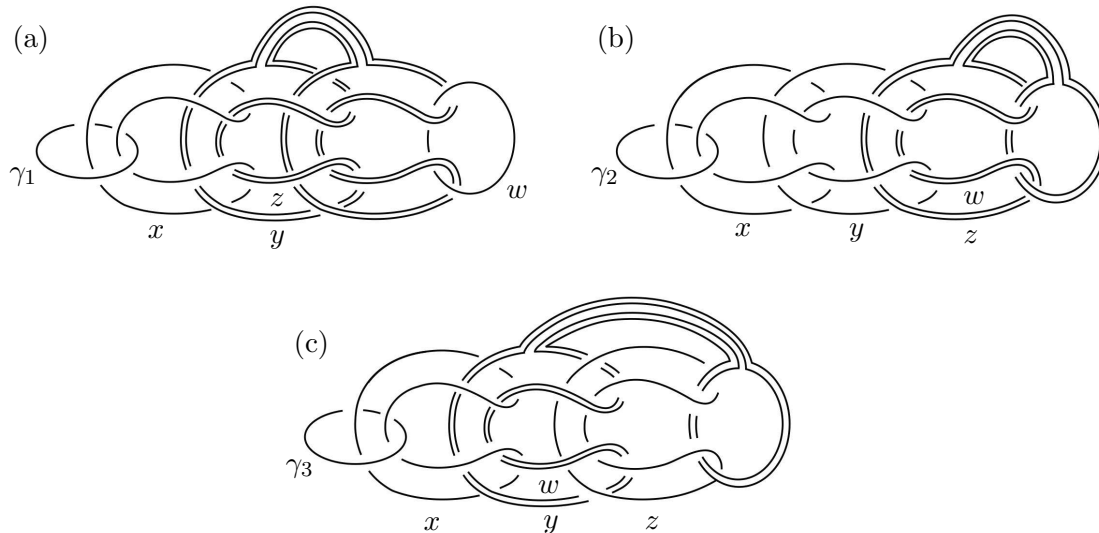
FIGURE 2.4. (a): $\gamma_1 = [x, yz, yz, w]$, (b): $\gamma_2 = [x, y, zw, zw]$, (c): $\gamma_3 = [x, yw, z, yw]$.

The "elementary" weakly null-homotopic links, geometric analogues of the commutators $[h_1, \ldots, h_4]$ that come up in the proof of corollary 2.3 for $k = 4$, are obtained by band-summing two components at a time and then taking a parallel copy, as shown in figure 2.4.

3. The A-B slice problem

We start by recalling the definition of an AB slice link from [10]. Section 3.1 summarizes the relative-slice formulation of the AB slice problem, and section 3.2 defines the notion of a homotopy AB slice link, used in theorem 1.

Definition 3.1. A *decomposition* of D^4 is a pair of compact codimension zero smooth submanifolds with boundary $A, B \subset D^4$, satisfying conditions (1)-(3) below. Denote

$$\partial^+ A = \partial A \cap \partial D^4, \ \ \partial^+ B = \partial B \cap \partial D^4, \ \ \partial A = \partial^+ A \cup \partial^- A, \ \ \partial B = \partial^+ B \cup \partial^- B.$$

(1) $A \cup B = D^4$,
(2) $A \cap B = \partial^- A = \partial^- B$,
(3) $S^3 = \partial^+ A \cup \partial^+ B$ is the standard genus 1 Heegaard decomposition of S^3.

Each side A, B of a decomposition has an attaching circle (a distinguished curve in the boundary: $\alpha \subset \partial A, \beta \subset \partial B$) which is the core of the solid torus $\partial^+ A$, respectively $\partial^+ B$. The two curves α, β form the Hopf link in $S^3 = \partial D^4$. Figure 3.1 illustrates the notion of a decomposition in 2 dimensions. The "trivial" decomposition of D^4 is given by $(A, \alpha) =$ unknotted 2-handle: $(D^2 \times D^2, \partial D^2 \times 0)$ and $(B, \beta) =$ collar: $(S^1 \times D^2 \times I, S^1 \times 0 \times 0)$. See [12], [23], and section 4 below for interesting examples of decompositions.

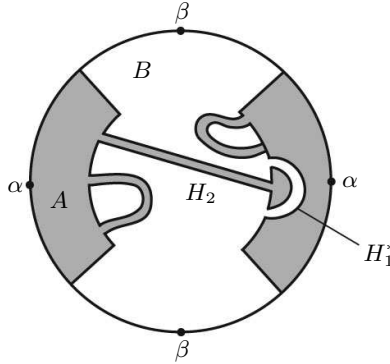FIGURE 3.1. A decomposition in two dimensions: $D^2 = A \cup B$, B is shaded. α, β are linked 0-spheres in ∂D^2. The notation for handles is discussed in section 3.1.

Given an n-component link $L = (l_1, \ldots, l_n) \subset S^3$, consider its untwisted parallel copy $L' = (l'_1, \ldots, l'_n)$.

Definition 3.2. The link L is *A-B slice* if there exist decompositions (A_i, B_i), of D^4 and self-homeomorphisms ϕ_i, ψ_i of D^4, $i = 1, \ldots, n$ such that all sets in the collection $\phi_1 A_1, \ldots, \phi_n A_n, \psi_1 B_1, \ldots, \psi_n B_n$ are disjoint and satisfy the boundary data: $\phi_i(\partial^+ A_i)$ is a tubular neighborhood of l_i and $\psi_i(\partial^+ B_i)$ is a tubular neighborhood of l'_i, for each i.

The restrictions $\phi_i|_{A_i}$, $\psi_i|_{B_i}$ give disjoint embeddings of the entire collection of $2n$ manifolds $\{A_i, B_i\}$ into D^4. Moreover, these re-embeddings are *standard*: they are restrictions of self-homeomorphisms of D^4, so the complement $D^4 \smallsetminus \phi_i(A_i)$ is homeomorphic to B_i, and $D^4 \smallsetminus \psi_i(B_i) \cong A_i$. Analogues of this condition in the homotopy context are introduced in definitions 3.7, 3.9.

[9, 10] reformulated 4-dimensional topological surgery conjecture for free groups in terms of the AB slice problem for GBR's (the Generalized Borromean rings). Figure 4.1 shows a representative link from this family. The proof of theorem 1 will use the relative-slice formulation of the AB slice problem, discussed next.

3.1. **The relative slice problem.** Our summary of this approach to the AB slice problem follows [12], the reader is referred to this reference for further details.

Given a decomposition $D^4 = A \cup B$, without loss of generality it may be assumed [12] that each side A, B has a handle decomposition (rel. collar $S^1 \times D^2 \times I$) with only 1- and 2-handles. Fix the notation: $A = (\partial^+ A) \times I \cup \mathbf{H}_1 \cup \mathbf{H}_2$. As usual in Kirby calculus [16], the 1-handles will be considered as standard 2-handles $\mathbf{H}_1^*$ removed from the collar, $A = (\partial^+ A \times I \setminus \mathbf{H}_1^*) \cup \mathbf{H}_2$. In the illustration in figure 3.1 the side A has three 2-handles and a single 1-handle.

Consider a slightly smaller 4-ball D', equal to the original D^4 minus the collar $(\partial^+ A) \times I$. The removed handles $\mathbf{H}_1^*$ may be considered as 2-handles attached (with zero framing) to D'. Note that in figure 3.1 none of the 2-handles $\mathbf{H}_2$ go over $\mathbf{H}_1^*$. This condition does not have to be satisfied for an arbitrary handle decomposition of a given submanifold of D^4, but it will hold for all decompositions constructed in this paper, as stated in Condition 3.4.

Remark 3.3. The details of the embeddings of $\{A_i, B_i\}$ into D^4 are important in the A-B slice problem. It was shown in [23] that any link $L = (l_1, \ldots, l_n)$ with trivial linking numbers is *weakly* A-B slice: there exist n decompositions $D^4 = A_i \cup B_i$ and disjoint embeddings of the entire collection of $\{A_i, B_i\}$ into D^4 with the boundary data given by L and its parallel copy. To be relevant to the surgery conjecture, these disjoint embeddings have to be *standard*, as discussed in the paragraph following definition 3.2. We record the relevant information about embeddings in condition 3.4; analogous statements in the relative-slice setting and in the homotopy context are given in conditions 3.6, 3.8 respectively.

Condition 3.4. For each side $C = A$, B of a decomposition $D^4 = A \cup B$, the 2-handles $\mathbf{H}_2$ of C do not go over the handles $\mathbf{H}_1^*$ corresponding to the 1-handles of C.

Moreover, in our decompositions each 2-handle of C is embedded in a standard way (i.e. is unknotted) in $D^4 \setminus$ collar on $\partial^+ C$. It follows that (except for a single 2-handle) the 1-handles of each side are in one-to-one correspondence with the 2-handles of

the complement. In the decompositions in section 4 the A-side has a zero framed 2-handle attached to the core of the solid torus $\partial^+ A$, this is the "distinguished" handle of A which does not have a counterpart on the B-side. On the 3-manifold level, the zero-framed surgery on this core transforms $\partial^+ A$ into $\partial^+ B$. A Kirby diagram for B may be obtained by taking a Kirby diagram in the solid torus for A, performing the surgery as above, and replacing all zeros with dots, and conversely all dots with zeros. (Note that the 2-handles in all our decompositions are zero-framed.) To fix the notation, denote the distinguished 2-handle by H_2 (as in figure 3.1), and the rest of the 2-handles by $\overline{\mathbf{H}}_2$.

Suppose an n-component link L is AB slice, with decompositions $D^4 = A_i \cup B_i$, $i = 1, \ldots, n$. Denote by D_0^4 the smaller 4-ball obtained by removing from D^4 the collars on the attaching regions $\phi_i(\partial^+ A_i), \psi_i(\partial^+ B_i)$ of all $2n$ submanifolds $\{\phi_i(A_i), \psi_i(B_i)\}$. Let $\mathcal{H}_2$ denote the 2-handles of all these submanifolds, and $\mathcal{H}_1^*$ the 2-handles removed from the collars, corresponding to the 1-handles. As above, consider $\mathcal{H}_1^*$ as zero-framed 2-handles attached to D_0^4.

A more precise description of $\mathcal{H}_2$, $\mathcal{H}_1^*$ may be given as follows. Add superscript i to the handle notation: the 2-handles of A_i are $\mathbf{H}_2^i = H_2^i \cup \overline{\mathbf{H}}_2^i$, where H_2^i is a "distinguished" 2-handle of A_i, and 1-handles of A_i correspond to $\mathbf{H}_1^{i*}$. Then

$$(3.1) \qquad \mathcal{H}_2 = \bigcup_{i=1,\ldots,n} \phi_i(\mathbf{H}_2^i) \bigcup_{i=1,\ldots,n} \psi_i(\mathbf{H}_1^{i*}), \quad \mathcal{H}_1^* = \bigcup_{i=1,\ldots,n} \phi_i(\mathbf{H}_1^{i*}) \bigcup_{i=1,\ldots,n} \psi_i(\overline{\mathbf{H}_1^i}).$$

Consider the following two links J, K in $S^3 = \partial D_0^4$, which may be read off from the Kirby diagrams of the $\{A_i, B_i\}$. Let J denote the attaching curves of the 2-handles $\mathcal{H}_2$, and K the attaching curves of the 2-handles $\mathcal{H}_1^*$. (Note that $\mathcal{H}_2 \subset D_0^4$, and $\mathcal{H}_1^*$ are attached with zero framings to D_0^4 along K.) The distinction between the drawings of the two links J, K will be clear in each instance when they come up, in particular K will be drawn red (visible online). We refer to the pair (J, K) as the "stabilization" of the original link L. (Note that L is included in J as the attaching curves of the distinguished 2-handles $\{H_i\}$.) The structure of the stabilization links which is a consequence of the duality between the 1- and 2-handles of the two sides of each decomposition is shown in figure 3.2.

Definition 3.5. A link pair (J, K) in $S^3 = \partial D_0^4$ is called *relatively slice* if the components of J bound disjoint, smoothly embedded disks in the handlebody

$$H_K := D_0^4 \cup \text{zero-framed 2-handles attached along } K.$$

If a link L is AB slice, by construction the associated link pair (J, K) is then relatively slice. Moreover, since the embeddings $\phi_i(A_i), \psi_i(B_i)$ are restrictions of self-homeomorphisms $\phi_i(A_i), \psi_i(B_i)$ of the 4-ball, the following analogue of condition 3.4 holds for the relative slicing (J, K).
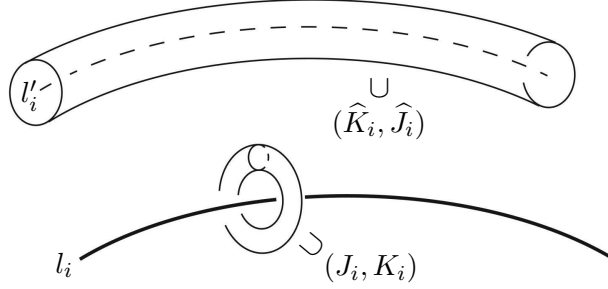
FIGURE 3.2. "Stabilization" associated to an A-B slice link $L = \{l_i\}$: link pairs $(J_i, K_i) \subset$ neighborhood of a meridian to l_i, $(\widehat{K}_i, \widehat{J}_i) \subset$ solid torus neighborhood of a parallel copy l_i'. (The parallel copy l_i' is not part of the link.) A diffeomorphism between the solid tori exchanging their meridian and longitude takes K_i to $\widehat{K}_i$ and J_i to $\widehat{J}_i$.

