NOTES ON ENDS OF HYPERBOLIC 3-MANIFOLDS

Michael H. Freedman Vyacheslav S. Krushkal

Lecture 1. The central theme of these notes is a question of Marden [M]: Do all hyperbolic 3-manifolds with finitely-generated fundamental group have product ends?

An end is said to have a product structure (or to be tame) if it is homeomorphic to $(\partial \text{ end}) \times \mathbb{R}_+$. Marden's question may be reformulated as follows: Is every hyperbolic 3-manifold M with finitely-generated fundamental group homeomorphic to the interior of a compact 3-manifold?

By work of Canary [C], the affirmative answer to Marden's question for a finitely-generated Kleinian group $G \cong \pi_1(M)$ implies Ahlfors' conjecture [A] for G: The limit set Λ_G is either the whole 2-sphere at infinity or has Lebesgue measure zero.

Note that hyperbolic 3-manifolds with finitely-generated fundamental group have finitely many ends. (This is not true in general for dimensions greater than three.) To prove this, recall

Scott core theorem [S]. Let M be a hyperbolic 3-manifold with finitely-generated fundamental group. Then there exists a compact, codimension zero submanifold C of M such that the inclusion map $C \hookrightarrow M$ induces an isomorphism of fundamental groups.

In particular, $\pi_1(M)$ is finitely presented, and $H_2(M;\mathbb{Z}) \cong H_2(\pi_1(M);\mathbb{Z})$ is finitely generated. Let $(H_1^{l,f}(M))$ denote the homology of locally-finite chains on M. By Poincaré duality $H_1^{l,f}(M) \cong H^2(M)$ is also finitely generated. Considering lines exiting through various ends of M shows that k ends of M provide (k-1) linearly independent elements of $H_1^{l,f}(M)$, hence M has finitely many ends.

The classes of groups for which Marden's conjecture has been verified are:

(1) Fuchsian groups (2-dimensional case): $G \subset PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$.

A fundamental domain for the action of G on \mathbb{H}^3 (Poincaré model) is an "apple core", bounded by a finite number of totally geodesic planes orthogonal to the

standard $\mathbb{H}^2 \subset \mathbb{H}^3$. It is easy to visualize a product structure on \mathbb{H}^3/G :

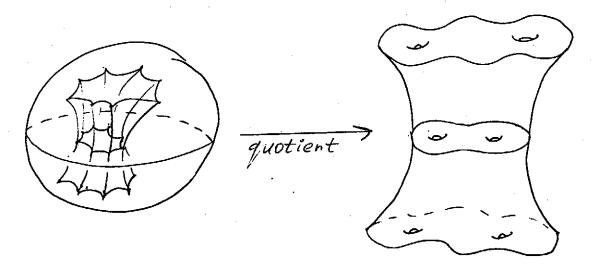


FIGURE 1.1

- (2) Quasi-Fuchsian groups. The limit sets of these groups are quasi-circles; fundamental domains have more bites taken out of the "apple core" but are still finite-sided convex polyhedra.
- (3) Certain limits of geometrically-finite groups analyzed by Thurston [Th].
- (4) Indecomposable groups (Bonahon [B]). Recall that group G is called indecomposable if $G \not\cong \mathbb{Z}$, and $G \cong A * B$ implies one of the groups A and B is trivial.

Notice that if $\pi_1(M)$ is indecomposable, Scott core C of M has a special feature: $\overline{M-C}$ is a homotopy product $(\partial C) \times \mathbb{R}_+$. To prove this, first observe that ∂C is incompressible in C. We may assume, using induction, that M-C is connected. Suppose the homomorphism induced by inclusion $\pi_1(\partial C) \to \pi_1(C)$ is not injective. By the loop theorem and Dehn's lemma there exists an embedded disk $(D, \partial D) \subset (C, \partial C)$ with ∂D essential in ∂C . The disk D cannot be separating, since it would provide a free decomposition of $\pi_1(C) \cong \pi_1(M)$. In the non-separating case, since $\pi_1(M) \not\cong \mathbb{Z}$, we can take a band sum of two parallel copies of D to get a separating disk $D' \subset C$ with $\partial D'$ still essential in ∂C . This contradicts the assumption that $\pi_1(\partial C) \to \pi_1(C)$ is not injective.

