EMBEDDING OBSTRUCTIONS IN R^d FROM THE GOODWILLIE-WEISS CALCULUS AND WHITNEY DISKS

GREGORY ARONE AND VYACHESLAV KRUSHKAL

Abstract. Given a finite CW complex K, we use a version of the Goodwillie-Weiss tower to formulate an obstruction theory for embeddings of K into a Euclidean space $\mathbb{R}^d$. For 2-dimensional complexes in $\mathbb{R}^4$, a geometric analogue is also introduced, based on intersections of Whitney disks and more generally on the intersection theory of Whitney towers developed by Schneiderman and Teichner. The focus in this paper is on the first obstruction beyond the classical embedding obstruction of van Kampen. In this case we show the two approaches lead to essentially the same obstruction. We also relate it to the Arnold class in the cohomology of configuration spaces. The obstructions are shown to be realized in a family of examples. Conjectures are formulated, relating higher versions of these homotopy-theoretic, geometric and cohomological theories.

1. Introduction

Let K be a finite CW complex. In this paper we investigate, and compare, two approaches to constructing obstructions to the existence of a topological embedding $K \hookrightarrow \mathbb{R}^d$, with special focus on the case of 2-dimensional complexes in $\mathbb{R}^4$.

Our first approach is inspired by the embedding calculus of Goodwillie and Weiss [35, 14], which provides a systematic framework for studying embedding spaces. The difference between the two settings is that their theory was developed for studying smooth embeddings of smooth manifolds, while we adapt their ideas to the context of topological (or PL) embeddings of finite complexes.

The embedding calculus of Goodwillie-Weiss works particularly well for the study of manifold embeddings in codimension at least 3; in this case the tower converges to the embedding space. In some instances the Goodwillie-Weiss tower is known to give rise to highly nontrivial invariants in codimension 2 as well; for example in the case of long knots in $\mathbb{R}^3$ this theory is closely related to Vassiliev invariants, cf. [33], [6], [16]. We use a weaker version of the Goodwillie-Weiss tower to formulate obstructions to embeddability of a complex K into $\mathbb{R}^d$. In particular, we apply this theory to 2-complexes in $\mathbb{R}^4$, another instance of codimension 2 embeddings. The (weak) convergence of this tower to the embedding space is an open problem, see Section 8.

For 2-complexes in $\mathbb{R}^4$ we also consider an alternative, geometric approach based on the failure of the Whitney trick in this dimension. Some instances of this approach are well-known, for example in the study of Milnor's invariants [21]. More generally, Schneiderman and Teichner [26] developed the intersection theory of *Whitney towers* in 4-manifolds. We use these ideas to formulate embedding obstructions for 2-complexes in $\mathbb{R}^4$.

Focusing on the first new obstruction, we show that these a priori unrelated approaches in fact give the same result (Theorem 5.1). This provides a useful perspective on both of them: the homotopy-theoretic obstruction is manifestly well-defined but lacks an immediate geometric interpretation; the Whitney tower approach has a clear geometric meaning but establishing its well-definedness directly is a challenging problem.

Before outlining our work in more detail, we recall some of the classical results in this subject. When $2m < d$, an m-dimensional simplicial complex K embeds in $\mathbb{R}^d$ by general position. If $2\dim(K) \geq d$ there is a classical obstruction, proposed by van Kampen [32], based on the following idea. Suppose there exists a topological embedding $f\colon K \hookrightarrow \mathbb{R}^d$. Then the map $f\times f\colon K\times K \to \mathbb{R}^d\times\mathbb{R}^d$ restricts to a Σ_2-equivariant map, which we call *the deleted square* of f:

$$f^2_\Delta\colon \mathrm{C}\,(K,2) \to \mathrm{C}\,(\mathbb{R}^d,2). \tag{1.1}$$

Here $\mathrm{C}\,(X,2) = X\times X\setminus X$ is the configuration space of ordered pairs of distinct points in X, and Σ_n denotes the symmetric group of degree n; in particular $\Sigma_2 \cong \mathbb{Z}/2\mathbb{Z}$.

Thus the existence of a Σ_2-equivariant map $\mathrm{C}\,(K,2) \to \mathrm{C}\,(\mathbb{R}^d,2)$ is a necessary condition for the existence of a topological embedding $f\colon K\hookrightarrow\mathbb{R}^d$. The van Kampen obstruction is a cohomological obstruction to the existence of such a map. It is an element of the equivariant cohomology group $H^d_{\Sigma_2}(\mathrm{C}\,(K,2);\mathbb{Z}[(-1)^d])$, where the notation indicates the action of Σ_2 by $(-1)^d$ on the integers. (The original formulation of van Kampen [32] predated a formal definition of cohomology, and it was based on the geometric approach discussed in Section 4. Moreover, van Kampen's formulation focused on the case $2\mathrm{dim}(K) = d$.)

The van Kampen obstruction is known to be complete when $2\dim(K) = d \neq 4$. For $\dim(K) > 2$, this follows from the validity of the Whitney trick [30, 36]; a modern treatment may be found in [12]. For 1-complexes in $\mathbb{R}^2$ this follows from the Kuratowski graph planarity criterion [18] and the naturality of van Kampen's obstruction under embeddings. When K is a 2-dimensional complex and $d = 4$, it was shown in [12] that the existence of a Σ_2-equivariant map $\mathrm{C}\,(K,2) \to \mathrm{C}\,(\mathbb{R}^d,2)$ is insufficient for embeddability, and thus the van Kampen obstruction is incomplete. The underlying geometric reason, the failure of the Whitney trick in 4 dimensions, is well-known. However, as in many other aspects of 4-manifold topology, it is a non-trivial problem to formulate an invariant that captures this geometric fact. In this paper, as we discuss below, we formulate such an invariant in the context of 2-complexes in $\mathbb{R}^4$.

Building on work of Haefliger [15], Weber [34] extended the embeddability result to the "metastable range" of dimensions. More precisely, it is shown in [34] that given an m-dimensional simplicial complex K and a Σ_2-equivariant map $f_2\colon\ \mathrm{C}\,(K,2) \longrightarrow \mathrm{C}\,(\mathbb{R}^d,2)$ with $2d \geq 3(m+1)$, there exists a PL embedding $f\colon\ K \longrightarrow \mathbb{R}^m$ such that the induced map f^2_Δ is Σ_2-equivariantly homotopic to f_2.

In this paper we produce a new obstruction to the existence of an embedding of a finite complex in $\mathbb{R}^d$ beyond the metastable range, which is defined when the van Kampen obstruction vanishes. Our first new obstruction $\mathcal{O}_3(K)$, where the index refers to 3-point configuration spaces used to define it, depends on a choice of a Σ_2-equivariant map f_2 as in (1.1), and it really is an obstruction to f_2 being Σ_2-equivariantly homotopic to the deleted square f^2_Δ of some embedding $f\colon K\hookrightarrow\mathbb{R}^d$.

We will give several topological, geometric and algebraic interpretations of $\mathcal{O}_3(K)$. On the topological side, we interpret $\mathcal{O}_3(K)$ as the fundamental class of the subspace of points

$(k_1, k_2, k_3) \in \mathrm{C}\,(K, 3)$ for which the vectors $f_2(k_1, k_2)$, $f_2(k_2, k_3)$ and $f_2(k_3, k_1)$ are co-directed (Section 3.1). On the geometric side, we show that it counts intersections of K with the Whitney disks that arise from the vanishing of the van Kampen obstruction (Sections 4, 5). On the algebraic side, we interpret it as the kernel of the Arnold relation (Lemma 6.5). We use this algebraic interpretation to verify that $\mathcal{O}_3(K)$ is non-zero in a family of examples (Section 6).

For concreteness, in this paper we focus on embeddings of 2-dimensional complexes in $\mathbb{R}^4$, and just on the first obstruction beyond van Kampen's. But the general approach underpinning our construction is applicable in other dimensions, and leads to an infinite sequence of obstructions. Moreover, our methods should lead to a formulation of an obstruction theory for embedding complexes into more general manifolds, not just $\mathbb{R}^d$. We intend to pursue this elsewhere. In this paper we will only give an outline of the general approach, and state a few conjectural claims.

Work in progress [2], joint with Danica Kosanović, Rob Schneiderman and Peter Teichner, formulates the non-repeating version of the Goodwillie-Weiss tower in a different context, for *link maps*. In this setting the results of [2], which apply when the ambient manifold has an arbitrary fundamental group, establish an equivalence between the homotopy lifting obstructions from the non-repeating version of the Goodwillie-Weiss tower and the higher geometric intersection theory of Whitney towers.