Condition 3.6. Let S be any submanifold in the collection $\{\phi_i(A_i), \psi_i(B_i)\}$. Then after an isotopy (depending on S) of the handlebody $H_K = D_0^4 \cup_K$ (2-handles) the slices for the components of J corresponding to S do not go over the 2-handles attached to D_0^4 along the components of K corresponding to the same submanifold S.
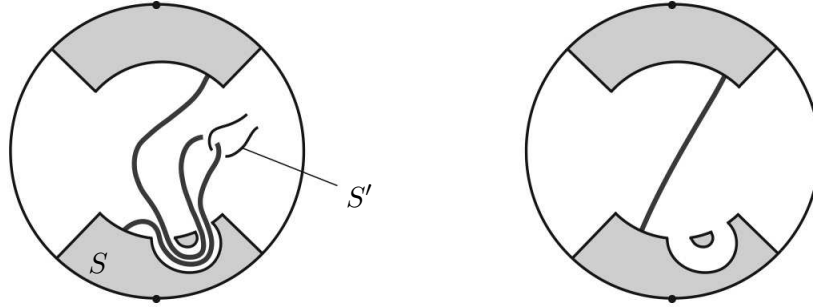
FIGURE 3.3. An illustration of condition 3.6: the 2-handles of a submanifold S may "go over its 1-handles" and link another submanifold S' in D^4 (left). However they do not go over its 1-handles after an isotopy (right), where other submanifolds are disregarded.

Note that the statement of condition 3.6 in general indeed requires an isotopy: as illustrated in figure 3.3, 2-handles of S may link other submanifolds S'. There is an isotopy "straightening out" the 2-handles of S as shown on the right in the figure, but the condition may not be achieved simultaneously for all submanifolds $\{\phi_i(A_i), \psi_i(B_i)\}$.

3.2. **Homotopy A-B slice problem.** We now turn to the definition of a homotopy AB slice link, referred to in the statement of theorem 1. In fact, we state two natural versions of the definition. It will be shown in section 4 that theorem 1 holds in both contexts. The first notion is motivated by link-homotopy theory (section 2.1):

Definition 3.7. (**Link-homotopy A-B slice**) An n-component link L is *link-homotopy A-B slice* if there exist decompositions $D^4 = A_i \cup B_i$, $i = 1, \ldots, n$ and handle decompositions of the submanifolds A_i, B_i so that the corresponding relative-slice problem (J, K) has a link-homotopy solution. That is, in the notation of definition 3.5 the components of J bound disjoint maps of disks Δ in the handlebody H_K. Moreover, the disks Δ are subject to condition 3.8.

Recall that the free group action in the context of the AB slice problem is encoded in condition 3.6. A stronger version of that condition is to omit a reference to an isotopy and require that for no S do its 2-handles pass over the dual representation of its 1-handles. We use this stronger version to define an analogue for a link-homotopy A-B slice link:

Condition 3.8. Let S be any submanifold in the collection $\{\phi_i(A_i), \psi_i(B_i)\}$. Then the maps of disks Δ for the components of J corresponding to S do not go over the 2-handles attached to D_0^4 along the components of K corresponding to the same submanifold S.

Theorem 1 will be established for a stronger version of a homotopy solution which is defined next. Suppose a possibly disconnected codimension zero submanifold $(C, \partial^+ C) \subset (D^4, S^3)$ is given, together with a $1, 2$-handle decomposition of $(C, \partial^+ C)$. As in the beginning of section 3.1, consider the 4-ball $D' = D^4 \setminus (\partial^+ C) \times I$, and consider the 1-handles of C as standard slices $\mathbf{H}_1^*$ removed from the collar. We say that a map $f \colon (C, \partial^+ C) \longrightarrow (D^4, S^3)$ is *homotopy standard* if there exists a 1-parameter family of maps f_t connecting f and id $\colon C \subset D^4$ such that

(1) the restriction of f_t to $(\partial^+ C) \times I \setminus \mathbf{H}_1^*$ is the identity map for all t, and
(2) the images of the 2-handles are contained in D' and are disjoint for all t.

Definition 3.9. (**Homotopy A-B slice**) An n-component link L is *homotopy A-B slice* if there exist decompositions $D^4 = A_i \cup B_i$, a $1, 2$-handle structure for each submanifold A_i, B_i, and disjoint maps of all $2n$ submanifolds $\{A_i, B_i\}$ into D^4 with the boundary data corresponding to L and its parallel copy (as in definition 3.2), such that the restriction to each A_i, B_i, $i = 1, \ldots, n$ is homotopy standard.

Here is a brief outline of the way (singular) slices will be found for a homotopy solution to the relative-slice problem in theorem 1. One may band sum the components of J with (an arbitrary number of) parallel copies of the components of K. These bands correspond to index 1 critical points of the slices with respect to the radial

Morse function on D_0^4. Parallel copies of each component K_i bound disjoint copies of the core of the 2-handle attached to K_i. For a suitable choice of band sums, the resulting band-summed link J' will be null-homotopic. Then the construction of singular slices is completed by capping off the components of J' with disjoint maps of disks in D_0^4. Of course the crucial part of the proof is the construction of the decompositions $D^4 = A_i \cup B_i$ enabling this strategy to succeed.

4. PROOF OF THEOREM 1: A HOMOTOPY SOLUTION TO THE AB SLICE PROBLEM.

As discussed in the introduction, the generalized Borromean rings $\{Bing(Hopf)\}$ form a collection of model surgery problems. We start by noting that highly Bing doubled links in $\{Bing(Hopf)\}$ are still universal for surgery. In the setting of capped gropes this follows from grope height raising: for any $n \geq 2$ the attaching curve of a capped grope g^c of height 2 bounds a capped grope of height n in the (untwisted) thickening of g^c [13].

Proposition 4.1. *Let $n \geq 5$ be fixed. Denote by $\{Bing(Hopf)\}_n$ the links L in $\{Bing(Hopf)\}$ satisfying $ML/(ML)^n \cong MF/(MF)^n$. Then $\{Bing(Hopf)\}_n$ forms a collection of model surgery problems.*
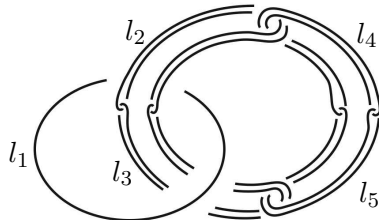

FIGURE 4.1. A link $L \in \{Bing(Hopf)\}_5$

The Milnor group condition picks out the class $\{Bing(Hopf)\}_n$, obtained from the Hopf link by (ramified) Bing doubling performed at least $n - 2$ times. The usually referenced [8] class of model problems is, in this notation, $\{Bing(Hopf)\}_3$, but grope height raising [13] allows one to restrict to any coinitial segment, such as $\{Bing(Hopf)\}_n$, $n \geq 5$. For our purposes, in light of lemma 2.4, it suffices to consider $n = 5$. In the proof of theorem 1 first consider the case where $L \in \{Bing(Hopf)\}_5$ is almost homotopically trivial (in this collection of links this is equivalent to L being Brunnian). This means that L is obtained from the Hopf link by iterated Bing doubling without ramification. This case captures both the complexity of the problem and the idea of the proof; at the end of this section we show what adjustments need to be made in the general case. To be very specific, consider one of the smallest representatives of $\{Bing(Hopf)\}_5$, the 5-component link in figure 4.1.
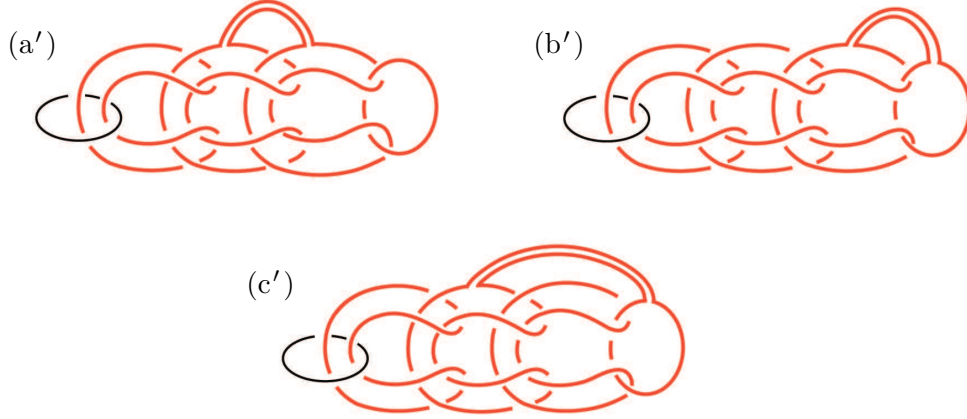
FIGURE 4.2.

To define the decompositions giving a homotopy A-B slice solution for L, consider the links in figure 4.2. These links should be compared with the elementary weakly null-homotopic links in figure 2.4. The crucial difference is that now the band-summed curves are taken *without* a parallel copy, and unlike the links in figures 2.4 each of the three links in figure 4.2 is homotopically *trivial*. The color coding (black/red, visible online) is indicated in preparation for the role these curves are going to play in the relative-slice problem.

We are now in a position to define the relevant decompositions of the 4-ball. For the components l_i, $2 \leq i \leq 5$ consider the trivial decomposition $D^4 = A_i \cup B_i$ where $A_i =$ unknotted 2-handle, $B_i =$ collar on the attaching curve. For the first component consider the decomposition determined by the side A_1 shown in figure 4.3. There is one zero-framed 2-handle and $12 \times 3 = 36$ 1-handles. (This very specific handle description is given for the 5-component link in figure 4.1 in part to take a note of the complexity of our homotopy A-B slice solution. The definition for a general $L \in \{\text{Bing(Hopf)}\}_5$ is given at the end of the proof.) The curves representing 1-handles are dotted (standard notation in Kirby calculus [16]), and they are drawn red (online) due to the role they will play in the relative-slice setting.

A precise definition of the construction in figure 4.3 is as follows. The Kirby diagram is drawn in the solid torus neighborhood of the attaching curve ($\partial^+ A$ in the notation of definition 3.1). Start with a single curve, the zero-framed attaching curve for the 2-handle of A_1, given by the core of the solid torus. Consider a total of 12 links embedded in disjoint 3-balls in (solid torus $\smallsetminus$ its core): six copies of the link (a′) in figure 4.2, four copies of (b′) and two copies of (c′). The proof below shows how the number and types of links are determined by the algebraic structure of the 2-Engel relation (seen in the proof of lemma 2.1). In each of these 12 links the left-most component is band-summed with the previously chosen core curve. It is convenient to use the bands shown in figure 4.3, so that the resulting band-summed curve c is
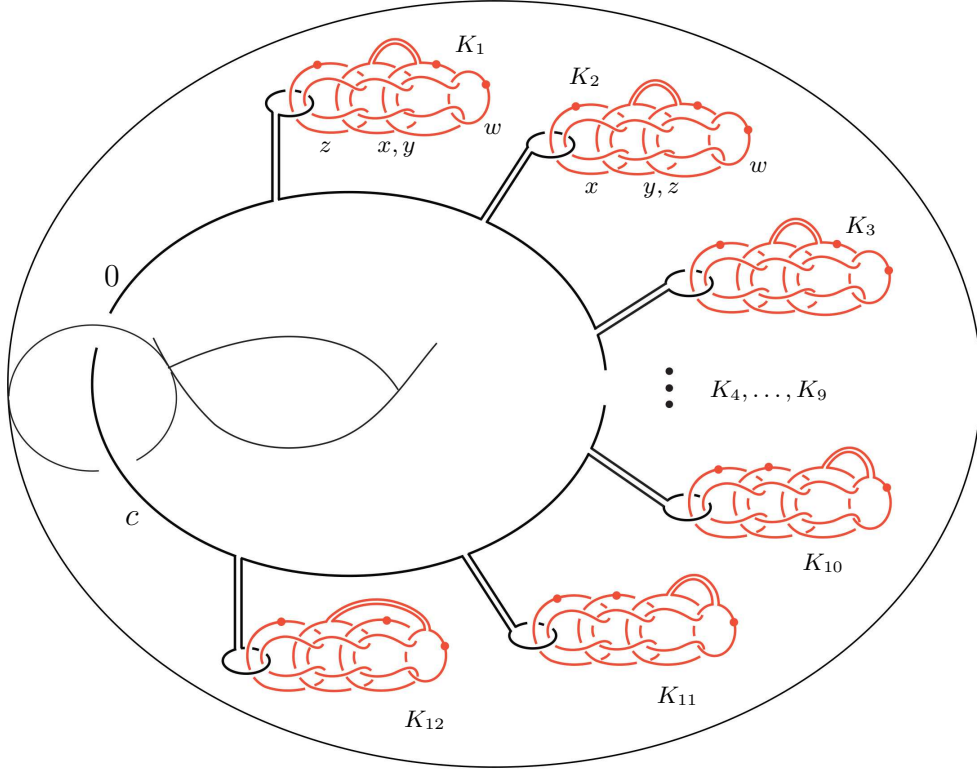
FIGURE 4.3. Part A_1 of the decomposition $D^4 = A_1 \cup B_1$.

isotopic to the original core curve of the solid torus. To fix the embedding of this handlebody A_1 into D^4 we specify that the 2-handle is embedded into the 4-ball in a standard (unknotted) way. This determines the decomposition $D^4 = A_1 \cup B_1$.