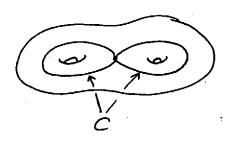
By Seifert-van Kampen's theorem $\pi_1(M)$ is the pushout:

$$\begin{array}{ccc} \pi_1(\partial C) & \stackrel{\alpha}{-\!\!\!-\!\!\!-\!\!\!-} & \pi_1(C) \\ \downarrow^{\gamma} & & \downarrow^{\beta} \\ \pi_1(\overline{M \smallsetminus C}) & \longrightarrow & \pi_1(M) \end{array}$$

Since α is injective and β is an isomorphism, γ is also an isomorphism. This shows that $\overline{M-C}$ is a homotopy product $(\partial C) \times \mathbb{R}_+$.

This is not necessarily the case when $\pi_1(M)$ is decomposable. For example, let M be a genus two handlebody. Then both the standard wedge of two circles and a knotted one

are Scott cores of M.



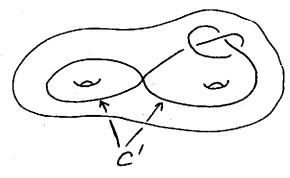


FIGURE 1.2

The idea of Thurston and Bonahon in the indecomposable case—to find a sequence of pleated surfaces, homeomorphic to ∂C , exiting the end of M—does not work in the decomposable case for a knotted core as in the example above. One would like to find a "natural" geometric core to avoid this problem.

We will now state two geometric reformulations of the condition that the ends of M are tame.

Fix a finitely-generated Kleinian group $G \cong \pi_1(M)$ and let \mathcal{O} denote the orbit G(x) of a point $x \in \mathbb{H}^3$.

Theorem 1.1. [F] The following two conditions are equivalent:

- (a) For any r there exists a (sufficiently large) R such that every arc in $(\mathbb{H}^3 \setminus N_r(\mathcal{O}), \partial N_r(\mathcal{O}))$ can be deformed, relative to its ends, into $(N_R(\mathcal{O}) \setminus N_r(\mathcal{O}), \partial N_r(\mathcal{O}))$, where N_r denotes the neighborhood of radius r.
- (b) All ends of M are tame.

To understand condition (a), note that for a simple closed curve K in S^3 , knottedness is the failure of the relative fundamental group $\pi_1(S^3 \setminus K, \partial)$ to be trivial. From this perspective the negation of (a) means that the orbit of G is coarsely knotted at all size scales.

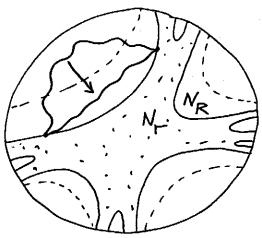


FIGURE 1.3

Before giving the proof of Theorem 1.1, we recall

Tucker's Theorem. [T] Let M be a hyperbolic 3-manifold with finitely-generated fundamental group. Then all ends of M are tame if and only if $\pi_1(M \setminus K)$ is finitely generated for all compact submanifolds $K \subset M$.

Proof of Theorem 1.1. Let \bar{x} be the image of x in $M = \mathbb{H}^3/G$. Pick a basepoint $p \in \partial(M \setminus N_r(\bar{x}))$ and fix a preimage $\tilde{p} \in \partial(N_r(\mathcal{O}))$ of p. The fundamental group $\pi_1(M \setminus N_r(\bar{x}), p)$ may be interpreted as classes of paths in the universal cover, starting at \tilde{p} and ending at various lifts of p. The multiplication is given by composition of paths, using covering translations. By Tucker's theorem M has product ends if and only if $\pi_1(M \setminus N_r(\bar{x}))$ is finitely generated for all r. This is equivalent to finding a set of generators in the universal cover within a bounded distance from $N_r(\mathcal{O})$. \square

Another sufficient condition for the existence of a product structure is the following "Elder Sibling Property".