We now proceed to outline our construction; see Sections 2, 7 for more details. Let $\mathrm{Emb}(K, \mathbb{R}^d)$ be the space of topological embeddings of K into $\mathbb{R}^d$. We are not going to use the topology of this space (except when K is a finite set), so the reader can think of $\mathrm{Emb}(K, \mathbb{R}^d)$ as just a set. Our sole concern is whether this set is empty or not. Our obstruction theory is based on a tower of spaces under $\mathrm{Emb}(K, \mathbb{R}^d)$, which we denote as follows

$$\mathrm{Emb}(K, \mathbb{R}^d) \to \cdots \to T_n \,\mathrm{Emb}(K, \mathbb{R}^d) \to T_{n-1} \,\mathrm{Emb}(K, \mathbb{R}^d) \to \cdots \to T_2 \,\mathrm{Emb}(K, \mathbb{R}^d). \tag{1.2}$$

Note that we make no claim that the induced map

$$\mathrm{Emb}(K, \mathbb{R}^d) \longrightarrow \operatorname{holim} T_n \,\mathrm{Emb}(K, \mathbb{R}^d)$$

is an equivalence. Nevertheless, since there is a map $\mathrm{Emb}(K, \mathbb{R}^d) \to T_n \,\mathrm{Emb}(K, \mathbb{R}^d)$, a necessary condition for $\mathrm{Emb}(K, \mathbb{R}^d)$ to be non-empty is that $T_n \,\mathrm{Emb}(K, \mathbb{R}^d)$ is non-empty for all n.

It is easy to see that

$$T_2 \,\mathrm{Emb}(K, \mathbb{R}^d) \simeq \mathrm{map}(\mathrm{C}\,(K, 2), \mathrm{C}\,(\mathbb{R}^d, 2))^{\Sigma_2}$$

As we mentioned above, there is a well-known cohomological obstruction $\mathcal{O}_2(K)$ for this space to be non-empty, and this is the van Kampen obstruction.

From this point on, our strategy is to look for an obstruction for a path component of $T_n \,\mathrm{Emb}(K, \mathbb{R}^d)$ to be in the image of a path component of $T_{n+1} \,\mathrm{Emb}(K, \mathbb{R}^d)$. We will now describe explicitly how to do it in the case $n = 2$. This case is our main focus in this paper.

Suppose K is a finite-dimensional complex for which the van Kampen obstruction vanishes. Then there exists a Σ_2-equivariant map

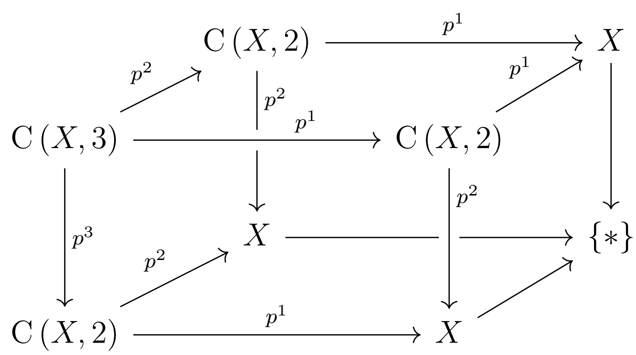
$$f_2 \colon \mathrm{C}\,(K, 2) \longrightarrow \mathrm{C}\,(\mathbb{R}^d, 2)$$

Our goal is to give an effective necessary condition for the existence of an embedding $f\colon K \hookrightarrow \mathbb{R}^d$ such that $f^2_\Delta\colon \mathrm{C}\,(K,2) \to \mathrm{C}\,(\mathbb{R}^d,2)$ is equivariantly homotopic to f_2. Let

$$\mathrm{C}\,(X,3) = \left\{(x_1,x_2,x_3) \in X^3 \mid x_i \neq x_j \text{ for } i \neq j\right\}$$

be the configuration space of ordered triples of distinct points in X. There is a cubical diagram of configuration spaces, where the projection p^i omits the i-th coordinate:

(1.3)



Now suppose we have a topological embedding $f\colon K \hookrightarrow \mathbb{R}^d$. Such an embedding induces a Σ_n-equivariant map of configuration spaces $\mathrm{C}\,(K,n) \longrightarrow \mathrm{C}\,(\mathbb{R}^d,n)$ for each n; moreover it induces a map of cubical diagrams (1.3) for K, $\mathbb{R}^d$. In the diagram for $\mathbb{R}^d$ the space $\mathrm{C}\,(\mathbb{R}^d,1) = \mathbb{R}^d$ is contractible, and (up to homotopy) the map of cubical diagrams may be replaced by a smaller diagram (1.4) below. Denote by p_X the canonical Σ_3-equivariant map

$$\begin{array}{rccccccc} p_X\colon & \mathrm{C}\,(X,3) & \longrightarrow & \mathrm{C}\,(X,2) & \times & \mathrm{C}\,(X,2) & \times & \mathrm{C}\,(X,2) \\ & (x_1,x_2,x_3) & \mapsto & (x_1,x_2) & , & (x_2,x_3) & , & (x_3,x_1) \end{array}$$

Then f induces a commutative diagram

$$(1.4)\qquad \begin{array}{ccc} \mathrm{C}\,(K,3) & \xrightarrow{\quad f^3_\Delta \quad} & \mathrm{C}\,(\mathbb{R}^d,3) \\ \downarrow{\scriptstyle p_K} & & \downarrow{\scriptstyle P_{\mathbb{R}^d}} \\ \mathrm{C}\,(K,2)\times\mathrm{C}\,(K,2)\times\mathrm{C}\,(K,2) & \xrightarrow{(f^2_\Delta)^3} & \mathrm{C}\,(\mathbb{R}^d,2)\times\mathrm{C}\,(\mathbb{R}^d,2)\times\mathrm{C}\,(\mathbb{R}^d,2) \end{array}$$

Therefore, given a Σ_2-equivariant map $f_2\colon \mathrm{C}\,(K,2) \longrightarrow \mathrm{C}\,(\mathbb{R}^d,2)$, a necessary condition for it being induced by an embedding, is that the lifting problem in the following diagram has a solution

$$(1.5)\qquad \begin{array}{ccccc} & & & \nearrow\!\!\text{-}\,\text{-}\,\text{-} & \mathrm{C}\,(\mathbb{R}^d,3) \\ & & & & \downarrow{\scriptstyle p_{\mathbb{R}^d}} \\ \mathrm{C}\,(K,3) & \xrightarrow{p_K} & \mathrm{C}\,(K,2)^{\times 3} & \xrightarrow{(f_2)^3} & \mathrm{C}\,(\mathbb{R}^d,2)^{\times 3} \end{array}$$

There exists a cohomological obstruction to the existence of a Σ_3-equivariant dashed arrow that makes the diagram commute up to homotopy. We denote this obstruction by $\mathcal{O}_3(K)$. It turns out to be an element of the equivariant cohomology group $H^{2d-2}_{\Sigma_3}(\mathrm{C}\,(K,3);\mathbb{Z}[(-1)^{d-1}])$. Details can be found in Section 2.

Remark 1.1. In terms of the tower $T_n \operatorname{Emb}(K, \mathbb{R}^d)$, $\mathcal{O}_3(K)$ is precisely the obstruction for the path component of f_2 in $T_2 \operatorname{Emb}(K, \mathbb{R}^d)$ to be in the image of the map $T_3 \operatorname{Emb}(K, \mathbb{R}^d) \to T_2 \operatorname{Emb}(K, \mathbb{R}^d)$.

Remark 1.2. The analogue of $\mathcal{O}_3(K)$ in the context of smooth embeddings was studied by Munson [23]. In fact, Munson did not consider the cohomological obstruction, but its lift to equivariant framed cobordism (a.k.a. stable cohomotopy). We will also investigate a lift of the cohomology class $\mathcal{O}_3(K)$ to a framed cobordism class $\mathcal{O}_3^{\mathrm{fr}}(K)$ in Sections 2 and 3.

Focusing on the case of simplicial 2-complexes in $\mathbb{R}^4$, geometrically (as we recall in Section 4) the vanishing of the van Kampen obstruction implies that a general position map $f\colon K \longrightarrow \mathbb{R}^4$ may be found such that for any two non-adjacent 2-simplices σ_i, σ_j of K, the algebraic intersection number $f(\sigma_i) \cdot f(\sigma_j)$ is zero. In higher dimensions in this setup the Whitney trick enables one to find an actual embedding, cf. [12, Theorem 3]. In dimension 4 one may still consider Whitney disks W_{ij} pairing up the intersections points $f(\sigma_i) \cap f(\sigma_j)$ but the Whitney disks themselves have self-intersections and intersect other 2-cells, see [13, Section 1.4] and also Figure 2 in Section 4 below.