Proposition 4.2. *There exist disjoint, homotopy standard maps (in the sense of definition 3.9) of $A_1, \ldots, A_5$ into D^4 with the boundary data corresponding to the link L in figure 4.1.*

Proof. The relative-slice problem corresponding to this embedding question is obtained by embedding the link in figure 4.3 in a neighborhood of l_1 in figure 4.1. We identify the curve c with the component l_1. Then the relative-slice problem concerns the pair of links (J, K) where $J = L$ is the link in figure 4.1, and K consists of the 36 (red) dotted curves in figure 4.3, embedded in a neighborhood of l_1. More precisely, $K = \cup_i K_i$ consists of 12 links K_i.

Using the commutator notation from section 2.2, the component l_1 of the link L in figure 4.1 represents the element

$$(4.1) \qquad l_1 = [[m_2, m_3], [m_4, m_5]] = [m_2, m_3, m_4, m_5] \cdot [m_3, m_2, m_4, m_5]^{-1}$$

in the free Milnor group $M\pi_1(S^3 \smallsetminus (l_2 \cup \ldots \cup l_5)) \cong MF_{m_2,\ldots,m_5}$. The second equality above follows from the Hall-Witt identity (2.4) where, as usual in the Milnor group setting, conjugation may be omitted. The choice of a basepoint for l_1 also does not affect the expression (4.1) since these commutators are of maximal length in the Milnor group, this is discussed further in remark 4.2. Detailed calculations of this type may be found, for example, in [24].

Apply corollary 2.3 to $g = l_1 \in (MF_{m_2,\ldots,m_5})^4$, interpreting the commutators $[h_1, \ldots, h_4]$ appearing in the corollary as the elementary weakly-homotopic links in figure 2.4. It is not difficult to see the number and the types of such commutators $[h_1, \ldots, h_4]$ that come up, going through the proof of lemma 2.1. Specifically, consider each of the two basic commutators in (4.1), for example start with $[m_2, m_3, m_4, m_5]$. To match the current notation with that of lemma 2.1, set

(4.2) $$x = m_2, \ y = m_3, \ z = m_4, \ w = m_5.$$

Lemma 2.1 establishes that $A := [x, y, z, w] \equiv 1$ (is trivial mod the 2-Engel relation) by showing $A^3 \equiv 1$ and $A^4 \equiv 1$. $A^3 \equiv 1$ is proved in (2.8) - (2.11) by representing A in the free Milnor group as a product of two commutators $[h_1, \ldots, h_4]$ corresponding to the link (a) in figure 2.4. Then $A^4 \equiv 1$ is proved in the two paragraphs following (2.12) using commutators corresponding to one copy of (a), two copies of (b) and one copy of (c). Establish a 1-1 correspondence between the commutators $[h_1, \ldots, h_4]$ appearing in the proof and six of the links K_i in the definition of A_1.

We implement the algebraic argument above geometrically as follows. In the relative-slice setting the slices for J may go multiple times over the 2-handles attached to K; we exploit this by band-summing J to the components of K and their parallel copies. Each link K_i consists of three (dotted) components; denote by K_i' the 4-component link obtained by adding to it a parallel copy of the "long" band-summed curve in K_i. (Note that this "reconstructs" the links in figure 2.4.) Every time a commutator $[h_1, \ldots, h_4]$ is used in the proof of lemma 2.1, perform a band-sum joining $l_2, \ldots, l_5$ with the corresponding link K_i', paying a careful attention to the order of indices discussed next.

The first such commutator that comes up in the proof (line (2.6), understood as a 4-fold commutator as in (2.11)) is $[z, xy, xy, w]$. Keeping in mind the notation (4.2), take a band sum of l_4 with the component of K_1 labeled z in figure 4.3. Then band sum l_2 (resp. l_3) with the "long component" of K_1 labeled x, y (resp. its parallel copy). Finally band sum l_4 with the component of K_1 labeled w. There is a $\pm$ choice for each band sum depending on orientations, this choice is discussed in remark 4.1.

The next commutator appearing in the proof is $[x, yz, yz, w]$, and there is a corresponding link K_2 reserved for band-summing into, as indicated in figure 4.3. Proceeding in this manner, perform band-summing into $K_1', \ldots, K_6'$ corresponding to

the proof of lemma 2.1 for $[m_2, \ldots, m_5]$. There is another elementary commutator, $[m_3, m_2, m_4, m_5]$ in the expression (4.1) for l_1. The proof of its weak null-homotopy similarly gives raise to band sums into $K'_6, \ldots, K'_{12}$.

Denote the result of band-summing $l_2, \ldots, l_5$ with all $\{K'_i\}$ by $l'_2, \ldots, l'_5$. The link $(l_2, \ldots, l_5)$, as well as each K'_i, is an unlink. For a suitable choice of bands, $l'_2, \ldots, l'_5$ is also the unlink. (But even with an arbitrary choice of bands, its Milnor group is free: $M\pi_1(S^3 \smallsetminus (l'_2 \cup \ldots \cup l'_5)) \cong MF_{m_2,\ldots,m_5}$.)

Remark 4.1. *(Orientations)* One aspect of commutator calculus and of band-summing was implicit in the argument above. The choice of a meridian (generator of the Milnor group) to each link component depends on the orientation of the based loop representing it. For example, we used expressions for commutators, such as (4.1) and the commutators in figure 2.4, which do not involve negative exponents of the meridians. These expressions assumed a particular choice of orientations. Similarly, different choices of orientations of the link components result in different orientation-preserving band-sums. The commutator identity

(4.3) $$[x^{-1}, y] = [y, x]^{x^{-1}}$$

(cf. [27]) is useful in this context. Conjugation may be omitted in the Milnor group (see remark 4.2), so an iterated application of this identity implies that changing the orientation of any one meridian m_{i_j} in a commutator $[m_{i_1}, \ldots, m_{i_k}]$ inverts the commutator. Therefore various choices of orientations for each such commutator have two possible outcomes overall: $[m_{i_1}, \ldots, m_{i_k}]^{\pm 1}$. We make a choice of orientations so that the commutators $[h_1, \ldots, h_4]$ in the paragraph following remark 4.2 have the correct exponent to match the calculation in the proof of lemma 2.1.

Remark 4.2. *(Conjugation)* It is useful to note another basic fact that conjugations that come up at various points in the proof do not affect calculations in the Milnor group in our setting. The key point is that the component l_1 of the link L (figure 4.1) is in the 4th term of the lower central series $(MF_4)^4 := (MF_{m_2,\ldots,m_5})^4$. The same comment applies to each curve γ_i in the links in figure 2.4. The Milnor group MF_4 is nilpotent of class 4, that is $(MF_4)^5 = \{1\}$. All calculations take place in the *abelian* group $(MF_4)^4$, so conjugation does not have any effect. In particular, the conjugations (of the commutators $[h_1, \ldots, h_4]$) that appear in the statement of corollary 2.3 may be omitted in our present case.

The link $(l_1, l'_2, \ldots, l'_5)$ defined before remark 4.1 may be viewed as a band-sum of the 5-component link L with 12 five-component links contained in disjoint 3-balls: the component l_1 is band-summed with the left-most component of each link as in figure 4.3, and the components $l_2, \ldots, l_5$ are band-summed with the dotted curves and their parallel copies as described above. The algebraic outcome is that the original element (4.1) representing l_1 in $MF_{m_2,\ldots,m_5}$ is multiplied by the inverse of the product of the commutators $[h_1, \ldots, h_4]$ that appear in the proof of corollary 2.3

for $g = l_1$. This may be seen directly by reading off the element represented by l_1 in $M\pi_1(S^3 \smallsetminus (l'_2 \cup \ldots \cup l'_5))$; this is also a special case of the additivity of $\bar{\mu}$-invariants of links under band-summing [22]. By corollary 2.3 the link $(l_1, l'_2, \ldots, l'_5)$ is then homotopically trivial. Capping it off with disjoint null-homotopies in D_0^4 gives the desired disjoint singular slices for the link L.

It is immediate from the construction that condition 3.8 is satisfied: the slice for l_1 does not go over the 2-handles attached along K. (And there is nothing to check for $A_2, \ldots, A_5$ since they do not have any 1-handles.) Moreover, each constructed map $f_i \colon A_i \longrightarrow D^4$, $i = 1, \ldots, 5$, is homotopy standard. Indeed, the singular disks completing the construction of the slices are contained in D_0^4 (the original 4-ball minus collars on the attaching regions), a contractible space. The null-homotopy completing the construction is homotopic within D_0^4 to the unknotted disks corresponding to the original embedding $A_i \subset D^4$, and since each A_i has a single 2-handle there is no disjointess to keep track of. This concludes the proof of proposition 4.2. $\square$

To complete the proof of theorem 1 for the link L we need to consider all 10 submanifolds $\{A_i, B_i\}$. Since $B_2, \ldots, B_5$ are collars, they do not affect the embedding problem. The handlebody B_1 has 36 two-handles and no 1-handles. A Kirby diagram for B_1 is obtained from that of A_1 (figure 4.3) by performing zero-framed surgery on the curve c and replacing the dots with zeros. Denote the zero-framed link corresponding to K_i by $\overline{K}_i$. It is clear that the resulting link $\overline{K} := (\overline{K}_1 \cup \ldots \cup \overline{K}_{12})$ is null-homotopic (in the sense of Milnor) in the solid torus $\partial^+ B_1$. The null-homotopies give raise to a map $(B_1, \partial^+ B_1) \longrightarrow (D^4, S^3)$ whose image is contained in a collar on the attaching solid torus, and where the 2-handles of B_1 are mapped in disjointly. The result thus far shows that the link L is *link-homotopy A-B slice* (definition 3.7).

The missing ingredient in establishing the *homotopy A-B slice* condition is checking that the constructed map $f \colon B_1 \longrightarrow D^4$ is homotopy standard, as required in definition 3.9. Since there are no 1-handles, one has to check only that there is a homotopy from f to the original embedding $B_1 \subset D^4$ (corresponding to the decomposition $D^4 = A_1 \cup B_1$), keeping all 2-handles disjoint. This original embedding consists of disjoint, unknotted 2-handles attached to $\overline{K}$. (The link $\overline{K} \subset$ (solid torus $\partial^+ B_1$) is the unlink when considered in the ambient 3-sphere$= \partial D^4$.)

For example, the link $\overline{K}_{10}$ is shown in figure 4.4. This link is considered in the solid torus = complement of a neighborhood of the curve d, drawn dashed. The obvious Whitney disk W for the null-homotopy of c_3, seen in the picture, intersects c_2. The link (d, c_1, c_2, c_3) is homotopically essential, so there is no Whitney disk disjoint from the rest of the link. However, *after d is omitted*, W is isotopic rel boundary to a Whitney disk W' whose interior is disjoint from c_1, c_2, c_3. Therefore c_1, c_2 bound disjoint embedded, unknotted disks D_1, D_2 in D^4 and c_3 bounds a disk D_3 with self-intersections which are paired up with a Whitney disk W' whose interior is disjoint from each D_i. A Whitney move on D_3 along W' gives a homotopy
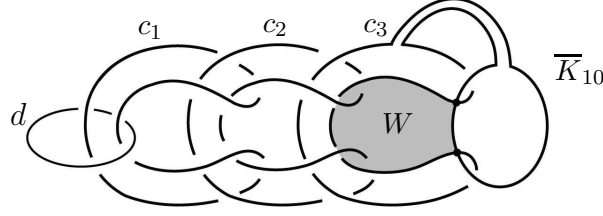
FIGURE 4.4. The link $\overline{K}_{10}$ in the solid torus $S^3 \setminus$ (neighborhood of d). W is a Whitney disk for a null-homotopy.

from the constructed map $\coprod D_i \longrightarrow D^4$ to three disjoint unknotted disks bounding (c_1, c_2, c_3). This is a homotopy rel boundary keeping the disks disjoint at all times, as required in the definition of "homotopy standard".

Analogous arguments apply to each link $\overline{K}_i$, and moreover all resulting disks and Whitney disks in D^4 are disjoint from each other. It follows that the map $f \colon B_1 \longrightarrow D^4$ is homotopy standard, establishing that the link L in figure 4.1 is homotopy A-B slice.

The proof for an arbitrary link $L = (l_1, \ldots, l_n) \in \{\mathrm{Bing}(\mathrm{Hopf})\}_5$, defined by Bing doubling without ramification, is directly analogous. The decompositions $D^4 = A_i \cup B_i$, $i = 2, \ldots, n$ may be taken to be the trivial decomposition, 2-handle $\cup$ collar. The decomposition $D^4 = A_1 \cup B_1$ has one 2-handle and its 1-handles are defined by the links K_i whose number and type are determined, as above, by the proof of corollary 2.3.