Consider the case when the domain of discontinuity Ω_G is non-empty. Let $x \in \Omega_G$ and let $h \subset \mathbb{H}^3$ be a horoball based at x. Consider the orbit G(h) in the upper-half space model where x is the point at infinity. Each horoball in the orbit meets only finitely many of its G-translates, and for h sufficiently large the union of the orbit $\cup G(h)$ is path-connected.

1.2. We say that G satisfies the Elder Sibling Property (ESP) if there exists a horoball h based at $x \in \Omega_G$ such that in the upper-half space model with x the point at infinity, each horoball in G(h) of finite (Euclidean) size meets a larger or infinite one.

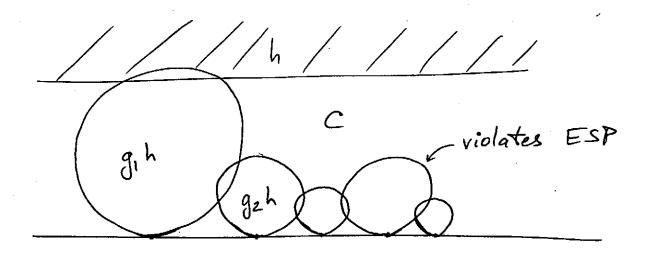


FIGURE 1.4

Let C denote $\mathbb{H}^3 \setminus (\cup G(h))$.

Theorem 1.3. [F] Suppose $G \cong \pi_1(M)$ satisfies ESP. Then $\pi_1(C, \partial C)$ is trivial, and all ends of M are tame.

The idea of the proof is to use Tucker's theorem with the image of h in M thought of as an almost compact set. In fact, h touches S^2_{∞} at a point of discontinuity where its behavior is well understood.

Sketch of the proof. Consider the Morse function on C, given by the height in the upper-half space model. There are six types of critical points that may occur on ∂C —each one has one of three Morse indices 0, 1, 2 and one of two signs + or -, according to whether the interior of C lies above or below ∂C at the critical point. It follows from the geometry of C, that points of index (+, 2) do not occur. The only possibility for $\pi_1(C, \partial C)$ to be non-trivial is that there are critical points of index (+, 0), which introduce at these respective height levels new components of C, which are then connected in a non-trivial way at points of index (-, 1). This possibility is eliminated by the ESP assumption, since each (+, 0) point is canceled by a corresponding saddle point. \Box

Lecture 2. A new extension of the Kneser-Haken finiteness principle to manifolds with boundary.

Let $G \cong \pi_1(M)$ be a free finitely-generated Kleinian group, for example the free group on two generators. Let x, y be a free basis of G. Similarly to representing an essential element of a fundamental group by a least-length geodesic, the chosen basis of G may be represented by a least length web (wedge of two circles) W. Let $\widetilde{W} \subset H^3$ denote the preimage of W in the universal cover.

Lemma 2.1. $\pi_1(\mathbb{H}^3 \setminus \widetilde{W})$ is a free group.

Proof. Let x be a vertex of \widetilde{W} , and let v_1, \ldots, v_4 denote the unit tangent vectors at x to the four edges of \widetilde{W} containing x. It follows from the choice of W that $\sum_{i=1}^{4} v_i = 0$ —otherwise a first order variation would reduce the length of W.

Fix a point $p \in \mathbb{H}^3$ and let r denote the (hyperbolic) radius function on \mathbb{H}^3 centered at p. The restriction $r|_{\widetilde{W}}$ cannot have a local maximum at an interior point of an edge since W is a least length web, and it does not have one at a vertex by the "balanced" condition discussed above. It follows that the presentation of $\pi_1(\mathbb{H}^3 \setminus \widetilde{W})$ associated to r has no relations. \square

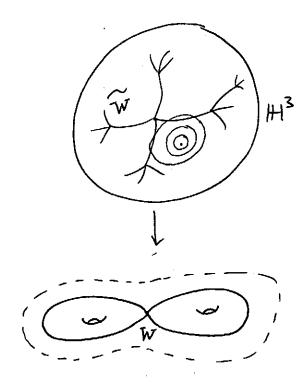


FIGURE 2.1

2.2. Examples of open 3-manifolds with free fundamental group on two generators may be built using an analogue of Whitehead's construction [Wh]. Let $(H_2)_i \hookrightarrow (H_2)_{i+1}$ be an

inclusion of a genus two handlebody into the interior of another one, which is homotopic but not isotopic to the identity map.