Our geometric obstruction $\mathcal{W}_3(K)$ is an element of the equivariant cohomology group

$$H^6_{\Sigma_3}(\mathrm{C_s}(K,3); \mathbb{Z}[(-1)]);$$

this is the same cohomology group as the one discussed above except that now $\mathrm{C_s}(K,3)$ denotes the *simplicial* configuration space, that is K^3 minus the simplicial diagonal consisting of products $\sigma_1 \times \sigma_2 \times \sigma_3$ of simplices where two of them have a vertex in common. The obstruction is defined on the cochain level by sending a 6-cell $\sigma_1 \times \sigma_2 \times \sigma_3$ (where σ_i is a 2-simplex of K) to the sum of intersection numbers $W_{ij} \cdot f(\sigma_k)$ over distinct indices i, j, k; see Section 4.4 for details. Informally, the obstruction may be thought of as measuring the failure of the Whitney trick in 4 dimensions. In the special case of disks in the 4-ball with a prescribed boundary – a link in the 3-sphere ∂D^4 – the analogous invariant equals the Milnor $\bar{\mu}$-invariant of a 3-component link, sometimes referred to as the triple linking number. For *knots*, a similar expression measuring self-intersections of a disk in D^4 equals the Arf invarint, see Remark 4.6 and references therein.

The obstruction $\mathcal{W}_3(K)$ depends on the map $f\colon K \longrightarrow \mathbb{R}^4$ and also on Whitney disks W_{ij}. In fact, we show in Lemma 4.3 that a choice of Whitney disks determines a Σ_2-equivariant map $\mathrm{C_s}(K,2) \longrightarrow \mathrm{C}(\mathbb{R}^4,2)$; in this sense the geometric setup is parallel to the homotopy-theoretic context discussed above.

In Theorem 5.1 we show that the two obstructions $\mathcal{O}_3(K)$, $\mathcal{W}_3(K)$ are in fact equal. The proof proceeds by localizing the problem, using subdivisions of the 2-complex K and splittings of Whitney disks, and identifying the Whitehead product in the homotopy fiber of the map $p_{\mathbb{R}^4}\colon \mathrm{C}(\mathbb{R}^d,3) \longrightarrow \mathrm{C}(\mathbb{R}^d,2)^{\times 3}$ in the notation of (1.5) using the Pontryagin-Thom construction; see Section 5 for details.

Finally, a cohomological interpretation in terms of the Arnold relation is given in Section 6. We use this interpretation to show that our obstruction is non-trivial for the examples constructed in [12]. In that reference 2-complexes were constructed that have trivial van Kampen's obstruction but which do not admit an embedding into $\mathbb{R}^4$. The proof of non-embeddability in [12] is group-theoretic in nature (using the Stallings theorem) and is quite different from the methods of this paper. Our work provides a general obstruction theory and gives a conceptual homotopy-theoretic and geometric framework for analyzing the embedding problem in this dimension.

The following is a brief outline of the structure of the paper. Section 2 starts with the discussion of van Kampen's obstruction and its properties, and proceeds to define the obstruction $\mathcal{O}_3(K)$ which is the main focus of this paper. In the case when $3\dim(K) = 2d-2$ this is a complete obstruction to the lifting problem (1.5). We also describe a lift of $\mathcal{O}_3(K)$ to a an equivariant framed coborodism class $\mathcal{O}_3^{\mathrm{fr}}(K)$, which is defined in terms of a *classifying map* $\mathrm{C}\,(\mathbb{R}^d, 2)^3 \to \widehat{\Omega}^2\Omega^\infty\Sigma^\infty\widehat{S}^{2d}$. The class $\mathcal{O}_3^{\mathrm{fr}}(K)$ is a complete obstruction to the lifting problem (1.5) whenever $\dim(K) + 2 \leq d$. An explicit construction of $\mathcal{O}_3^{\mathrm{fr}}(K)$ is given in Section 3; in particular it leads to a topological interpretation of $\mathcal{O}_3(K)$ in terms of the set of points satisfying a certain collinearity condition, see Section 3.1. Section 4 starts by recalling the geometric definition of van Kampen's obstruction and basic operations on Whitney disks in dimension 4. Lemma 4.3 establishes a relation between Whitney disks and maps of configuration spaces, which illustrates a key connection between geometry and homotopy theory explored in this paper. Section 4.4 defines $\mathcal{W}_3(K)$ and analyzes its properties. The construction of higher obstructions $\mathcal{W}_{\mathrm{n}}(K)$, in terms of intersection theory of Whitney towers of Schneiderman-Teichner, is outlined in Section 4.5. The main result of Section 5, Theorem 5.1, relates the obstructions $\mathcal{O}_3(K)$ and $\mathcal{W}_3(K)$. Section 6 recalls the examples of [12] and shows that the obstruction $\mathcal{O}_3(K)$ detects their non-embeddability in $\mathbb{R}^4$. In the process of doing this, $\mathcal{O}_3(K)$ is related to the Arnold class in Lemma 6.5. In fact, this point of view provides a uniform perspective on non-embeddability of 2-complexes in $\mathbb{R}^4$ from [12] and of the examples in other dimensions outside the metastable range from [29, 28], see Remark 6.6. Section 7 gives the construction of the tower $T_n\,\mathrm{Emb}(K, \mathbb{R}^n)$, formulates the higher obstructions $\mathcal{O}_{\mathrm{n}}(K)$, and discusses their properties including a conjectural framed cobordism lift. We conclude by stating a number of questions and conjectures motivated by our results in Section 8.

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2. The obstruction

We begin this section by reviewing the classical van Kampen obstruction $\mathcal{O}_2(K)$ from a homotopy-theoretic perspective. Then we introduce our main construction: a higher cohomological obstruction $\mathcal{O}_3(K)$, defined when $\mathcal{O}_2(K) = 0$, and depending on a choice of a Σ_2-equivariant map $f_2\colon\ \mathrm{C}\,(K, 2) \to \mathrm{C}\,(\mathbb{R}^d, 2)$. We also discuss certain refinements $\mathcal{O}_2^{\mathrm{fr}}(K)$ and $\mathcal{O}_3^{\mathrm{fr}}(K)$ of $\mathcal{O}_2(K)$ and $\mathcal{O}_3(K)$ respectively into classes that reside in framed cobordism rather than cohomology. We give a homotopy-theoretic description of $\mathcal{O}_3(K)$ as a map into an Eilenberg-Mac Lane space, and a geometric interpretation as a framed cobordism class.

Let K continue denoting an m-dimensional CW (or simplicial) complex. We are interested in the question whether there exists a topological (or PL) embedding of K in $\mathbb{R}^d$. As we saw in the introduction, a necessary condition for the existence of an embedding, is the existence of a Σ_2-equivariant map $f_2\colon \mathrm{C}(K,2) \to \mathrm{C}(\mathbb{R}^2,2)$. Or, equivalently, a Σ_2-equivariant map $\mathrm{C}(K,2) \to \widetilde{S}^{d-1}$, where $\widetilde{S}^{d-1}$ denotes the sphere with the antipodal action of Σ_2. Recall that there is a Σ_2-equivariant homotopy equivalence $\mathrm{C}(\mathbb{R}^d,2) \xrightarrow{\simeq} \widetilde{S}^{d-1}$ that sends (x_1,x_2) to $\frac{x_2-x_1}{|x_2-x_1|}$. We will occasionally switch back and forth between these spaces.

There is a well-known homotopical/cohomological obstruction to the existence of a Σ_2-equivariant map $f_2\colon \mathrm{C}(K,2) \to \mathrm{C}(\mathbb{R}^d,2)$, which we will now review. Let $\widehat{\mathbb{R}}^d$ be the d-dimensional sign representation of Σ_2. Let $\widehat{S}^d$ be the one-point compactification of $\widehat{\mathbb{R}}^d$, considered as a space with an action of Σ_2. Equivalently, $\widehat{S}^d$ is the unreduced suspension of $\widetilde{S}^{d-1}$. Note that $\widehat{S}^d$ has two points fixed by Σ_2, corresponding to 0 and ∞ in the compactificaton of $\widehat{\mathbb{R}}^d$. By convention, ∞ is the basepoint of $\widehat{S}^d$. The following elementary lemma gives several conditions for the existence of a Σ_2-map $K\times K\setminus K \to \widetilde{S}^{d-1}$.

Lemma 2.1. *Conditions* (1) *and* (2) *below are equivalent*

(1) There exists a Σ_2-equivariant map $K\times K\setminus K \to \widetilde{S}^{d-1}$.
(2) The vector bundle

$$(K\times K\setminus K)\times_{\Sigma_2}\widehat{\mathbb{R}}^d \to (K\times K\setminus K)_{\Sigma_2} \tag{2.1}$$

has a nowhere vanishing section.