Finally consider an arbitrary n-component link L in $\{\mathrm{Bing}(\mathrm{Hopf})\}_5$. In this general setting L is not assumed to be almost homotopically trivial. By assumption all $\bar{\mu}$-invariants of L of length ≤ 4 vanish. It is convenient to consider the proof of lemma 2.4 as a sequence of weakly null-homotopic links $L = L_4, L_5, \ldots, L_n =$ unlink, where all $\bar{\mu}$-invariants of L_i of length $\leq i$ are trivial, $4 \leq i \leq n$. Specifically, the proofs of lemma 2.1, corollary 2.3, and the additivity of Milnor's invariants under band sums [22] show that band-sums of L with links in figure 2.4 may be performed so that the result is L_5 as above. Recall that the 5-component elementary weakly null-homotopic in figure 2.4 correspond to 4-fold commutators $h_1, \ldots, h_4$ in corollary 2.3. Then band-sums with elementary links corresponding to 5-fold commutators yield L_6, etc.

The number and types of elementary weakly null-homotopic links that come up in this process for L are determined by the proof of lemma 2.1. Each of the elementary links has a *null-homotopic* counterpart, illustrated in figure 4.2. The decompositions $D^4 = A_i \cup B_i, i = 1, \ldots, n$ are then defined analogously to figure 4.3: each A_i has a single zero-framed 2-handle, and the $\{A_i\}$ incorporate all red (dotted) links that are needed in the algebraic argument. The proof of proposition 4.2 then goes through to

give disjoint, homotopy standard maps of $A_1, \ldots, A_n$ into D^4. The proof that the submanifolds B_i admit homotopy standard maps into the collar is directly analogous to that in the almost trivial case considered above. $\qquad \square$

Remark 4.3. The proof of theorem 1 essentially relied on the asymmetry of the roles played by the 1- and 2-handles of the submanifolds A_i, B_i: many parallel copies of each dual 1-handle core are used in the relative slicing whereas a single copy of each 2-handle must be mapped disjointly. This should be compared with the statement of the Round Handle Problem in section 5 where the analogous links are in fact symmetric, see remark 5.6.

4.1. **A link-homotopy solution for the Borromean rings.** In light of proposition 4.1, in the proof of theorem 1 it sufficed to consider links in $\{\mathrm{Bing}(\mathrm{Hopf})\}_5$. The Borromean rings (Bor) is the simplest and best known example of a link in $\{\mathrm{Bing}(\mathrm{Hopf})\}$, however since Bor is not in $\{\mathrm{Bing}(\mathrm{Hopf})\}_5$, the proof did not apply directly to this link. We sketch a modification needed to give a link-homotopy solution of the A-B slice problem for Bor.

Consider the decomposition $D^4 = A_1 \cup B_1$ as in the proof of theorem 1. For $i = 2, 3$ let $D^4 = A_i \cup B_i$ where $B_i = T \times D^2$ is a thickening of the torus T with a single boundary component, embedded in a standard way in D^4. Then A_i is obtained from the collar on the solid torus $\partial^+ A_i$ by attaching two zero-framed 2-handles to the Bing double of the core, figure 4.5.
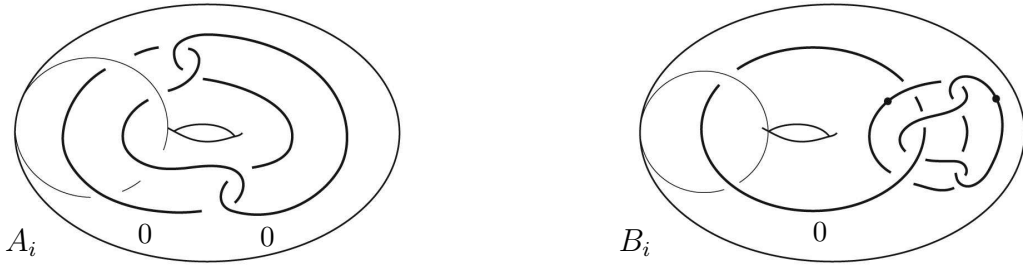

FIGURE 4.5. $D^4 = A_i \cup B_i$, $i = 2, 3$.

The link in figure 4.1 is obtained from Bor by Bing doubling two of the components. This Bing doubling is incorporated in the definition of A_2, A_3, so proposition 4.2 applies directly to yield a link-homotopy solution for A_1, A_2, A_3: disjoint maps $f_i \colon A_i \longrightarrow D^4$ so that the attaching curves $\{\alpha_i\}_{i=1,2,3}$ form the Borromean rings, all 2-handles of $\{A_i\}$ are mapped in disjointly, and condition 3.8 is satisfied.

Since each component of Bor bounds a genus one surface in the complement of the other components (figure 4.6), the submanifolds B_2, B_3 admit disjoint embeddings in the complement of $\coprod f_i(A_i)$. Since B_1 is a collar, it does not affect the embedding problem. This completes a link-homotopy solution for Bor.
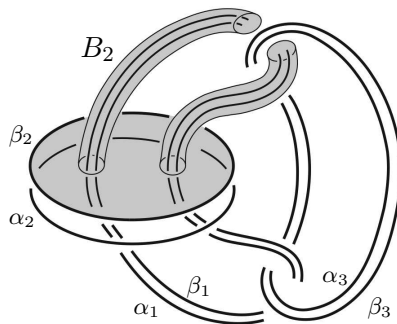
FIGURE 4.6.

5. The Round Handle Problem, 5-dimensional s-cobordisms, and general doubles.

We call attention to an explicit construction of a smooth 4-manifold M with $\partial M \cong \mathcal{S}_0(Wh(L))$, the zero-framed surgery on the Whitehead double[2] of an initial k-component link $L \subset S^3$. Whitehead doubling replaces each component l_i of L with a satellite:
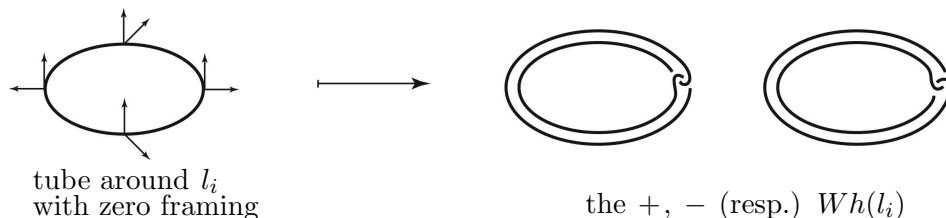

tube around l_i
with zero framing   the $+$, $-$ (resp.) $Wh(l_i)$

FIGURE 5.1. Whitehead doubling

M is the source of an F_k (free group) - surgery problem, rel boundary, with target $\natural_{i=1}^k S^1 \times D^3$. The problem has vanishing (Wall) surgery obstruction iff all the linking numbers $(l_i, l_j) = 0$, $1 \leq i, j \leq k$. Solving this surgery problem constructs a slice complement for $Wh(L)$. The well-known universal surgery problems [8] arise when L is some ramified Bing double of the Hopf link: . It is routine to build the maps and cover them with required normal data, so we will only describe the construction of M.

By definition $M = M(L)$ is obtained (see figure 5.2) by attaching k pairs of plumbed 2-handles to D^4. The attaching circles are $2L$, i.e. the link L and a parallel copy with all framings equal to zero. There is a sign choice, $\pm$, at each plumbing point.

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[2]As seen in figure 5.1 $Wh(L)$ is not defined until the clasp signs ± 1 on each component are specified. We drop this detail from our notation, but point out where in the construction of M these signs are seen. Assertions about $Wh(L)$ apply to *all* sign choices.
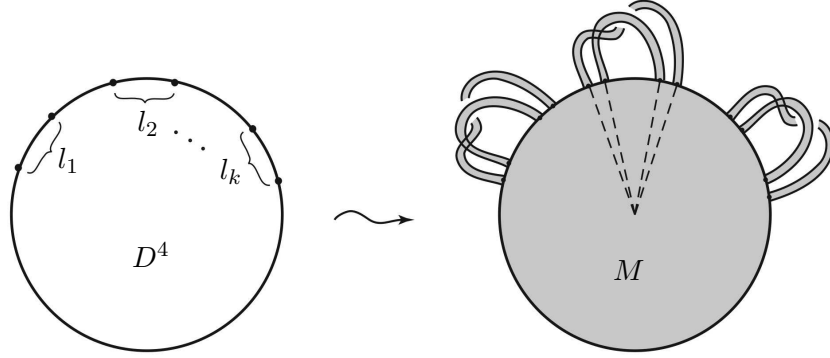
FIGURE 5.2. Schematic of M

This is the place where the sign of the clasp in the Whitehead double is determined and will not be commented on again.

Lemma 5.1. $\partial M \cong \mathcal{S}_0(Wh(L))$.

The proof is given below (following definition 5.5). Henceforth assume all linking numbers $(l_i, l_j) = 0$. From this we see k hyperbolic pairs (one displayed using dotted lines in figure 5.2) over $\mathbb{Z}[F_k]$, i.e. $2k$ spherical classes of the form (2-handle core)$\cup$cone to origin (∂core), representing $\oplus_k \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$. F_k is the free fundamental group of M generated by the plumbings. The nonsingularity of this form is equivalent to the natural map $\alpha \colon \partial M \longrightarrow \natural_k S^1 \times S^2$ being a $\mathbb{Z}[F_k]$-homology isomorphism.

A consequence of the (still open) topological surgery conjecture is that there exist a topological 4-manifold N, $\partial N = \partial M$, with a homotopy equivalence β extending α:

$$\beta \colon (N, \partial) \longrightarrow (\natural_k\, S^1 \times D^3, \natural_k\, S^1 \times S^2).$$

The entire thrust of the A-B slice discussion was to find a way of contradicting the existence of N using "low-tech"[3] nilpotent invariants of L. The philosophy was that $Wh(L)$ has little to grab onto, certainly no nilpotent invariants so it was preferable to "undouble" the problem and work directly with L. In this section we describe a variant of this approach which we call the "round handle problem" (RHP). Like the A-B slice problem RHP can be translated into a question about slicing some "stabilized" version of L. The advantage of the RHP variant is that the stabilization is better controlled - the ramification of dotted (red) curves featured in the homotopy A-B slicing (section 4) cannot occur in the RHP context. The "disadvantage" of the new context is that an obstruction to "stable slicing" - which could be based on $\bar{\mu}$-invariants - contradicts the union (4D surgery conjecture $\wedge$ 5D s-cobordism

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[3]It is an interesting question whether there is a "high-tech" approach to refuting the surgery conjecture. We know no useful reformulation in gauge theory.

conjecture) and we would not gain any information on which fails[4] - merely that something goes seriously wrong.

Remark 5.2. *Aside on proper s-cobordism.* It is an old observation (see the next paragraph) that the "proper 5D s-cobordism" conjecture implies both 4D surgery and 5D s-cobordism conjectures. So, if one insists, a specific failure could be pointed to (if the RHP has no solution).

The proper or p-s-cobordism theorem was established by L. Taylor (Ph.D. thesis, UC Berkeley, 1972) for p-s-cobordisms of dimension 6 and higher. It is an open question whether his thesis result extends in the topological category to dimension 5. The algebraic setting for the general obstructions is complicated a bit by properness but the case of greatest interest is when the global fundamental group is free and the fundamental group of the end also (the same) free group. In [7] a variant of the low dimensional surgery sequence is established. It is straightforward that a successful extension of Taylors thesis (in the above free case) would convert the published variant to the full topological surgery sequence in these dimensions. It was considered so unlikely that this extension existed when [7] was written that this point is not explicitly made in the paper.

Again given L define $R = D^4 \cup k$ round 1-handles. In this dimension a round 1-handle is $(D^1 \times D^2 \times S^1, S^0 \times D^2 \times S^1)$. R is built by attaching the ith round 1-handle r_i to a meridian m_i to l_i *and* a parallel copy l_i' of l_i (lying beyond the meridian), figure 5.3.
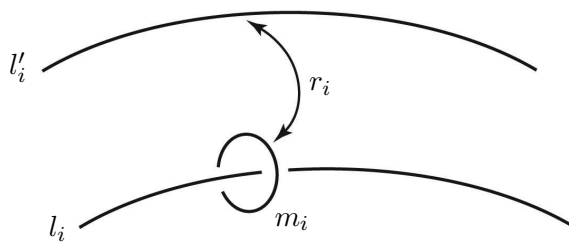

FIGURE 5.3. Attaching a round handle

Suppose the link L, considered as lying in ∂R, is slice in R, meaning L bounds k disjoint, topologically flat 2-disks in R, equivalently bounds k disjoint topological 2-handles in R. Let T be the "slice complement", i.e. the manifold with boundary obtained by deleting the interior of those k 2-handles, $T = R \smallsetminus \text{int}(\coprod_k 2\text{-handles})$. The proofs of the following two lemmas are postponed until after the definition of the Round Handle problem (definition 5.5).

---

[4]However see remark 5.2.

Lemma 5.3. $\partial T \cong \partial M$.

Actually N is a candidate for the slice complement T. By this we mean, if N exists we can reconstruct a manifold R' very much like R by attaching k 2-handles to N.