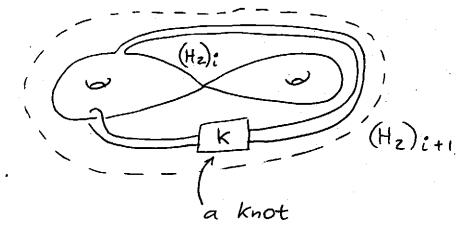


FIGURE 2.2

Let M denote the infinite nested union $\bigcup_{i=1}^{\infty} (H_2)_i$. For Marden's conjecture to hold, these manifolds should not admit a hyperbolic structure. This can be proved in many cases by finding a closed incompressible surface in $\mathbb{H}^3 \setminus \widetilde{W}$, showing that $\pi_1(\mathbb{H}^3 \setminus \widetilde{W})$ is not free. A technique for finding incompressible surfaces is described below. (See Theorem 2.6.)

It turns out that this method does not always work, since many 3-manifolds have locally free but not free fundamental group, and hence they do not contain incompressible surfaces. Recall that a group G is called *locally free* if every finitely-generated subgroup of G is free.

2.3. Example of a locally free but not free group. Let N be a thickening in \mathbb{R}^3 of the following infinite 2-complex:

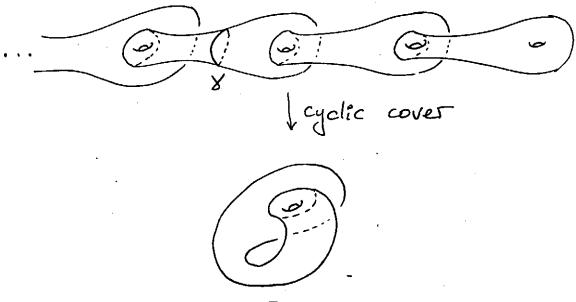


FIGURE 2.3

 $\pi_1(N)$ is locally free, since the union of a finite number of consequent fundamental domains has free fundamental group, and these are joined along incompressible annuli. $\pi_1(N)$ is not free, since the loop γ bounds in N an infinite half-grope, showing that $[\gamma] \in (\pi_1(N))_{\omega}$ —the ω -term of the lower-central series. γ represents a non-trivial element of $\pi_1(N)$, since a null-homotopy would lie in a compact piece of N; on the other hand, the ω -term of free groups is trivial.

Question 2.4. Give a general geometric method for showing that a fundamental group is not free.

Lemma 2.5. Let M be an irreducible non-compact 3-manifold without boundary. The following conditions are equivalent:

- (1) $\pi_1(M)$ is locally free.
- (2) M contains no closed incompressible surface.
- (3) M is exhausted by handlebodies.

We will now describe a technique for finding closed incompressible surfaces in non-compact 3-manifolds.

Theorem 2.6. [F-F] Let C be a curve in the solid torus $M=D^2\times S^1$ which is homotopic but not isotopic to the core $\{0\}\times S^1$ of M, and let $\widetilde{C}\subset D^2\times \mathbb{R}$ denote its preimage in the universal cover. Then $(D^2\times \mathbb{R})\setminus \widetilde{C}$ contains a closed incompressible surface.

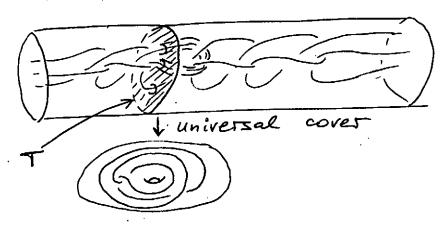


FIGURE 2.4

In this case, for a genus one handlebody, the existence of a hyperbolic structure is not an issue, but this is a good test case for finding incompressible surfaces.