Furthermore, conditions (1) *and* (2) *above imply conditions* (3) *and* (4) *below. Under the assumption $d \geq \dim(K)+2$, the conditions (1)-(4) are equivalent.*

(3) The constant map $K\times K\setminus K \to \widehat{S}^d$ that sends $K\times K\setminus K$ to zero is Σ_2-equivariantly null-homotopic.
(4) The constant map $K\times K\setminus K \to \Omega^\infty\Sigma^\infty\widehat{S}^d$ which is the map of part 3 followed by the suspension map $\widehat{S}^d \to \Omega^\infty\Sigma^\infty\widehat{S}^d$ is Σ_2-equivariantly null-homotopic.

Proof. The vector bundle (2.1) has a nowhere vanishing section if and only if the sphere bundle

$$(K\times K\setminus K)\times_{\Sigma_2}\widetilde{S}^{d-1} \to (K\times K\setminus K)_{\Sigma_2}$$

has a section. It is well-known that sections of this bundle are in bijective correspondence with Σ_2-equivariant maps $K\times K\setminus K \to \widetilde{S}^{d-1}$, which is why (1) and (2) are equivalent.

Suppose there is a Σ_2-equivariant map $K\times K\setminus K \to \widetilde{S}^{d-1}$. It induces Σ_2-equivariant maps

$$(K\times K\setminus K)\times I \to \widetilde{S}^{d-1}\times I \to \widehat{S}^d$$

where the latter map is the obvious quotient. This composite map is a null homotopy of the constant zero map $K\times K\setminus K \to \widehat{S}^d$. This is why (1) implies (3). It is obvious that (3) implies (4).

For the reverse implication in the last statement of the lemma, let $\widetilde{\Omega}\widehat{S}^d$ be the space of paths in $\widehat{S}^d$ from the basepoint ∞ to 0. There is a canonical Σ_2-equivariant map $\widetilde{S}^{d-1} \to \widetilde{\Omega}\widehat{S}^d$. It follows from the Blakers-Massey theorem that this map is $2d-3$-connected. It follows that the induced map of mapping spaces

$$\mathrm{map}(K\times K\setminus K, \widetilde{S}^{d-1})^{\Sigma_2}\to \mathrm{map}(K\times K\setminus K, \widetilde{\Omega}\widehat{S}^d)^{\Sigma_2}$$

is $2d-2\dim(K)-3$-connected. So if $d-\dim(K)\geq 2$ this map is at least 1-connected, and therefore induces a bijection on π_0. But a Σ_2-equivariant map $K\times K\setminus K\to\widetilde{\Omega}\widehat{S}^d$ is the same thing as a Σ_2-equivariant null homotopy of the constant zero map from $K\times K\setminus K$ to $\widehat{S}^d$. Thus, under the assumption $d\geq \dim(K)+2$, condition (3) implies (1).

Finally, the map $\widehat{S}^d\to\Omega^\infty\Sigma^\infty\widehat{S}^d$ is $2d-1$-connected by the Freudental suspension theorem. It follows that (4) implies (3) when $d\geq\dim(K)+1$, which is a weaker condition than stated in the lemma. □

Lemma 2.1 points to several (equivalent) ways to define a cohomological obstruction to the existence of a Σ_2-equivariant map $K\times K\setminus K\to\widetilde{S}^{d-1}$. To begin with, the map given in part (4) of the lemma can be interpreted as an element of an equivariant stable cohomotopy group, or equivalently an equivariant framed cobordism group of $K\times K\setminus K$. We denote this element by $\mathcal{O}_2^{\mathrm{fr}}(K)$. Lemma 2.1 says that $\mathcal{O}_2^{\mathrm{fr}}(K)$ is a complete obstruction to the existence of a Σ_2-equivariant map $K\times K\to\widetilde{S}^{d-1}$ when $\dim(K)+2\leq d$.

The natural map of spectra $\Sigma^\infty S^0\to H\mathbb{Z}$ induces a Σ_2-equivariant map

$$\Omega^\infty\Sigma^\infty\widehat{S}^d\to\Omega^\infty H\mathbb{Z}\wedge\widehat{S}^d\simeq K(\mathbb{Z}[(-1)^d],d). \tag{2.2}$$

Here $K(\mathbb{Z}[(-1)^d],d)$ denotes the Eilenberg-Mac Lane space with an action of Σ_2, that on the non-trivial homotopy group realizes the representation $Z[(-1)^d]$, which is the trivial representation if d is even and the sign representation if d is odd. Any two such Eilenberg-Mac Lane spaces are weakly equivariantly equivalent.

Composing the maps in Lemma 2.1(4) and (2.2), we obtain a Σ_2-equivariant map

$$K\times K\setminus K\to K(\mathbb{Z}[(-1)^d],d).$$

This map defines an element in the equivariant cohomology group $\mathcal{O}_2(K)\in H^d_{\Sigma_2}(K\times K\setminus K;\mathbb{Z}[(-1)^d])$. This is the classical van Kampen obstruction. It is the same as the Euler class of the vector bundle (2.1). The classical van Kampen obstruction is a complete obstruction to the existence of a Σ_2-equivariant map $K\times K\setminus K\to\widetilde{S}^{d-1}$ when $d=2\dim(K)$. We are going to focus on the case when $4=d=2\dim(K)=\dim(K)+2$. In this case, the cohomological obstruction is a complete obstruction to the existence of an equivariant map (but not to the existence of an embedding $K\hookrightarrow\mathbb{R}^d$), and using the framed cobordism version does not add information. But in other situations $\mathcal{O}_2^{\mathrm{fr}}(K)$ contains more information than $\mathcal{O}_2(K)$.

Remark 2.2. The framed cobordism viewpoint points to a geometric interpretation of the van Kampen obstruction. It is perhaps even more convincing in the context of smooth manifolds. In that context, the analogue of the van Kampen obstruction is the obstruction for lifting from the first to the second stage of the Goodwillie-Weiss tower. In other words, it is the first obstruction to an immersion of a smooth manifold M into $\mathbb{R}^d$ being regularly homotopic to an embedding. This obstruction is an element in the relative equivariant cobordism group $\Omega_{fr}^{\widehat{\mathbb{R}}^d}(M\times M, M)$, and it can be interpreted as the framed cobordism class of the double points manifold of an immersion. This is explained, for example, in the introduction to [23]. In the case of topological embeddings of a 2-dimensional complex in $\mathbb{R}^4$, the van Kampen obstruction also can be interepreted as a double points obstruction. Of course this interpretation is well-known, and indeed it was how van Kampen thought about it. We review this in Section 4.1.

Now let us consider the next step. Suppose we have a finite complex K for which $\mathcal{O}_2(K)$ (or $\mathcal{O}_2^{\mathrm{fr}}(K)$) vanishes, and suppose we choose a Σ_2-equivariant map $f_2\colon \mathrm{C}(K,2) \to \mathrm{C}(\mathbb{R}^d, 2)$. We want to know if f_2 is Σ_2-equivariantly homotopic to the deleted square of some embedding $f\colon K \hookrightarrow \mathbb{R}^d$.

Suppose W is a space with an action of Σ_2. Then we endow the space $W \times W \times W$ with an action of Σ_3 via the homeomorphism $W \times W \times W \cong \mathrm{map}_{\Sigma_2}(\Sigma_3, W)$. In particular, the spaces $\mathrm{C}(X,2)^3$ (for any space X) and $(\widetilde{S}^{d-1})^3$ are equipped with a natural action of Σ_3 in this way.