Lemma 5.4. *If N exists we can form $R' = N \cup k$ (2-handles) so that there exists a $5D$ s-cobordism W, which is a smooth product over the boundary, joining R' to R.*

Thus is we assume 4D surgery and 5D s-cobordism conjectures (we call this package the surgery sequence conjecture (SSC)) then L is slice in R. The slice disks S may be taken to be topologically transverse [13] to the k cocores $(D^2 \times S^1)_i$ of r_i, the round 1-handles. Cutting R open along the cocores recovers the 4-ball D^4 with the promised "stabilization" $\widehat{L}$ of L, i.e. two copies of the 1-manifolds $=$(slice disks) $\cap$ $\coprod_{i=1}^{k}(D^2 \times S^1)_i$. The components of $\widehat{L}$ are now seen to co-bound some disconnected planar surface P made from fragments of the slice disks. To summarize: $L = \partial S \subset R$ yields $\widehat{L} = \partial P \subset D^4$. One may understand the combinatorial possibilities for $\widehat{L}$ and P and attempt to see if any are compatible with known properties of $\bar{\mu}$-invariants. The chief feature of $\widehat{L}$ is that $\widehat{L} = L \cup Q \cup \widetilde{Q}$ where Q is contained in the meridional solid tori M_i with core circles m_i, and $\widetilde{Q}$ is identical to Q but transported by the zero-frame preserving homeomorphism from M_i to L'_i, the parallel solid torus with core l'_i (see figure 5.3). In practice the components of Q "help" with the existence of P by canceling the $\bar{\mu}$-obstructions but they beget harmful $\widetilde{Q}$ with new $\bar{\mu}$-obstruction to bounding P.

Definition 5.5. The Round handle problem RHP is to determine whether non-trivial $\bar{\mu}$ of L survive all possible stabilization processes to prevent any $\widehat{L}$ bounding P as described above.

Remark 5.6. We summarize the difference between the link stabilization formulations of the A-B slice problem and of the RHP. They have similar set-ups: for each link component l_i of L both problems consider two solid tori M_i, L'_i, neighborhoods of a meridian m_i to l_i and of a parallel copy l'_i, compare figures 3.2, 5.3. To begin with, in both setups there are identical "stabilization" links in the solid tori $\{M_i, L'_i\}$, and the question is whether the link L bounds planar surfaces in D^4 whose other boundary components correspond to the stabilization links. The distinction between the two is that in the AB slice problem one is allowed to take an arbitrary number of parallel copies of the "helping" red curves (corresponding to the passage of the slices over the 2-handles attached along these curves), while their counterpart curves in the dual solid tori do not have to be ramified. In the RHP the curves in M_i, L'_i match precisely, corresponding to the passage of the slices over the round handles. The proof of theorem 1 in section 4, using the 2-Engel relation, crucially relies on taking parallel copies, so it does not go through in the RHP setting. This is the basis for our comment in the introduction that the non-repeating $\bar{\mu}$-invariants remain a

candidate for an obstruction to the surgery sequence conjecture, leading to definition 5.5.

Proof of Lemma 5.1. We use the usual conventions (cf. [16]) for handle diagrams (Kirby calculus). The argument is a local handle computation inside each of the k solid torus neighborhoods L_i of l_i. We draw dual circle marked d to define - via its complement - the solid torus L_i. The plumbed pair is diagrammed as two zero-framed 2-handles and one 1-handle (circle with dot) in L_i, figure 5.4.
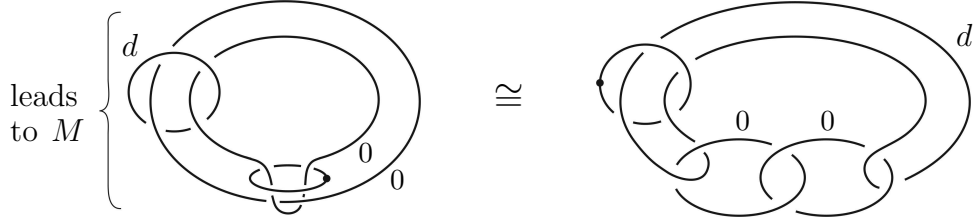

FIGURE 5.4.

To understand only the boundary we may cancel the hyperbolic pair and replace the dot with a zero, obtaining figure 5.5.
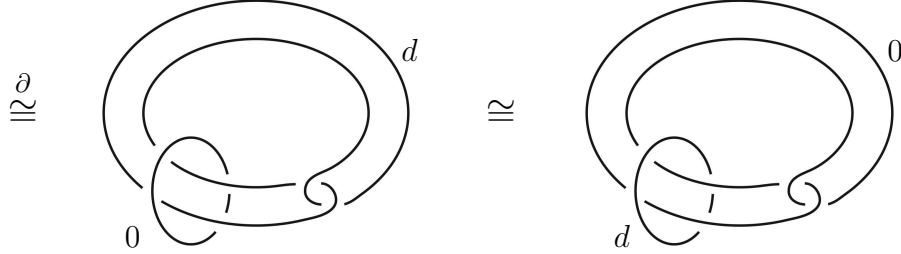
FIGURE 5.5.

Notice that the calculation did *not* assume unknottedness of l_i, it takes place in the solid torus L_i, reembedded as unknotted for convenience only. $\square$

Proof of lemma 5.3. Again the calculation can be localized to the solid tori L_i. The round handle attachment indicated in figure 5.3 is shown in terms of ordinary handles in figure 5.6.

Slicing l_i has the effect, in so far as the boundary is concerned, of placing either a dot or a zero on that component, returning us to the first panel of figure 5.4, i.e. a diagram of ∂M. $\square$

Proof of lemma 5.4. We have defined a diffeomorphism $d\colon \partial R \longrightarrow \partial R'$. The first step is to extend this to a simple homotopy equivalence $h\colon (R, \partial R) \longrightarrow (R', \partial R')$.
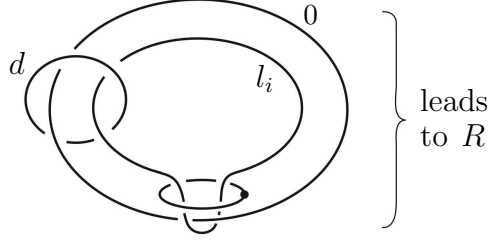
FIGURE 5.6.

R' is homotopy equivalent to $(\vee_k S^1) \vee (\vee_k S^2)$, so the only possible obstruction to extending d as a map is $\mathcal{O} \in H^3(R, \partial R; \pi_2 R')$.

The boundary of a basis of these relative 3-cocycles are the 2-sphere S_i factors, $1 \leq i \leq k$, in $\partial R \cong \mathcal{S}_0(L) \sharp (\sharp_k S^1 \times S^2)$. N has k distinguished tori T_i, the natural genus one Seifert surface for $Wh(l_i)$ capped off by surgery on l_i. In $R' = N \cup k$ 2-handles, the 2-handles h_i surger (along a copy of l_i) T_i into a 2-sphere S_i' and $d(S_i) = S_i'$. We use the notation:

$$\partial(T_i \times [0,1] \cup h_i^-) = T_i \times 0 \cup S_i', \text{ where}$$

$$h_i^- = (D^2 \times D^1, \partial D^2 \times D^1) \subset (D^2 \times D^1 \times D^1, \partial D^2 \times D^1 \times D^1) = h_i.$$

Since each T_i is null-homotopic in N, S_i' bounds a singular 3-ball b_i in N lying in $(T_i \times [0,1] \cup h_i^-) \cup_{T_i \times 0} (\text{Cone}(T_i))$, showing $\mathcal{O} = 0$.

Since R' is also homotopy equivalent to $(\vee_k S^1) \vee (\vee_k S^2)$ and $d \colon \partial R \longrightarrow \partial R'$ lines up the generators bijectively, h is automatically a homotopy equivalence. Since $Wh_1(F_k) = 0$ [30], h is also a simple homotopy equivalence.

Covering each map with (arbitrary) normal data we obtain two structures on (R', ∂):

$$\text{id} \colon (R', \partial) \longrightarrow (R', \partial), \text{ and } h \colon (R, \partial) \longrightarrow (R', \partial).$$

The possible obstructions to a relative normal cobordism between id and h lies in $[(R', \partial), (G/TOP, *)]$. The Postnikov tower for G/TOP begins with a $K(\mathbb{Z}_2, 2)$ and then a $K(\mathbb{Z}, 4)$ [21]. But $H^3(R', \partial; \mathbb{Z}_2) \cong 0$ so the first obstruction vanishes, and the second obstruction may be identified as signature(R)−signature(R')=0, so it vanishes as well. Thus id and h are normally cobordant, rel boundary, via W^5. According to Wall [33] there is a surgery obstruction $\sigma(W) \in L_s^5(F_k) \cong L^5(\{e\}) \oplus_{i=1}^k L^4(\{e\})$, by the splitting principle [2]. $L^5(\{e\}) \cong 0$ and $L^4(\{e\})_i \cong 8\mathbb{Z}$ given by the signature of the spin 4-manifold dual to each free generator. It is possible to modify our choice of W to W' to kill these k surgery obstructions. To change the ith obstruction by ± 8 replace an embedded $S^1 \times D^4$ parallel to the ith free group generator with an embedded $S^1 \times (E^8\text{-manifold} \smallsetminus \dot{D}^4)$. Knowing $\sigma(W') = 0 \in L_s^5(F_k)$, W' is normally cobordant, rel its boundary, to an s-cobordism W'' from R' to R. $\square$

5.1. **General doubles.** In the early days of 4-manifold topology decomposition theoretic properties of Whitehead doubling played a key role [6]. But in the current study of non-simply connected surgery we are completely divorced (and perhaps it is our loss) from point set topology so it becomes a hindrance to adhere to the literal meaning of "Whitehead double". We suggest a more algebraic generalization (in fact two) which will be exploited in section 6. Surface genus is a natural parameter in the generalization and we will only exploit the genus one case so the reader may restrict the definition below accordingly. The purpose of Whitehead doubling, from our current perspective is to weaken a link L so that the problem of slicing its replacement $Wh(L)$ can be expressed as an (unobstructed) 4D surgery problem (which if solved would produce a candidate manifold for the link slice complement.) The construction of M (see figure 5.2) illustrates this strategy. But if this is all we want then we may define a "general double" as follows.

Consider a (usually) disconnected surface $S = \coprod_{i=1}^{k} S_i \subset S^3$, with ∂S_i a simple closed curve, $\partial S =: K$ is a k-component link. The main case (used in section 6) where $\text{genus}(S_i) = g_i = 1$ is considered first.

We assume that in some basis of simple closed curves on S the Seifert form is:

$$
\bigoplus_{j=1}^{J} \quad \begin{array}{c c} & \begin{array}{cc} x_j & y_j \end{array} \\ \begin{array}{c} x_j \\ y_j \end{array} & \left| \begin{array}{cc} 0 & \pm 1 \\ 0 & 0 \end{array} \right| \end{array}, \qquad J = \sum_{i=1}^{k} g_i.
$$

This makes K a "good boundary link" [7] and when L has vanishing linking numbers $K = Wh(L)$ is of this type. Good boundary links are known to admit unobstructed surgery problems for constructing a slice complement. (It is not known, in general, when these surgery problems have topological solutions.)

Assume $g_i = 1$, then S_i is a $\pm$ plumbing of the untwisted bands which by a pushoff, corresponding to the lower left Seifert matrix entry, are disjoint and with linking number $<x_i, y_i^- >= 0$. ($<x_i, y_i^+ >= \pm 1$.)

Lemma 5.7. $\mathcal{S}_0(K) \cong \partial\overline{M}$ where $\overline{M}$ is obtained from D^4 by attaching k $\pm$ plumbed pairs of 2-handles. $\overline{M}$ is the source of an unobstructed surgery problem for building a slice complement for K. Each pair is attached to zero framed x_i and y_i^-.

Proof of lemma 5.7. The schematic for $\overline{M}$ is very similar to figure 5.2, except now x_i, y_i^- are not necessarily l_i and a parallel copy l_i' as implied in that figure. As in the proof of lemma 5.1 we can localize the Kirby calculation, this time to a genus two handlebody H (rather than a solid torus) containing a component K_i of K. We show initial and final Kirby diagrams in H are boundary equivalent, see figure 5.7.

The statement about surgery is immediate: π_2 is a free module over $\mathbb{Z}[F_k]$ and the intersection form is manifestly hyperbolic. $\square$
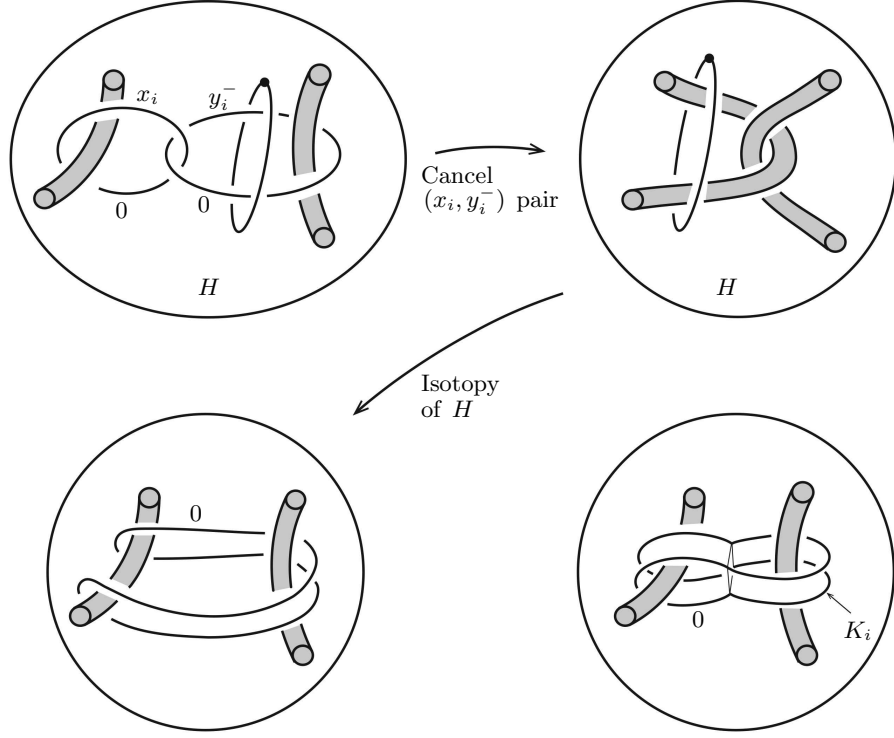
FIGURE 5.7.