Proof of Theorem 2.6. Let M_n denote the union of n consequent fundamental domains in $D^2 \times \mathbb{R}$, with \widetilde{C} cut out. Notice that a surface which is incompressible in M_n is also incompressible in $D^2 \times \mathbb{R} \setminus \widetilde{C}$ since $\pi_1(M_n)$ injects into $\pi_1(D^2 \times \mathbb{R} \setminus \widetilde{C})$ as a factor in the free product with amalgamation. Let Σ_n denote the boundary of M_n —a genus three surface, pushed slightly into the interior of M_n . It will be shown that M_n is not a genus three handlebody for sufficiently large n. This will prove that Σ_n does not compress to a 2-sphere and will complete the proof of Theorem 2.6. \square

Note that the argument given below gives n exponential in the complexity of the curve C in M. An easier argument was later found by C. Gordon and by A. Reid-D. Cooper-D. Long.

Let T denote the twice-punctured torus shown in FIGURE 2.4 with $\partial T \subset (D^2 \times \mathbb{R}) \setminus \widetilde{C}$. T and its translates provide (n+1) non-parallel incompressible surfaces in M_n . If M_n was a genus three handlebody, this would give a contradiction for large n with the bounded case of Kneser-Haken finiteness theorem. (See Theorem 2.7 below.)

Recall the Kneser-Haken finiteness principle for a closed 3-manifold M: There exists an integer c(M) which bounds from above the number of disjoint closed non-parallel incompressible surfaces in M. The idea of the proof is the observation that the intersection of a collection S of surfaces in normal form with each 3-simplex Δ in a triangulation of M provides at most six non-product regions of $\Delta \setminus S$ (FIGURE 2.5). The product regions of various tetrahedra glue up, modulo the homological correction term, to (surfaces) $\times I$.

This proof does not extend to the bounded case in the absence of the boundary-incompressibility assumption on surfaces: $S \cap \Delta$ may contain tunnels providing an uncontrolled number of non-product regions. (FIGURE 2.6.)

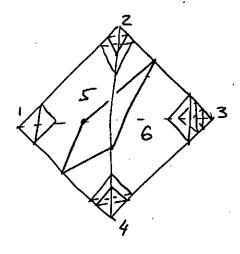


FIGURE 2.5

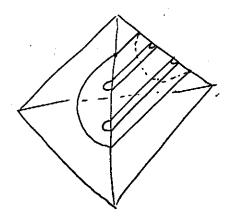


FIGURE 2.6

W. Sherman [Sh] exhibited arbitrarily many non-parallel incompressible surfaces in (genus two surface) $\times I$. His example has the property that the complexity of surfaces increases with their number.

Theorem 2.7. [F-F] Let M be a compact 3-manifold with boundary and b an integer greater than zero. There is a constant c(M,b) such that if S_1, \ldots, S_k , k > c, is a collection of incompressible surfaces such that all Betti numbers, $b_1(S_i) < b$, $1 \le i \le k$, and no S_i , $1 \le i \le k$, is a boundary-parallel annulus or a boundary-parallel disk, then at least two members of S_i and S_j are parallel.

Remarks. 1. The bound in the closed case is roughly $6 \cdot (\text{number of tetrahedra in a triangulation of <math>M)$, while in the bounded case Theorem 2.7 gives the bound which is exponential in the complexity of the data.

2. The proof of Theorem 2.6 does not apply to handlebodies of genus greater that one. The difference is that a free group with more than one generator has exponential growth, and the number of fundamental domains in the ball of radius R in the universal cover is comparable with the genus of the bounding surface.

Lecture 3. Before formulating a topological approach to the product-end conjecture, we first recall the observation (essentially contained in [C]) that in order to detect a wild end of a hyperbolic 3-manifold it suffices to check the infinite-generation condition in Tucker's theorem for just one of the "algebraically disk-busting" geodesics:

Lemma 3.1. Let γ be a simple closed geodesic in an open hyperbolic 3-manifold M satisfying

- (1) $[\gamma] = 0 \in H_1(M; \mathbb{Z}_2),$
- (2) no conjugate of γ lies in a factor of a non-trivial free product $\pi_1(M) \cong A * B$. If M has a wild end, then $\pi_1(M \setminus \gamma)$ is infinitely generated.