For any space X, a Σ_3-equivariant map $\mathrm{C}(X,3) \to \mathrm{C}(X,2)^3$ is the same thing as a Σ_2-equivariant map $\mathrm{C}(X,3) \to \mathrm{C}(X,2)$, where $\Sigma_2 \subset \Sigma_3$ is identified with the subgroup permuting $1, 2$. There is an obvious Σ_2-equivariant projection map $\mathrm{C}(X,3) \to \mathrm{C}(X,2)$ which sends (x_1, x_2, x_3) to (x_1, x_2). This map induces a canonical Σ_3-equivariant map

$$\begin{array}{ccccccccc} p_X & : & \mathrm{C}(X,3) & \to & \mathrm{C}(X,2) & \times & \mathrm{C}(X,2) & \times & \mathrm{C}(X,2) \\ & & (x_1,x_2,x_3) & \mapsto & (x_1,x_2) & , & (x_2,x_3) & , & (x_3,x_1) \end{array} \tag{2.3}$$

This map is natural with respect to embeddings of X. Therefore, an embedding $f\colon K \hookrightarrow \mathbb{R}^d$ induces a commutative square as we saw in the introduction (1.4). Conversely, if $f_2\colon \mathrm{C}(K,2) \to \mathrm{C}(\mathbb{R}^d,2)$ is a Σ_2-equivariant map, then a necessary condition for f_2 to be equivarintly homotopic to the deleted square of an embedding is that the lifting problem in the following diagram has a Σ_3-equivariant solution

$$\begin{array}{ccc} & \nearrow & \mathrm{C}(\mathbb{R}^d,3) \\ & & \downarrow p_{\mathbb{R}^d} \\ \mathrm{C}(K,3) & \xrightarrow{(f_2)^3 \circ p_K} & \mathrm{C}(\mathbb{R}^d,2)^{\times 3} \end{array} \tag{2.4}$$

At this point we want to bring obstruction theory into play. For this, we need to examine the map $p_{\mathbb{R}^d}\colon \mathrm{C}(\mathbb{R}^d,3) \to \mathrm{C}(\mathbb{R}^d,2)^3$ a little more closely.

Let us recall some well-known facts about the effect of this map on cohomology. We shall introduce the following slight refinement of the notation for configurations spaces:

$$\mathrm{C}(\mathbb{R}^d, \{i,j\}) := \mathrm{Emb}(\{i,j\}, \mathbb{R}^d). \tag{2.5}$$

Next, let us give names to some cohomology classes. Let $u_{12} \in H^{d-1}(\mathrm{C}(\mathbb{R}^d, \{1,2\}))$ be a generator. More generally, for any two points i, j let $u_{ij} \in H^{d-1}(\mathrm{C}(\mathbb{R}^d, \{i,j\}))$ be the generator corresponding to u_{12} under homeomorphism induced by the bijection $\{1,2\} \cong \{i,j\}$.

Definition 2.3. The *Arnold class* is the following cohomological element.

$$u_{12}\otimes u_{23}\otimes 1+1\otimes u_{23}\otimes u_{31}+(-1)^{d-1}u_{12}\otimes 1\otimes u_{31} \in H^{2d-2}(\mathrm{C}(\mathbb{R}^d,\{1,2\})\times \mathrm{C}(\mathbb{R}^d,\{2,3\})\times \mathrm{C}(\mathbb{R}^d,\{3,1\})).$$

The following lemma is well-known

Lemma 2.4.

$$p_{\mathbb{R}^d}\colon \mathrm{C}\,(\mathbb{R}^d, 3) \to \mathrm{C}\,(\mathbb{R}^d, 2)^3$$

is surjective in cohomology, and its kernel in cohomology is the ideal generated by the Arnold class.

We refer to the statement of this lemma as the Arnold relation. The original reference is [1], where it is proved for configuration spaces in $\mathbb{R}^2$. The general result is proved in [8, Lemma 1.3 and Proposition 1.4]). The following corollary is an an easy consequence of the lemma, and is also well-known

Corollary 2.5. *The map $p_{\mathbb{R}^d}$ is $2d-3$-connected, and moreover it induces an isomorphism in homology and cohomology in degrees up to an including $2d-3$. In degree $2d-2$ there is an isomorphism of abelian groups $H_{2d-2}(\mathrm{C}\,(\mathbb{R}^d, 3)) \cong \mathbb{Z}^2$ and an isomorphism of Σ_3-modules*

$$H_{2d-2}(\mathrm{C}\,(\mathbb{R}^d, 2)^3) \cong \mathbb{Z}[\Sigma_3] \otimes_{\mathbb{Z}[\Sigma_2]} \mathbb{Z}[(-1)^{k-1}].$$

Moreover, the homomorphism in H_{2d-2} induced by $p_{\mathbb{R}^d}$ fits in a short exact sequence of Σ_3-modules

$$0 \to H_{2d-2}(\mathrm{C}\,(\mathbb{R}^d, 3)) \to H_{2d-2}(\mathrm{C}\,(R^d, 2)^3) \to Z[(-1)^{d-1}] \to 0$$

where the second homomorphism can be identified with the canonical surjection of Σ_3-modules

$$\mathbb{Z}[\Sigma_3] \otimes_{\mathbb{Z}[\Sigma_2]} \mathbb{Z}[(-1)^{k-1}] \to \mathbb{Z}[(-1)^{k-1}].$$

It is worth noticing that the short exact sequence splits, but not Σ_3-equivariantly.

Let F_d be the homotopy fiber of the map $p_{\mathbb{R}^d}$. It follows from the corollary that the first non-trivial homotopy group of F is $\pi_{2d-3}(F) \cong \mathbb{Z}[(-1)^{d-1}]$. The following result is a straightforward application of equivariant obstruction theory.

Proposition 2.6. *The first obstruction to the lifting problem in figure* (2.4) *is an element of the equivariant cohomology group $\mathcal{O}_3(K) \in H^{2d-2}_{\Sigma_3}(\mathrm{C}\,(K, 3); \mathbb{Z}[(-1)^{d-1}])$. This obstruction is complete if* $3\dim(K) = 2d-2$

It follows in particular that $\mathcal{O}_3(K)$ is a complete obstruction to the lifting problem in (2.4) if $\dim(K) = 2$ and $d = 4$.

One common way to think of cohomological obstruction to lifting a map is via the Postnikov tower. Now that we are looking at spaces with an action of Σ_3, let $K(\mathbb{Z}[(-1)^d], 2d-2)$ denote an Eilenberg-Mac Lane space with an action of Σ_3 that acts by $\mathbb{Z}[(-1)^d]$ on the non-trivial homotopy group. Lemma 2.8 below is, again, an easy consequence of Corollary 2.5. Before stating the lemma, let us review the definition of a k-(co)cartesian square diagram.

Definition 2.7. Suppose that we have a commutative diagram

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X_{12} \end{array}$$

One says that the diagram is *k-cartesian* if the induced map from X_0 to the homotopy pullback of

$$X_2 \to X_{12} \leftarrow X_1$$

is k-connected. Dually, the diagram is *k-cocartesian* if the induced map from the homotopy pushout

$$X_2 \leftarrow X_0 \rightarrow X_1$$

to X_{12} is k-connected.

Notice that if, say, $X_2 \simeq *$ then k-cartesian means that the map from X_0 to the homotopy fiber of the map $X_1 \to X_{12}$ is k-connected.

Lemma 2.8. *For a suitable model of $K(\mathbb{Z}[(-1)^d], 2d-2)$, there is a Σ_3-equivariant map*

$$\mathrm{C}\,(R^d, 2)^3 \to K(\mathbb{Z}[(-1)^{d-1}], 2d-2),$$

such that the composite map

$$\mathrm{C}\,(\mathbb{R}^d, 3) \xrightarrow{p_{\mathbb{R}^d}} \mathrm{C}\,(R^d, 2)^3 \to K(\mathbb{Z}[(-1)^{d-1}], 2d-2)$$

is equivariantly null-homotopic, and the following diagram is $2d-2$-cartesian

$$(2.6) \qquad \begin{array}{ccc} \mathrm{C}\,(\mathbb{R}^d, 3) & \longrightarrow & \mathrm{C}\,(\mathbb{R}^d, 2)^3 \\ \downarrow & & \downarrow \\ * & \longrightarrow & K(\mathbb{Z}[(-1)^{d-1}], 2d-2) \end{array} .$$

In terms of this lemma, the obstruction class $\mathcal{O}_3(K)$ is represented by the following composition

$$(2.7) \qquad \mathrm{C}\,(K, 3) \xrightarrow{p_K} \mathrm{C}\,(K, 2)^3 \xrightarrow{f_2^3} \mathrm{C}\,(\mathbb{R}^d, 2)^3 \to K(\mathbb{Z}[(-1)^{d-1}], 2d-2).$$

We saw earlier that the classical, cohomological van Kampen obstruction has a natural lift to a potentially stronger obstruction that lives in equivariant stable cohomotopy, a.k.a equivariant framed cobordism. The obstruction $\mathcal{O}_3(K)$ has a similar lift, which we denote $\mathcal{O}_3^{\mathrm{fr}}(K)$.