Note. The proof shows that the diffeomorphism type of $\mathcal{S}_0(K)$ does not depend on the choice of arcs which join x_i to y_i^- and define the plumbing, similarly for $\overline{M}$ in lemma 5.7.

Now drop the genus restriction. In this case $\overline{M}$ has some further 2-handles. Beyond the $\pm$ plumbed pairs for each (x_i, y_i^-) when more than one, say g_i, pairs lie on S_i we must add an additional $g_i - 1$ 2-handles to add relations collapsing the g_i free generators $e_1, \ldots, e_{g_i}$ dual to these plumbings to one. The relations can be taken to be a chain of loops in the simplest possible form (shown in figure 5.8) which read $e_1 e_2^{-1}, \ldots, e_{g_i-1} e_{g_i}^{-1}$. For $g_i > 1$ the required calculation takes place in a genus $2g_i$ handlebody H and is given in figure 5.8 as the proof of lemma 5.8 (stated below).

Lemma 5.8. *Now considering general doubling (without genus restriction), we have defined a 4-manifold $\overline{M}$ (the details of the attaching circles for the last 2-handles $\{h\}$ are as specified in figure 5.8). Again $\overline{M}$ is the source for an unobstructed surgery problem for building a slice complement (with π_1 freely generated by meridians). $\mathcal{S}_0(K) \cong \partial \overline{M}$.*
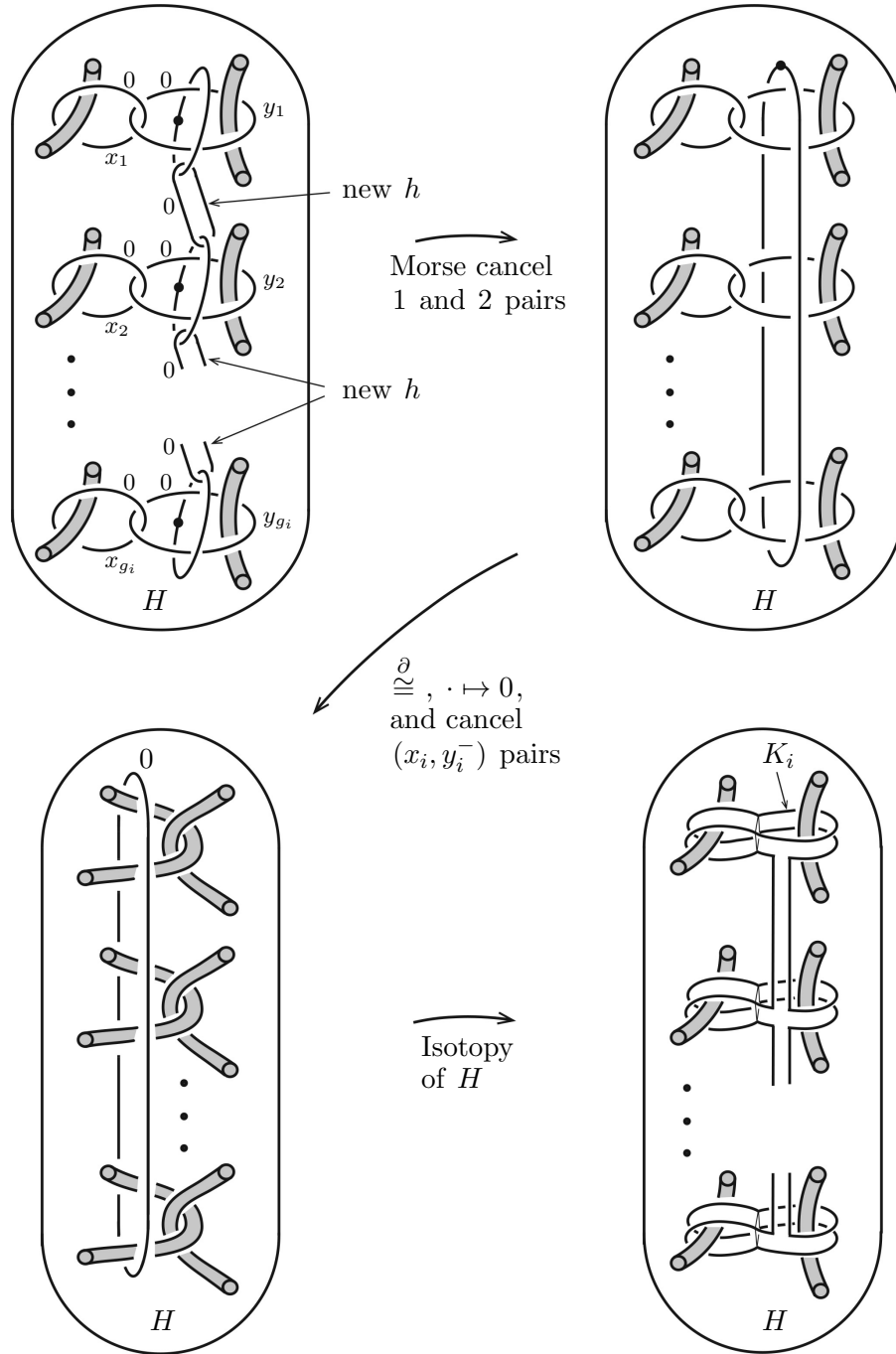
FIGURE 5.8.

Note on terminology: General doubles generalize Whitehead doubles but have little relation to Bing doubles. Bing doubles do not really weaken links but merely push nilpotent invariants further down the lower central series.

6. New universal surgery problems

For us surgery problems ("problems") are in 4D and, as studied in [33], non-singular over the integral group ring of the target. It is irrelevant whether they arise in a closed or bounded context, the set up being a normal map $f\colon M \longrightarrow P$ from a 4-manifold to a Poincaré space (or pair, but we drop the boundary from the notation). A "solution" is a normal cobordism to a simple homotopy equivalence. In attempting to solve a given problem one typically struggles to embed 2-complexes X in the source M of the surgery problem, where X in some sense approximates $S^2 \vee S^2$. There is considerable universality governing which 2-complexes turn up, e.g. $S^2 \vee S^2$-like capped gropes[5] [13], and the neighborhoods $\mathcal{N}(X) = M'$ of these standard examples are then themselves sources of bounded surgery problems to a $P' \simeq \vee S^1$. This set $\{M' \longrightarrow P'\}$ or in abbreviation $\{M'\}$ constitutes a countable, but easily parametrized list of surgery problems, called "universal"[6] which if solvable imply *all* unobstructed problems are solvable. Solutions are transitive (see lemma 6.3): if $M \subset M'$ (and captures the surgery kernel) then a solution to M' solves M as well.
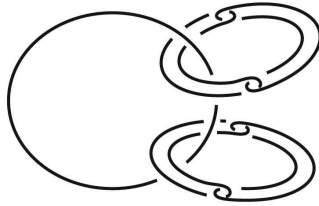

FIGURE 6.1.

The usual list of universal surgery problems is $\{Wh(Bing(Hopf))\}$. One starts with the Hopf link: , then does any (non-zero) amount of iterated ramified Bing doubling. For example, Bing doubling the right component with ramification two (and no iteration) yields the link in figure 6.1.

Finally, Wh means apply (without iteration) ramified $\pm$ Whitehead doubling to *each* component. See figure 6.2 for an example.

The surgery problems M are exactly the M as in figure 5.2 and lemma 5.1 where $L = Bing(Hopf)$ (possibly ramifying the components with parallel copies), and by lemma 5.1 solving M produces a "free" slice complement, i.e. one with free π_1, freely generated by meridians, for $Wh(Bing(Hopf))$, i.e. constructs a manifold $\simeq \vee S^1$ (generators = meridians) with boundary $\mathcal{S}_0(Wh(Bing(Hopf)))$.

---

[5]the right hand side of figure 6.2 is a height$=1$ $S^2 \vee S^2$-like capped grope.
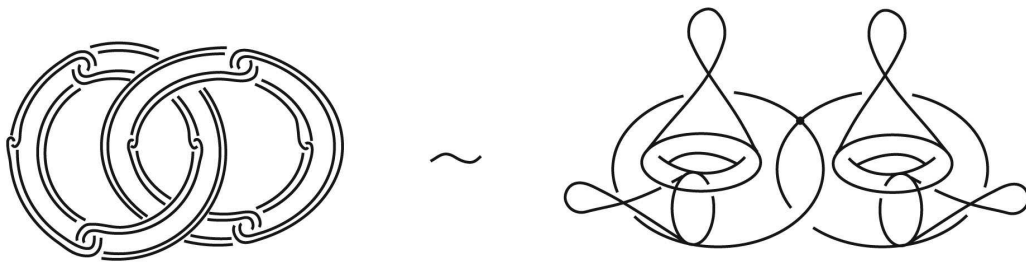[6]also called "model" or "complete"

FIGURE 6.2. The link on the left is obtained by Bing doubling both components of the Hopf link, and then Whitehead doubling each resulting component. The figure on the right illustrates the corresponding surgery kernel.

A consequence of a technique called "reembedding" is that under a natural partial order, where more ramification, more Bing doubling, and more double points means "smaller" or "subset" any co-initial segment of $\{M'\}$ is also universal.

There seems to be a hierarchy of links (with all linking numbers vanishing). The strongest - hardest to slice their Whitehead doubles - are the ramified iterated Bing doubles of Hopf, Bing(Hopf). These are universally hard. At the other extreme are the (homotopically trivial)$^+$ - links L (triv$^+$). By definition these are the k-component links L which if turned into L^+, a $(k+1)$-component link obtained by adding a parallel copy to any single component, L^+ is homotopically trivial in the sense of Milnor (see section 2.1). It is known [15] that for such L the associated problem M can be solved. To be very explicit consider three cases, figure 6.3.
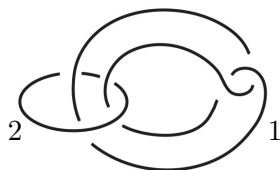
$\bar{\mu}_{1122} = 1$
Wh link is triv$^+$
M(Wh) solved.

$\bar{\mu}_{1122'} = 1$
Wh$_{\mathrm{ram}}$, slightly ramified,
not triv$^+$.
M(Wh$_{\mathrm{ram}}$): open question.
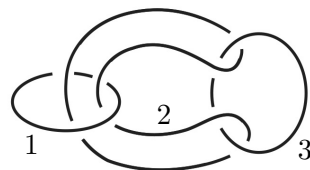
$\bar{\mu}_{123} = 1$
Bor. rings $\in \{$Bing(Hopf)$\}$.
M(Bor) a presumably typical
universal problem.

FIGURE 6.3.

The basic Whitehead link is "held together" by a nontrivial $\bar{\mu}_{1122}$ but it is triv$^+$ since both indices must be repeated to obtain a non-zero value. Wh$_{\mathrm{ram}}$, a temporary notation for the 3-component link obtained by ramifying one component of Wh, middle figure 6.3, still has no nontrivial $\bar{\mu}$ invariants with non-repeating indices, but it is no longer triv$^+$. Despite 15 years of work its status is still open: we do not

know if the surgery problem $M(Wh_{ram})$ is solvable or even if $Wh(Wh_{ram})$ is slice (with any fundamental group). Finally, the Borromean rings seems typical (though at the top of the partial order) of links whose associated surgery problems M form the universal family. Bor (and other members of $\{Bing(Hopf)\}$) are held together by a non-repeating $\bar{\mu}$ invariant.

In this section we exhibit new universal links which are general (see section 5) doubles of links L which are more like Wh_{ram} than Bor. For convenience of the reader we restate the theorem from the introduction:

Theorem 2. *There is a family of links $\{K\}$ for which the problem of constructing free slices constitutes a universal problem, where each $K \in \{K\}$ is of the form:*

$$D(Ram(h\text{-}triv)),$$

a general double of a ramified homotopically trivial link.

Remark 6.1. Any link L with a non-trivial $\bar{\mu}$ may be ramified to produce L_r with a nontrivial non-repeating $\bar{\mu}$, so theorem 2 does not yield a universal set of links based on general doubling of a link with all non-repeating $\bar{\mu}$ invariants vanishing. However it was a surprise to see any collection of universal problems tied so closely to homotopically trivial links. The new universal problems certainly focus attention on the impact of ramification. This is the second surprise and is perhaps the "other side of the coin" to section 4 where ramification is exploited to construct an unexpected homotopy solution to the A-B slice problem.

Remark 6.2. Part of what we have learned is that what 15 years ago was regarded as a minor technical distinction - homotopically trivial versus (homotopically trivial)$^+$ - may instead lie at the heart of the matter. (See [5] for recent results related to this problem.) At present, $Wh(Wh_{ram})$ is certainly the most interesting link slice problem.

The next lemma expresses the transitivity of surgery solutions; we will use it to pass from the original to the new universal problems.