Remarks. Note that the set of algebraically disk-busting (that is, satisfying condition (2) above) geodesics has full measure in the set of all geodesics ([K], [C]). If a geodesic γ is not simple, then the arguments below apply to any C^1 -perturbation of γ .

The algebraically disk-busting condition for a curve γ in a compact 3-manifold N is equivalent to the condition that γ meets every essential disk in N.

Proof of Lemma 3.1. Suppose $\pi_1(M \setminus \gamma)$ is finitely generated. We will show that the 2-fold branched cover \widetilde{M}^{γ} of M with the branching locus γ has finitely-generated indecomposable fundamental group. By a theorem of Gromov and Thurston [G-Th] \widetilde{M}^{γ} then admits a metric of pinched negative curvature, and Bonahon's proof [B] extends to this setting to show that \widetilde{M}^{γ} has product ends. Since branched coverings preserve the structure of ends, the proof of Lemma 3.1 will be complete once we show that $\pi_1(\widetilde{M}^{\gamma})$ is finitely generated and indecomposable.

By the relative Scott core theorem [Mc] there is a compact core $C \subset \overline{M \setminus N(\gamma)}$ with $\partial N(\gamma) \subset C$, where $N(\gamma)$ denotes a closed tubular neighborhood of C. By assumption (1) γ bounds a surface Σ in M, and \widetilde{M}^{γ} is constructed by gluing two copies of $\overline{M \setminus N(\Sigma)}$ to obtain a double cover \widetilde{M}^2 of $\overline{M \setminus N(\gamma)}$ and then regluing back $N(\gamma)$. Cores are functorial under these operations:

Now the problem is reduced to the compact setting, and as in [C] the equivariant Dehn's Lemma and Sphere Theorem, combined with the assumptions (1), (2) on γ , show that $\pi_1(\widetilde{C}^{\gamma}) \cong \pi_1(\widetilde{M}^{\gamma})$ is indecomposable. \square

Topological Conjecture 3.2. Let M be an open 3-manifold with finitely-generated fundamental group and with the universal cover $\widetilde{M} \cong \mathbb{R}^3$, and let $\gamma \subset M$ be a simple closed curve with $\pi_1(M \setminus \gamma)$ infinitely generated. Then $(\widetilde{M}, \pi^{-1}(\gamma)) \not\cong (\mathbb{R}^3$, standard countable collection of lines) where $\pi : \widetilde{M} \to M$ is the covering projection.

A model for the standard collection above is given by the vertical lines in \mathbb{R}^3 passing through points of the integral lattice in a horizontal plane.

This topological conjecture implies Marden's and Ahlfors' conjectures, since for M hyperbolic the preimage in \mathbb{H}^3 of a simple closed geodesic γ is standard by Morse theory.

Some evidence for the conjecture is provided by the fact that while in the higher-dimensional case there are various restrictions that one may impose on an end, such as π_1 -stability at infinity or vanishing of K_0 -obstruction, the known conditions in dimension three that restrict wildness already imply the existence of a product structure.

3.3. Various examples of wild ends may be produced by using Whitehead's construction [Wh]. (See also [S-T].)

Let $M = \bigcup_{i=1}^{\infty} T_i$ where $T_i \cong S^1 \times D^2$ and the embedding $T_i \hookrightarrow T_{i+1}$ is shown in Figure 3.1.