Till the end of this section, and in the next section, we focus on spaces with an action of Σ_3 and no other symmetric groups. Until the end of next section, let $\widehat{R}^2$ be the reduced standard representation of Σ_3, let $\widehat{\mathbb{R}}^{2d} = \widehat{R}^2 \otimes \mathbb{R}^d$, and let $\widehat{S}^{2d}$ be the one-point compactification of $\widehat{\mathbb{R}}^{2d}$. As a space, $\widehat{S}^{2d}$ is simply the $2d$-dimensional sphere. The 'hat' is there to indicate that it is a space with a specific action of Σ_3. In the same vein, let $\widehat{\Omega}^2\widehat{S}^{2d} = \mathrm{map}_*(\widehat{S}^2, \widehat{S}^{2d})$ be the double loop space $\Omega^2 S^{2d}$, on which Σ_3 acts via both S^2 and S^{2d}. Similarly define the space with Σ_3-action $\widehat{\Omega}^2\Omega^\infty\Sigma^\infty\widehat{S}^{2d}$. The following proposition is a refinement of Lemma 2.8

Proposition 2.9. *There is a $3d-5$-cartesian diagram of spaces with an action of Σ_3*

$$(2.8) \qquad \begin{array}{ccc} \mathrm{C}\,(\mathbb{R}^d, 3) & \longrightarrow & \mathrm{C}\,(\mathbb{R}^d, 2)^3 \\ \downarrow & & \downarrow \\ * & \longrightarrow & \widehat{\Omega}^2\Omega^\infty\Sigma^\infty\widehat{S}^{2d} \end{array} .$$

We will prove this proposition in the next section. For the rest of the section, we consider some consequences. It follows from the proposition that given a Σ_2-equivariant map $f_2\colon \mathrm{C}(K,2) \to \mathrm{C}(\mathbb{R}^d,2)$, a necessary condition for the lifting problem (2.4) to have a solution (and therefore also for f_2 to be equivariantly homotopic to the deleted square of some embedding) is that the following composition is Σ_3-equivariantly null-homotopic (compare with (2.7)):

$$\mathrm{C}(K,3) \xrightarrow{p_K} \mathrm{C}(K,2)^3 \xrightarrow{f_2^3} \mathrm{C}(\mathbb{R}^d,2)^3 \to \widehat{\Omega}^2\Omega^\infty\Sigma^\infty\widehat{S}^{2d}.$$

We interpret this composition as an element in the equivariant stable cohomotopy of $\mathrm{C}(K,3)$, or equivalently in the equivariant framed cobordism group $\mathcal{O}_3^{\mathrm{fr}}(K) \in \Omega_{fr}^{\widehat{\mathbb{R}}^{2(d-1)}}(\mathrm{C}(K,3))$. The class $\mathcal{O}_3^{\mathrm{fr}}(K)$ is an obstruction to a solution of the lifting problem (2.4). $\mathcal{O}_3^{\mathrm{fr}}(K)$ is a refinement of $\mathcal{O}_3(K)$ in the same way as $\mathcal{O}_2^{\mathrm{fr}}(K)$ is a refinement of $\mathcal{O}_2(K)$. $\mathcal{O}_3^{\mathrm{fr}}(K)$ is a complete obstruction to the lifting problem if $3\dim(K) \leq 3d-5$, while $\mathcal{O}_3(K)$ is a complete obstruction if $3\dim(K) \leq 2d-2$. Of course when $\dim(K) = 2$ and $d = 4$ both conditions hold, and $\mathcal{O}_3^{\mathrm{fr}}(K)$ does not really provide more information than $\mathcal{O}_3(K)$.

Remark 2.10. The obstruction $\mathcal{O}_3^{\mathrm{fr}}(M)$ in the context of smooth embeddings is the subject of Munson's paper [23]. In particular, Proposition 2.9 is proved there. We give a different proof in the next section. As a result, we will give a geometric interpretation of $\mathcal{O}_3^{\mathrm{fr}}(K)$ that is hinted at in [op. cit.].

3. Construction of a classifying map

In this section we prove Proposition 2.9. That is, we will construct a Σ_3-equivariant map $\mathrm{C}(\mathbb{R}^d,2)^3 \to \widehat{\Omega}^2\Omega^\infty\Sigma^\infty\widehat{S}^{2d}$ that makes the square 2.8 $3d-5$-cartesian. In fact, we will do something slightly stronger. Namely, we will construct a Σ_3-equivariant map (recall that $\widetilde{S}^{d-1}$ is Σ_2-equivariantly equivalent to $\mathrm{C}(\mathbb{R}^d,2)$)

$$f\colon (\widetilde{S}^{d-1})^3 \to \widehat{\Omega}^2\widehat{S}^{2d}$$

such that the following composition is Σ_3-equivariantly null-homotopic

$$\mathrm{C}(\mathbb{R}^3,3) \to (\widetilde{S}^{d-1})^3 \xrightarrow{f} \widehat{\Omega}^2\widehat{S}^{2d}$$

and moreover the following square diagram is $3d-5$-cartesian.

$$\begin{array}{ccc} \mathrm{C}(\mathbb{R}^d,3) & \longrightarrow & (\widetilde{S}^{d-1})^3 \\ \downarrow & & \downarrow f \\ * & \longrightarrow & \widehat{\Omega}^2\widehat{S}^{2d} \end{array} \quad . \tag{3.1}$$

We call a map f with these properties a *classifying map*. Since there is a natural map

$$\widehat{\Omega}^2\widehat{S}^{2d} \to \widehat{\Omega}^2\Omega^\infty\Sigma^\infty\widehat{S}^{2d}$$

that is $4d-3$-connected, it follows that the square (3.1) is $3d-5$-cartesian if and only if the square (2.8) is $2d-5$-cartesian.

The following lemma gives a practical way to verify that a given map is a classifying map.

Lemma 3.1. *Suppose that we have a Σ_3-equivariant map*

$$f\colon (\widetilde{S}^{d-1})^3 \to \widehat{\Omega}^2\widehat{S}^{2d}$$

satisfying the following conditions:

(1) The composite map

$$\mathrm{C}\,(\mathbb{R}^d, 3) \to (\widetilde{S}^{d-1})^3 \to \widehat{\Omega}^2\widehat{S}^{2d} \tag{3.2}$$

is equivariantly null-homotopic.

(2) f induces an epimorphism on H_{2d-2} (or, equivalently, a monomorphism on H^{2d-2}).

Then f is a classifying map.

Proof. By Lemma 2.4 and Corollary 2.5, the homology of the space $\mathrm{C}\,(\mathbb{R}^d, 3)$ is concentrated in degrees $0, d-1, 2(d-1)$. Similarly the homology of $(\widetilde{S}^{d-1})^3$ is concentrated in degrees $i(d-1)$, where $i \leq 3$. The map

$$\mathrm{C}\,(\mathbb{R}^d, 3) \to (\widetilde{S}^{d-1})^3$$

induces an isomorphism on H_{d-1} and a monomorphism on $H_{2(d-1)}$. The cokernel of this map in $H_{2(d-1)}$ is isomorphic to $\mathbb{Z}$, which is also isomorphic to $H_{2(d-1)}(\Omega^2S^{2d})$. Our assumption implies that the homomorphism from the cokernel of f in $H_{2(d-1)}$ to $H_{2(d-1)}(\Omega^2S^{2d})$ is an epimorphism from $\mathbb{Z}$ to $\mathbb{Z}$. Therefore it is an isomorphism. Since all the spaces in the diagram 3.1 have trivial homology in dimension above $2(d-1)$ and below $3(d-1)$, it follows that the square is $3d-4$-cocartesian. Furthermore, the maps from $\mathrm{C}\,(\mathbb{R}^d, 3)$ to $(\widetilde{S}^{d-1})^3$ and to $*$ are $2d-3$ and $d-1$-connected respectively. By the Blakers-Massey theorem, the square (3.1) is $3d-5$-cartesian. □

Now we are ready to construct a classifying map. We will use the Thom-Pontryagin collapse map associated with the diagonal inclusion $\widetilde{S}^{d-1} \hookrightarrow (\widetilde{S}^{d-1})^3$. To get a clean description of the Σ_3-equivariant properties of this collapse map, let us first consider a more general setting, where M is a manifold with a free action of Σ_2. The action of Σ_2 can be extended to an action of Σ_3 via the surjective homomorphism $\Sigma_3 \twoheadrightarrow \Sigma_2$. In this way, we consider M as a space with an action of Σ_3.

The group Σ_3 acts on M^3 via either one of the identifications

$$M^3 \cong \mathrm{map}_{\Sigma_2}(\Sigma_3, M) \cong \mathrm{map}(\Sigma_3/\Sigma_2, M).$$

The diagonal inclusion $\Delta\colon M \hookrightarrow M^3$ is a Σ_3-equivariant map (note again that the action of Σ_3 on M is not trivial). The normal bundle of this inclusion has an induced action of Σ_3. The normal bundle is Σ_3-equivariantly isomorphic to the quotient bundle $3\tau/\Delta(\tau)$. Here τ is the tangent bundle of M, $3\tau = \tau \oplus \tau \oplus \tau$, and $\Delta(\tau)$ is the diagonal copy of τ in 3τ. We denote the normal bundle by $\widehat{2\tau}$. It is the tensor product of τ with $\widehat{\mathbb{R}}^2$. Let $M^{\widehat{2\tau}}$ denote the Thom space of the normal bundle. The Thom-Pontryagin collapse map associated with Δ is a Σ_3-equivariant map $M^3 \to M^{\widehat{2\tau}}$.