Lemma 6.3. *(Transitivity of surgery) Let $f \colon M \longrightarrow P$ be an unobstructed problem with surgery below the middle dimension completed so that $f_\sharp$ is an isomorpshim on π_1 and so that the kernel module*

$$K(f) = \ker(H_2(M; \mathbb{Z}[\pi_1 P]) \longrightarrow H_2(P; \mathbb{Z}[\pi_1 P]))$$

is a free module with hyperbolic intersection form λ and standard μ. Suppose $K(f)$ is represented by an embedded (possibly disconnected) 4-manifold $(W, \partial) \subset$ interior(M). Here "represented" means that

$$H_2(W; \mathbb{Z}[\pi_1 P]) \xrightarrow{inc_*} H_2(M; \mathbb{Z}[\pi_1 P])$$

is an isomorphism. We assume that $\pi_1(W) \cong F_k$ is free and W itself is the source of an unobstructed problem, $g \colon W \longrightarrow Q$ where $Q = (\natural_k S^1 \times D^3, \sharp_k S^1 \times S^2)$.

Suppose $g'\colon V \longrightarrow Q$ *is a solution to* g, *then there is a solution to* f *of the form* $f'\colon M' := (M \smallsetminus W) \cup V \longrightarrow P$.

Note. We have used notations in the statement and proof appropriate to the case of orientable M. There is no difficulty extending to the non-orientable case.

Proof. Consider the two braided Mayer-Vietoris sequences below, where for ease of reading we have suppressed the functor $H_*(\quad; \mathbb{Z}[\pi_1 P])$ applied to each space and simply written a subscript for the value of $*$.

(6.1)

$$
\begin{array}{c}
(M \smallsetminus W)_2 \oplus W_2 \rightarrow M_2 \;\; \partial \qquad\qquad (M \smallsetminus W)_1 \oplus W_1 \rightarrow M_1 \;\; \partial \\
\ldots \rightarrow \partial W_2 \qquad\qquad \partial\;\; \partial W_1 \qquad\qquad \partial\;\; \partial W_0 \\
(M \smallsetminus W)_2 \oplus V_2 \rightarrow M'_2 \qquad\qquad (M \smallsetminus W)_1 \oplus V_1 \rightarrow M'_1
\end{array}
$$

A theorem of Wall [32] constructs 1-manifold 1-skeleton for Poincaré spaces P so we may embed $Q \subset P$ and thereby obtain maps from line (6.1) to line (6.2), the corresponding Mayer-Vietoris sequence of Poincaré spaces.

(6.2)
$$
\begin{aligned}
\ldots &\longrightarrow \sharp_k(S^1 \times S^2)_2 \longrightarrow (P \smallsetminus Q)_2 \oplus Q_2 \longrightarrow P_2 \xrightarrow{\partial} \sharp_k(S^1 \times S^2)_1 \\
&\longrightarrow (P \smallsetminus Q)_1 \oplus Q_1 \longrightarrow P_1 \longrightarrow \sharp_k(S^1 \times S^2)_0
\end{aligned}
$$

Thus we now reinterpret line (6.1) not as homology groups but the kernel groups, $K_*(\quad; \mathbb{Z}[\pi_1 P])$.

Since $V \simeq \vee S^1 \simeq Q$, regardless of the cover $\pi_1 P$ induces on V and Q the homotopy equivalence lifts to a homotopy equivalence $\widetilde{V} \simeq \widetilde{Q}$ implying $K_*(V; \mathbb{Z}[\pi_1 P]) \cong 0$. In fact, the *only* non-trivial kernel group in line (6.1) is $K_2(W; \mathbb{Z}[\pi_1 P])$. Thus $K_*(M'; \mathbb{Z}[\pi_1 P]) \cong 0 \;\; \forall *$. Finally, a calculation using van Kampen's theorem implies that $f_\sharp \colon \pi_1 M' \longrightarrow \pi_1 P$ is an isomorphism, so $f'\colon M' \longrightarrow P$ is a homotopy equivalence. $\square$

Armed with this lemma, and knowledge of the strength of the 2-Engel relation, our task is now to draw a detailed schematic "spaghetti picture" of a new class of problems W inside M and carrying its kernel. M is a typical member of (a coinitial) segment of the original universal problems based on *Wh(Bing(Hopf))* and W will be a new problem based on *D(Ram(h-triv))*. Our spaghetti picture is intended to capture all the important features. Once drawn we will convert it into a schematic Kirby link diagram about which we have enough precise knowledge to prove theorem 2.

We remember there is ramification but do not draw it in schematics. We start with any element of *Bing(Hopf)* which is weakly h-trivial (see section 2.3), the simplest of these has 5 components, e.g. see figure 4.1. We use this L for illustration but formally choose any $L \in \{Bing(Hopf)\}$ with all $\bar{\mu}$-invariants of length ≤ 4 vanishing.

This property that all $\bar{\mu}$-invariants of length ≤ 4 vanish is inherited by all links PL, the class of links where the initial k components of L are arbitrarily ramified with parallel copies. Using the weak h-triviality of P_2L, the link made from L by adding one parallel copy of each component, we may visualize the $S^2 \vee S^2$'s inside $M(L)$ (lemma 5.1 and figure 5.2) built from the cores of the plumbed 2-handle pairs union the weak null homotopies of P_2L. This is pictured in figure 6.4.
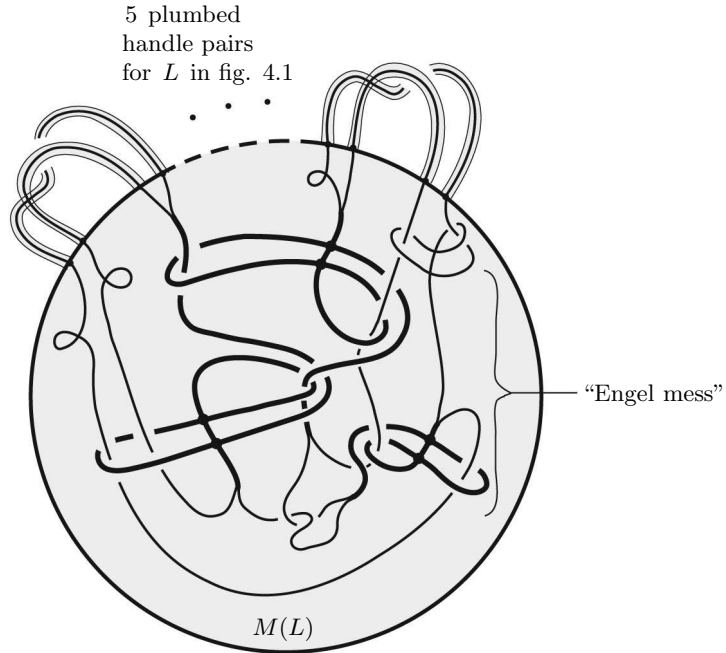
FIGURE 6.4.

The weak null-homotopy in figure 6.4 has (arbitrarily) 5 ordinary self-intersections pictured and 3 sections where two strands run together for a bit (they are called a "packet" in section 2.3, and drawn heavily in figure 6.4) and while fused have 6 (again arbitrary) self-intersections. Note two is not arbitrary but the number of strands in a packet required by corollary 2.3.

Remark 6.4. The commutator identity (4.3) implies that the commutators $[h_1, \ldots, h_k]$ in the statement of corollary 2.3 may be assumed to involve only the given normal generators $\{g_i\}$, and not their inverses. It follows that the two-strand "packets" may be assumed to be of a single type, where (for some choice of orientations) the orientations of the two strands match.

We call the 2-complex X pictured in figure 6.4 the "Engel mess". We need to make it still messier so that it contains dual spheres. Actually corollary 2.3 allows us to

a: an intersection point of
   caps (only one pictured)
b: a self-intersection point of caps
c: a cap
d: a double pt. of paired sheets
e: an ordinary capped double point
   of X (only one pictured)
f: the $2k$ essential 2-spheres
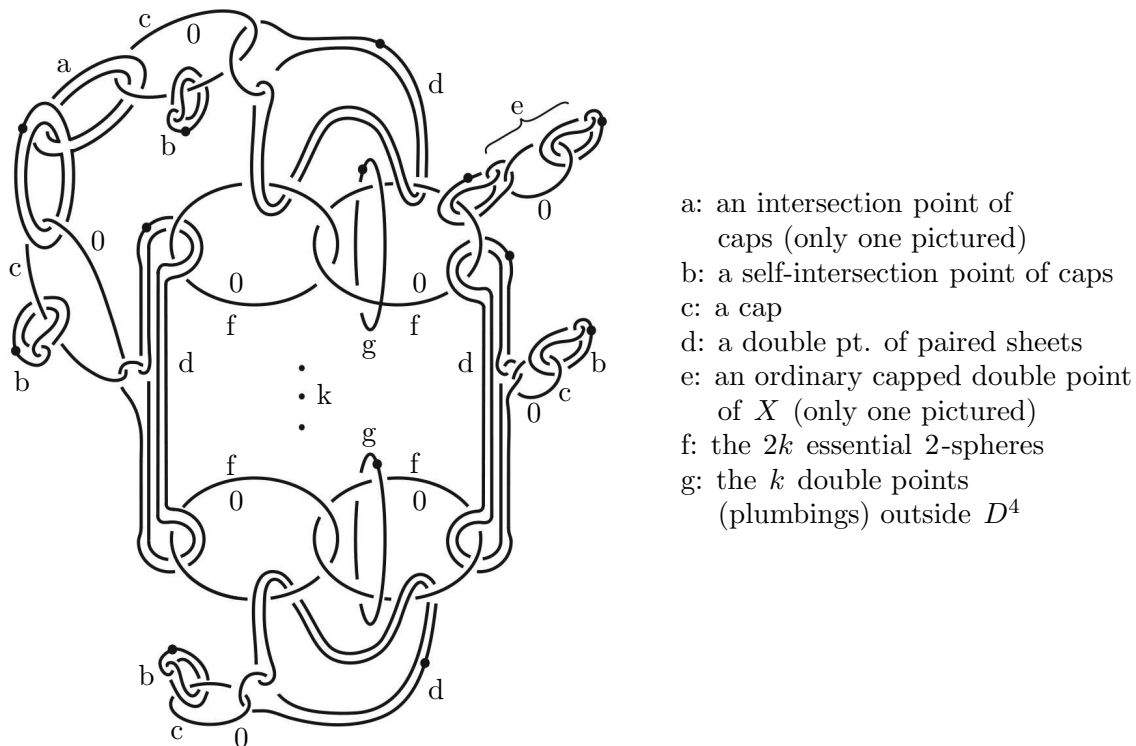g: the k double points
   (plumbings) outside D^4

FIGURE 6.5. Note: type d 1-handles occur (numerous) times between all $\binom{2k}{2}$ pairs of of types f curves.

build the weak null homotopy so that each of the $2k$ longitudes $\gamma_1, \ldots, \gamma_{2k}$ to $P_2 L$ lies in the kernel K:

$$(6.3) \qquad 1 \longrightarrow K \longrightarrow \pi_1(S^3 \smallsetminus P_2 L) \longrightarrow \pi_1(D^4 \smallsetminus X) \longrightarrow 1$$

This is because each $\gamma_i \in (\pi_1(S^3 \smallsetminus P_2 L))^4$, which follows from the formula [28] for the behavior of $\bar\mu$-invariants under ramification.

Thus each of the $2k$ basic spherical classes S_i in X contains a geometrically dual sphere $S_i^\perp$ in M (meeting X only in one point of the basis - sphere.) The null-homotopy of γ_i in $(D^4 \smallsetminus X)$ glued to the 2-handle core parallel to γ_i is one the geometric dual spheres. Of course these $2k$ dual spheres $\{S^\perp\}$ intersect and self-intersect.

It is now a standard construction [6, 13] that the $\{S^\perp\}$ can be exploited to add an additional layer of zero-framed caps (intersecting and self-intersecting) to "kill" all double points of X in D^4 (the k double points in the plumbed handles cannot be capped.) Let $X^+ = X \cup \text{caps}$ and $W = \text{Neighborhood}(X^+)$, figure 6.5 exhibits the essential features of a handle diagram for any such W.