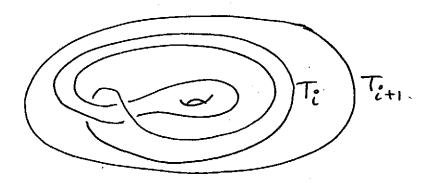
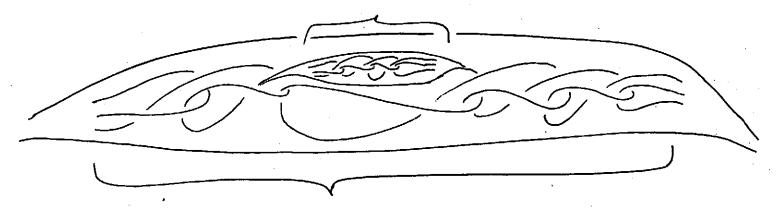


FIGURE 3.1

This 3-manifold has the properties that $\pi_1(M) \cong \mathbb{Z}$ and the universal cover $\widetilde{M} \cong \mathbb{R}^3$, but $M \not\cong S^1 \times \mathbb{R}^2$. Let γ denote the core curve of T_1 . An algorithm for drawing the preimage of γ in the universal cover is given in Figure 3.2.

3 fundamental domains of \widetilde{T}_1



5 fundamental domains of \widetilde{T}_1 and \widetilde{T}_2

FIGURE 3.2

Let \widetilde{T}_i denote the preimage of T_i in \widetilde{M} . One starts by drawing three fundamental domains of \widetilde{T}_1 . The second step is to draw five fundamental domains of \widetilde{T}_2 and extend the existing picture for \widetilde{T}_1 by one (rather stretched) domain to the left and to the right inside \widetilde{T}_2 . The next step is seven fundamental domains of \widetilde{T}_3 and an extension of \widetilde{T}_2 and then \widetilde{T}_1 to seven domains, each inside \widetilde{T}_3 Note that each step of the algorithm does not change the result of the previous step, hence providing an exhausting picture for the universal cover \widetilde{M} . It follows that $\pi_1(\mathbb{R}^3 \setminus \widetilde{\gamma})$ is infinitely generated, and the conjecture is true in this example.

In some cases the fact that a collection of lines in \mathbb{R}^3 is not standard may be detected, as in Lecture 2, by finding an incompressible surface in the complement, showing that the fundamental group of the complement is not free. However, there is again a possibility of a locally free but not free group.

Example 3.4. Let M be the infinite-nested union of genus two handlebodies, $M = \bigcup_{i=1}^{\infty} (H_2)_i$, where the inclusion $(H_2)_i \hookrightarrow (H_2)_{i+1}$ is shown in Figure 3.3.

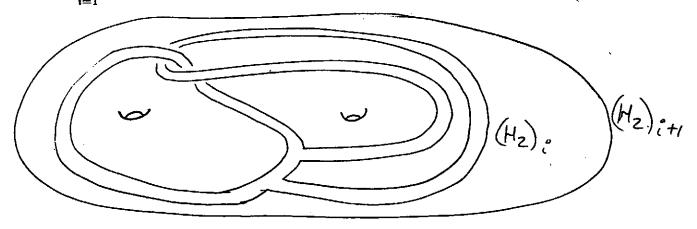


FIGURE 3.3

The map of fundamental groups $\mathbb{Z} * \mathbb{Z} \to \mathbb{Z} * \mathbb{Z}$ induced by inclusion sends the first factor isomorphically onto the first factor, and the second factor to the identity element, hence $\pi_1(M) \cong \mathbb{Z}$. Let γ denote a curve representing the sum of the two standard generators of $\pi_1((H_2)_1)$.

By M. Brown's criterion of exhaustion by balls [Br] the universal cover \widetilde{M} is \mathbb{R}^3 . The complement of $\widetilde{\gamma}$ in \widetilde{M} may be represented as a nested union of handlebodies, hence by Lemma 2.5, $\pi_1(\widetilde{M} \setminus \widetilde{\gamma})$ is locally free. The fact that it is not a free group may be established in this example by a rather special argument: by Alexander duality its Abelianization is isomorphic to the integers.

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University of California, San Diego, Department of Mathematics, La Jolla, CA 92093-0112, USA; email: mfreedman@ucsd.edu

MICHIGAN STATE UNIVERSITY, DEPARTMENT OF MATHEMATICS, EAST LANSING, MI 48824-1027, USA; EMAIL: KRUSHKAL@MATH.MSU.EDU