Now apply this to the case $M = \widetilde{S}^{d-1}$, the $(d-1)$-dimensional sphere, endowed with the antipodal action of Σ_2. The Thom-Pontryagin collapse map has the form

$$(\widetilde{S}^{d-1})^3 \to (\widetilde{S}^{d-1})^{\widehat{2\tau}}$$

Note that this is an unpointed map, as the space $(\widetilde{S}^{d-1})^3$ does not have an equivariant basepoint. Sometimes we like to think of the collapse map as a pointed map

$$(\widetilde{S}^{d-1})^3_+ \to (\widetilde{S}^{d-1})^{\widehat{2\tau}}$$

Let us take smash product of this map with $\widehat{S}^2$, to obtain the following Σ_3-equivariant map

$$(\widetilde{S}^{d-1})^3_+ \wedge \widehat{S}^2 \to (\widetilde{S}^{d-1})^{\widehat{2\tau}} \wedge \widehat{S}^2.$$

Now observe that there is a homeomorphism

$$(\widetilde{S}^{d-1})^{\widehat{2\tau}} \wedge \widehat{S}^2 \cong (\widetilde{S}^{d-1})^{\widehat{2(\tau\oplus\mathbb{R})}}.$$

Next, recall that there is an isomorphism $\tau\oplus\mathbb{R} \cong \mathbb{R}^d$. It follows that there is a Σ_3-equivariant homeomorphism

$$(\widetilde{S}^{d-1})^{\widehat{2(\tau\oplus\mathbb{R})}} \cong (\widetilde{S}^{d-1})^{\widehat{2(\mathbb{R}^d)}} \cong \widetilde{S}^{d-1}_+ \wedge \widehat{S}^{2d}.$$

Next we compose this homeomorphism with the collapse map $\widetilde{S}^{d-1}_+ \wedge \widehat{S}^{2d} \to \widehat{S}^{2d}$, and precompose with the (suspended) Thom-Pontryagin collapse map above. We obtain the map

$$(\widetilde{S}^{d-1})^3_+ \wedge \widehat{S}^2 \to \widehat{S}^{2d}.$$

Taking an adjoint, we obtain an unpointed Σ_3-equivariant map

$$(\widetilde{S}^{d-1})^3 \to \widehat{\Omega}^2\widehat{S}^{2d}. \tag{3.3}$$

This is our model for a classifying map.

Lemma 3.2. *The map* (3.3) *is a classifying map.*

Proof. We need to check that the map satisfies the hypotheses of Lemma 3.1. The first hypothesis is that the composite map

$$\mathrm{C}\,(\mathbb{R}^d, 3) \to (\widetilde{S}^{d-1})^3 \to \widehat{\Omega}^2\widehat{S}^{2d}$$

is equivariantly null homotopic. By construction, the second map factors through the Thom-Pontryagin collapse map associated with the inclusion of the thin diagonal of $(\widetilde{S}^{d-1})^3$. Clearly the space $\mathrm{C}\,(\mathbb{R}^d, 3)$, which is the complement of the fat diagonal of $(\widetilde{S}^{d-1})^3$, is contained in the complement of the thin diagonal, and therefore the restriction of the Thom-Pontryagin collapse to $\mathrm{C}\,(\mathbb{R}^d, 3)$ is (equivariantly) null homotopic.

The second hypothesis that we need to check is that the following homomorphism is an epimorphism

$$H_{2d-2}((\widetilde{S}^{d-1})^3) \to H_{2d-2}(\widehat{\Omega}^2\widehat{S}^{2d})$$

This is equivalent to showing that the adjoint map

$$S^2 \wedge (S^{d-1} \times S^{d-1} \times S^{d-1})_+ \to S^{2d}$$

Induces an epimorphism on H_{2d} (till the end of this proof we will omit the 'tilde' and 'hat' decorations, since we are not concerned with the action of Σ_3 at this point). Once again we recall that this map factors through the Thom-Pontryagin collapse as follows

$$S^2 \wedge (S^{d-1} \times S^{d-1} \times S^{d-1})_+ \to S^2 \wedge (S^{d-1})^{2\tau} \xrightarrow{\cong} S^{d-1}_+ \wedge S^{2d} \to S^{2d}.$$

We need to prove that this composite map induces an epimorphism on H_{2d}. To see this, choose a point $* \in S^{d-1}$ and consider the inclusion $S^{d-1} \times S^{d-1} \times \{*\} \hookrightarrow (S^{d-1})^3$. This inclusion intersects the thin diagonal transversely at a single point $(*, *, *) \in (S^{d-1})^3$. It follows quite easily that the composite map

$$S^2 \wedge S^{d-1} \times S^{d-1} \times \{*\}_+ \to S^2 \wedge (S^{d-1} \times S^{d-1} \times S^{d-1})_+ \to S^{2d}$$

is the double suspension of the Thom-Pontryagin collapse map associated with the inclusion of a point $(*, *) \hookrightarrow S^{d-1} \times S^{d-1}$. In other words, it is the double suspension of the map $S^{d-1} \times S^{d-1} \to S^{2d-2}$ that collapses the complement of a Euclidean neighborhood of $(*, *)$. Clearly this map is surjective on H_{2d}, and therefore the map $S^2 \wedge (S^{d-1} \times S^{d-1} \times S^{d-1})_+ \to S^{2d}$ is also surjective on H_{2d}. □

3.1. **A geometric interpretation.** Lemma 3.2 leads to a kind of a geometric interpretation of our obstruction, at least when K is a manifold, or if $2\dim(K) = d = 4$, which is the case we are focusing on. In the manifold case, such an interpretation was hinted at by Munson [23].

So suppose K is a 2-dimensional complex and we have a Σ_2-equivariant map

$$f_2 \colon K \times K \setminus K \to \widetilde{S}^3.$$

Consider the set

$$\{(k_1, k_2, k_3) \in \mathrm{C}\,(K, 3) \mid f_2(k_1, k_2) = f_2(k_2, k_3) = f_2(k_3, k_1)\}.$$

This set is the preimage of the diagonal under the map

$$\mathrm{C}\,(K, 3) \xrightarrow{f_2^3 \circ p_K} \widetilde{S}^3 \times \widetilde{S}^3 \times \widetilde{S}^3.$$

Under a transversality assumption, this set defines a an element of $H^6_{\Sigma_3}(\mathrm{C}\,(K, 3), \mathbb{Z}[-1])$ (or even an element of the appropriate cobordism group) and this class is precisely our obstruction $\mathcal{O}_3(K)$ (or even $\mathcal{O}_3^{\mathrm{fr}}(K)$).

To see why this is plausible, suppose that f_2 is a normalized deleted square of some embedding $f \colon K \hookrightarrow \mathbb{R}^4$. I.e., suppose that

$$f_2(k_1, k_2) = \frac{f(k_2) - f(k_1)}{|f(k_2) - f(k_1)|}$$

Then for all k_1, k_2, k_3, the three vectors $f(k_2) - f(k_1), f(k_3) - f(k_2), f(k_1) - f(k_3)$ sum up to zero, while our obstruction consists of triples (k_1, k_2, k_3) where these three vectors would be co-directed.

(It is interesting to compare this with the interpretation of the second coefficient of the Conway polynomial of a knot in terms of collinear triples in [6].)

4. Embedding obstructions from Whitney towers

This section starts by reviewing a geometric formulation of van Kampen's obstruction (Section 4.1) and operations on Whitney disks (Section 4.2) which are commonly used in 4-manifold topology. These techniques are then used to establish new results: a relation between Whitney disks and equivariant maps of configuration spaces (Section 4.3) and higher embedding obstructions for 2-complexes in $\mathbb{R}^4$ based on intersections of Whitney disks: $\mathcal{W}_3(K)$ in Section 4.4 and $\mathcal{W}_{\mathrm{n}}(K), n > 3$ in Section 4.5. The relation of $\mathcal{W}_3(K)$ to the obstruction $\mathcal{O}_3(K)$ defined above is the subject of Section 5.