A slightly novel identity needed to draw figure 6.5 is the effect of a double point on a paired sheet, calculated in figure 6.6.
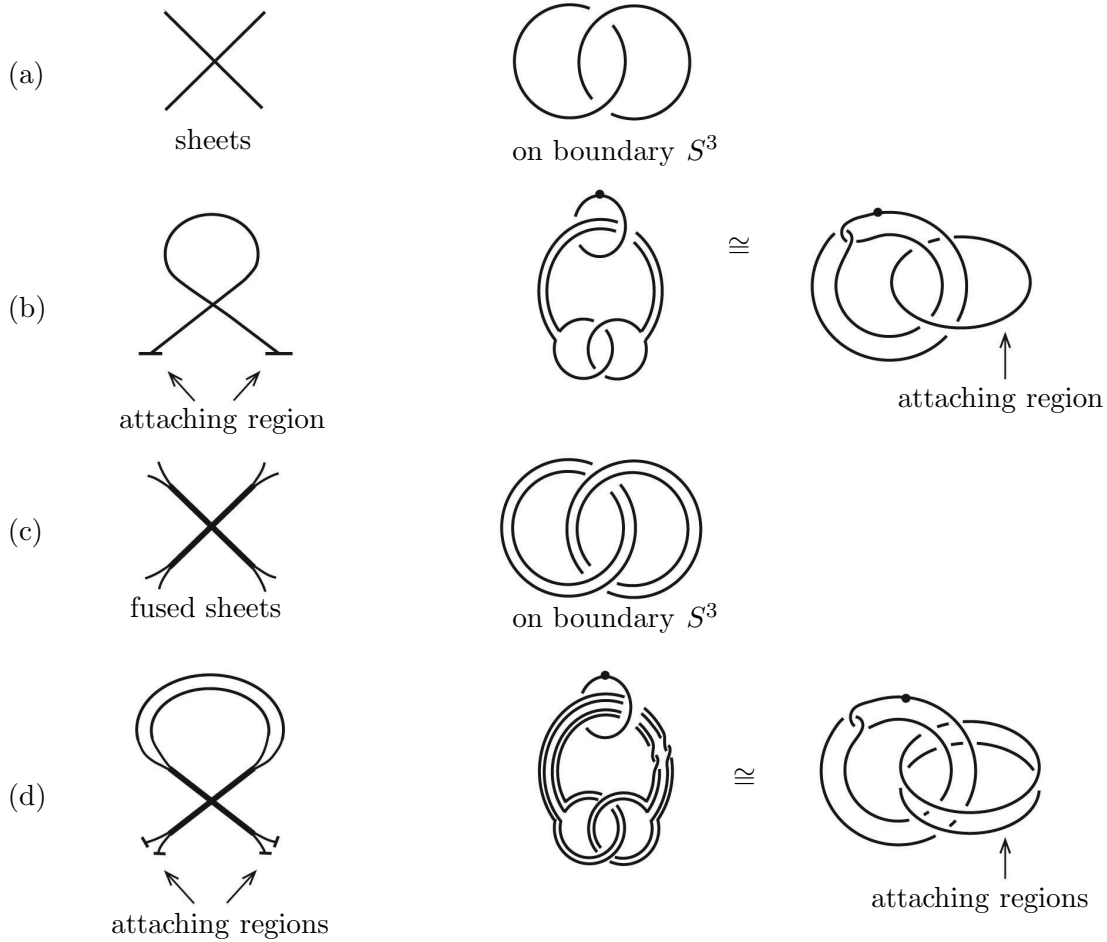

FIGURE 6.6. Four types of singularities and their associated Kirby diagrams

Next we should Morse cancel 1-handles and 2-handles whenever possible in figure 6.5. The result is a bit complicated to draw but we can use a short hand writing D whenever a component is "generalized double" (see section 5.1). In some cases (look at e in figure 6.5) this is precisely a Whitehead double - the curve drawn in figure 6.7 should be doubled with itself (i.e. Whitehead doubled) to produce the original curve (e). In other case, a, two separate curves are generalized doubled (indicated by the forked arrow in figure 6.7) to produce the curve (a) of figure 6.5. The b-curves in figure 6.5 account for the dotted curves labeled by D with a straight arrow. D^2 means "double twice". With this notation figure 6.5 becomes figure 6.7 with ramification permitted before each doubling step.
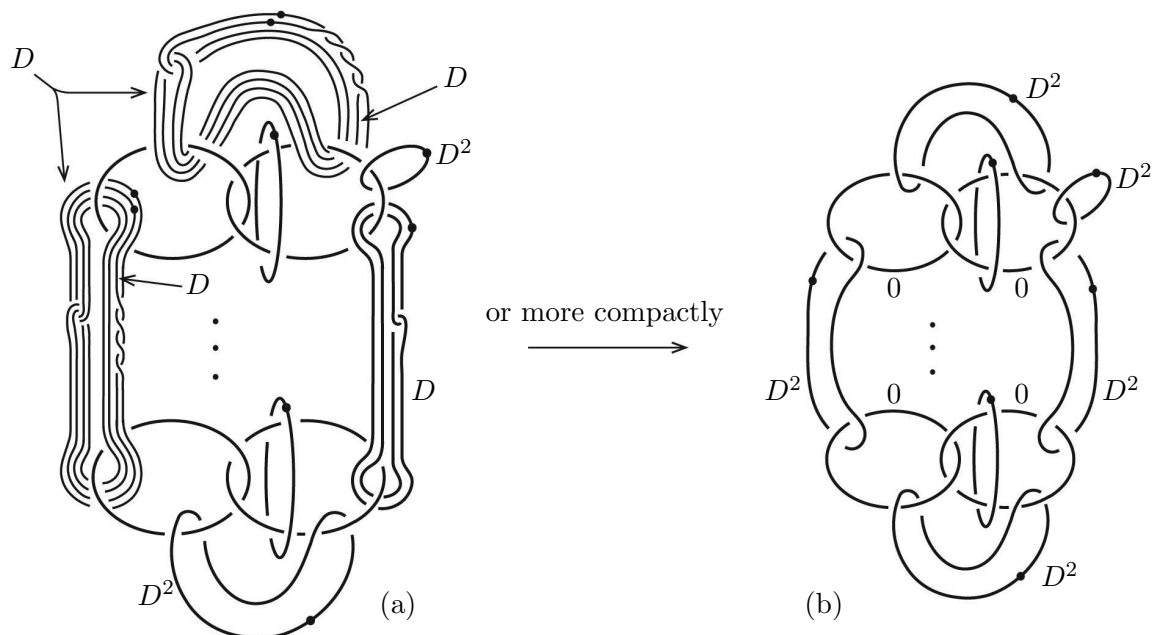
or more compactly $\longrightarrow$

FIGURE 6.7.

The only 1-handle components $p_1, \ldots, p_k$ (p for plumbing) not (twice) doubled in figure 6.7 (b) can be thought of as doubles of a pair x, y^- using the obvious Seifert surfaces these components bound disjoint from the rest of the link diagram, figure 6.8.
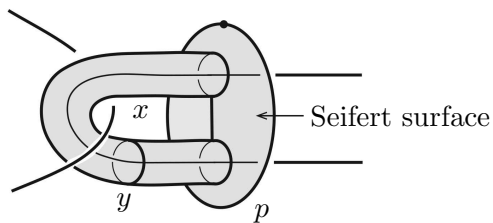

FIGURE 6.8.

Canceling all hyperbolic pairs and replacing p_i as Double (x_i, y_i^-) we get figure 6.9.

Figure 6.9 actually displays a link of the type claimed in theorem 2. Leave x_i and y_i^- as they are and double each component labeled by a D^2 *once*. The result is a ramification of a homotopically trivial link - but not better. Now *all* these components are singly doubled (which includes the possibility of ramifying before taking a generalized double) to obtain the link in figure 6.9, showing that $\partial W \cong \mathcal{S}_0(\mathrm{D}(\mathrm{Ram}(\text{h-triv})))$, as desired. $\qquad\square$
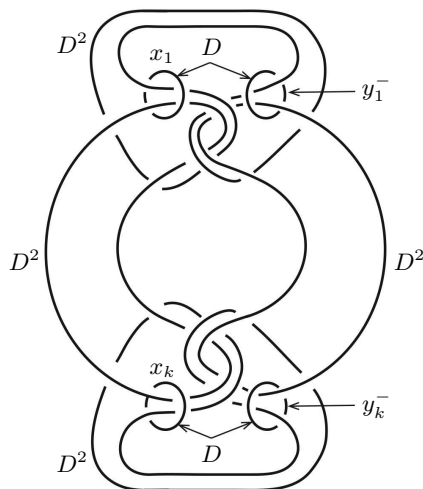
FIGURE 6.9.

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Appendix A. n-Engel relations

Since the 2-Engel relation is rather interesting in 4D topology, it is natural to wonder what use higher n-Engel relations might have. First we summarize a bit of what is known algebraically and raise some new questions natural when one works (as so often in 4D topology) in the free Milnor group, rather than in the free group.

The fact about 2-Engel groups that is crucially used in applications in this paper is that the free group F_k modulo the 2-Engel relation is nilpotent of a fixed class,

independent of the number of generators k. It is known that 3-Engel groups [19] and also 4-Engel groups [18] are locally nilpotent (i.e. every finitely generated subgroup is nilpotent). The question for n-Engel groups, $n > 4$ presently appears to be open. We do not know how the (local) nilpotency class of the free group F_k mod the 3- or 4-Engel relation depends on k. Recall (section 2.1) that the free Milnor group on k generators, MF_k, is nilpotent of class k.

Question A.1. *Fix $n \geq 3$. Is the nilpotency class of MF_k modulo the n-Engel relation less than k? More specifically, is it independent of k?*

An affirmative answer to this question could lead to an improvement of the results of geometric applications of the 2-Engel relation in this paper. Another possible way to refine the algebraic structure is to see if there is a way to restrict (for example to a certain term of the lower central series) the group elements y that come up in applications of the 2-Engel relation $[y, x, x]$.

Returning to the first sentence of the introduction, we should follow Casson's philosophy and determine the local singular disk structure which enforces $[y, x, \ldots, x]$, the n-Engel relation. This turns out to be rather easy beginning with the "kinky handle" and then elaborating. The correspondence is laid out in figure A.1. (The n-ary kinky handle gives raise to the $2n$-Engel relation.)

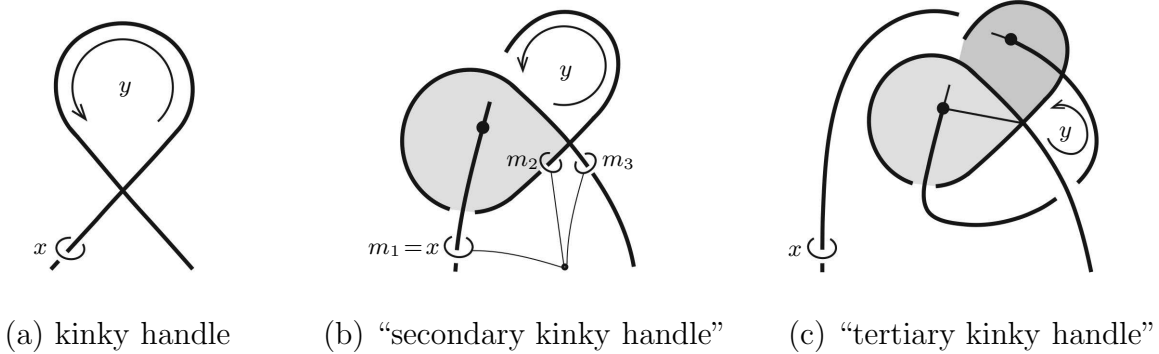(a) kinky handle   (b) "secondary kinky handle"   (c) "tertiary kinky handle"

FIGURE A.1.
(a) The double point implies $[x, x^y] = 1$ or equivalently $[y, x, x] = 1$.
(b) The second double point corresponds to $[y, x, x, x, x] = 1$.
(c) The third double point corresponds to $[y, x, x, x, x, x, x] = 1$, etc.

We sketch the calculation in (b). Denoting the meridian m_1 to the secondary kinky handle H in D^4 by x, the meridian m_2 equals x^y, and $m_3 = (x^y)^x$. Then a meridian m_w to the (shaded) Whitney disk reads off

$$m_\mathrm{w} = [m_2, m_3] = [x^y, (x^y)^x] = [x^y, [x, x^y]] = [x^y, [x, [y, x]]].$$

Finally, the 2-cell of the Clifford torus of the second double point gives raise to a relation $[m_w, m_1] = [[x^y, [x, [y, x]]], x]$ in $\pi_1(D^4 \smallsetminus H)$. Using the commutator identities (2.5), modulo higher order commutators this gives the 4-Engel relation $[y, x, x, x, x]$.

A more rigorous Kirby diagram description (or if you like "definition") of higher order kinky handles is given in figure A.2. As usual we suppress discussion of the $\pm$ sign at clasps, but we do pay careful attention to framings so that the two components of the final, canceled down, link diagram are each individually unknotted (as with the Whitehead link). This is a useful feature when dualizing link diagrams.
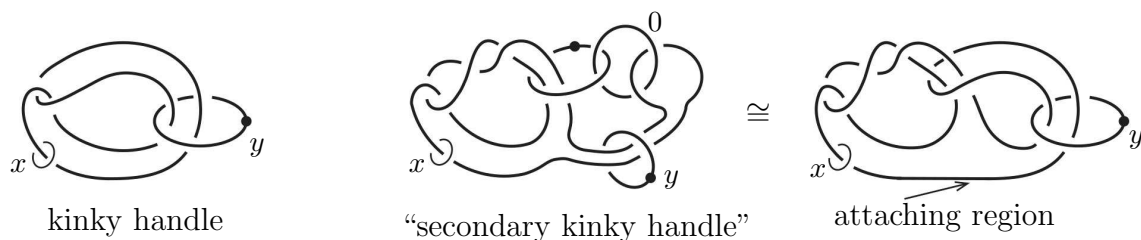

kinky handle   "secondary kinky handle"   attaching region

FIGURE A.2.

Continuing in this way the model n-ary kinky handle is shown in figure A.3.

FIGURE A.3.

Remark A.1. It looks very likely that the least area unknotting disk for the longer component as in [17] is exponential in n. Or put another way, if that component is made round the link diagram will necessarily have $\geq \text{const}^n$ crossings for some const≥ 1. The method of [17] looks relevant to this case as well, but we did not succeed in adapting the argument. The new feature here is that the bridge number of the diagram is not constant (as in [17]) but linear in n.

These links are similar to Milnor's family [29, Figure 1], which arise from thickenings of figure A.1 with less favorable choices of framings.

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