4.1. **The van Kampen obstruction.** The discussion in the paper so far concerned the general embedding problem for m-complexes in $\mathbb{R}^d$. Here we restrict to the original van Kampen's context where $d = 2m$. Later in this section we will specialize further to $m = 2$. We start by recalling a geometric description of the van Kampen obstruction

$$\mathcal{O}_2^{\mathrm{s}}(K) \in H^{2m}_{\Sigma_2}(\mathrm{C}_{\mathrm{s}}(K, 2); \mathbb{Z}) \tag{4.1}$$

to embeddability of an m-complex K into $\mathbb{R}^{2m}$. This was the construction outlined by van Kampen in [32]; the details were clarified in [30, 36], see also [12]. As in the introduction the notation $\mathrm{C}_{\mathrm{s}}(K, 2)$ denotes the "simplicial" configuration space $K \times K \smallsetminus \Delta$ where Δ consisting of all products of simplices $\sigma_1 \times \sigma_2$ having a vertex in common. The group Σ_2 acts on the configuration space $K \times K \smallsetminus \Delta$ by exchanging the factors; it may be seen from the description below that the action of Σ_2 on the coefficients is trivial, cf. [30, 22] (note that the sign was misstated as $(-1)^m$ in [12].)

The superscript in the notation $\mathcal{O}_2^{\mathrm{s}}(K)$ in (4.1) stands for "simplicial": it is an element of the cohomology group of $\mathrm{C}_{\mathrm{s}}(K, 2)$, while $\mathcal{O}_2(K)$ (considered in the introduction and in Section 2) is an element of the cohomology group of the configuration space $\mathrm{C}\,(K, 2)$ defined using the point-set diagonal. The invariant $\mathcal{O}_2(K)$ is the "universal" van Kampen obstruction, independent of the simplicial structure, and $\mathcal{O}_2^{\mathrm{s}}(K)$ may be recovered from it: $\mathcal{O}_2^{\mathrm{s}}(K) = i^*\,\mathcal{O}_2(K)$, where i is the inclusion map $\mathrm{C}_{\mathrm{s}}(K, 2) \subset \mathrm{C}\,(K, 2)$. A priori $\mathcal{O}_2^{\mathrm{s}}(K)$ could be a weaker invariant since it does not keep track of intersections of adjacent simplices. Nevertheless, it is a complete embedding obstruction for m-complexes in $\mathbb{R}^{2m}$ for $m > 2$: intersections of adjacent simplices may be removed using a version of the Whitney trick, cf. [12, Lemma 5].

Remark 4.1. The obstruction theory in Section 2 was developed for embeddings of finite CW complexes. The geometric approach presented here is based on intersection theory and it applies to finite simplicial complexes. We will interchangeably use the terms *cells* and *simplices* in the context of simplicial complexes; this should not cause confusion.

Consider any general position map $f : K \longrightarrow \mathbb{R}^{2m}$. Endow the m-cells of K with arbitrary orientations, and for any two m-cells σ_1, σ_2 without vertices in common, consider the algebraic intersection number $f(\sigma_1) \cdot f(\sigma_2) \in \mathbb{Z}$. This gives a Σ_2-equivariant cochain

$$o_f \colon C_{2m}(K \times K \smallsetminus \Delta) \longrightarrow \mathbb{Z}. \tag{4.2}$$

Since this is a top-dimensional cochain, it is a cocycle. Its cohomology class equals the van Kampen obstruction $\mathcal{O}_2^{\mathrm{s}}(K)$.

The fact that this cohomology class is independent of a choice of f may be seen geometrically as follows (see [12, Lemma 1, Section 2.4] for more details). Any two general position maps $f_0, f_1 \colon K \longrightarrow \mathbb{R}^{2m}$ are connected by a 1-parameter family of maps f_t where at a non-generic time t_i an m-cell σ intersects an $(m-1)$-cell ν. Topologically the maps $f_{t_i - \epsilon}$ and $f_{t_i+\epsilon}$ differ by a "finger move", that is tubing σ into a small m-sphere linking ν in $\mathbb{R}^{2m}$, Figure 1. The effect of this elementary homotopy on the van Kampen cochain is precisely the addition of the coboundary $\delta(u_{\sigma,\nu})$, where $u_{\sigma,\nu}$ is the Σ_2-equivaraint "elementary $(2m-1)$-cochain" dual to the $(2m-1)$-cells $\sigma \times \nu, \nu \times \sigma$.

This argument has the following corollary.

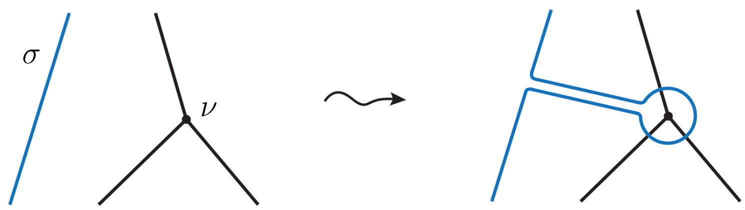

FIGURE 1. Finger move: homotopy of maps $f\colon K \longrightarrow \mathbb{R}^{2m}$

Lemma 4.2. *Any cocycle representative of the cohomology class* $\mathcal{O}_2^{\mathrm{s}}(K) \in H^{2m}_{\mathbb{Z}/2}(K \times K \smallsetminus \Delta; \mathbb{Z})$ *may be realized as the cocycle* o_f *for some general position map* $f\colon K \longrightarrow \mathbb{R}^{2m}$*. In particular, if the van Kampen obstruction* $\mathcal{O}_2^{\mathrm{s}}(K)$ *vanishes then there exists a general position map* $f\colon K \longrightarrow \mathbb{R}^{2m}$ *such that the cocycle* o_f *is identically zero. In other words, in this case for any two non-adjacent 2-cells* σ, τ *the algebraic intersection number* $f(\sigma)\cdot f(\tau)$ *is zero.*

4.2. **Operations on Whitney disks.** The rest of Section 4 concerns 2-complexes in $\mathbb{R}^4$. Assume the van Kampen class $\mathcal{O}_2^{\mathrm{s}}(K)$ vanishes. By Lemma 4.2, using finger moves on 2-cells as shown in Figure 1, a map f may be chosen so that $f(\sigma_i)\cdot f(\sigma_j) = 0$ for any non-adjacent 2-cells σ_i, σ_j. As usual, one groups intersection points $f(\sigma_i) \cap f(\sigma_j)$ into canceling pairs, chooses Whitney arcs connecting them in σ_i, σ_j, and considers Whitney disks W_{ij} for these intersections. Note that all Whitney arcs in each 2-cell may be assumed to be pairwise disjoint. Unlike the situation in higher dimensions where by general position a Whitney disk may be assumed to be embedded and to have interior disjoint from K, in 4-space generically W_{ij} will have self-intersections and also intersect the 2-cells of K. Moreover, the framing (the relative Euler number of the normal bundle of the Whitney disk) might be non-zero, but it may be corrected by boundary twisting [13, Section 1.3]. A detailed discussion of Whitney disks in this dimension is given in [13, Section 1.4]. This section summarizes the operations on Whitney disks and their relation with capped surfaces which will be used in the proofs in Section 5.

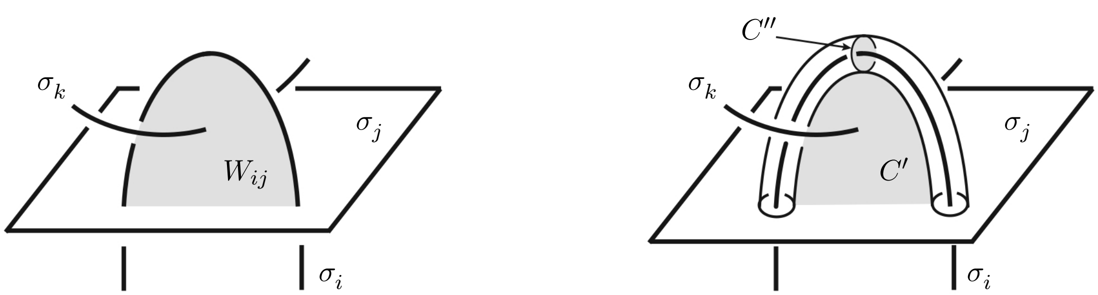

FIGURE 2. A Whitney disk and the associated capped surface

Convention. To avoid cumbersome notation, we will frequently omit the reference to a map f and keep the notation σ for the image of a cell σ under f.

A typical configuration is shown on the left in Figure 2. It is a usual representation in 3-space $\mathbb{R}^3 \times \{0\}$ (the 'present') of intersecting surfaces in $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ where the $\mathbb{R}$ factor is thought of as time. Here σ_j is pictured as a surface in $\mathbb{R}^3$ while σ_i, σ_k are arcs which extend as the product (arc$\times I$) into the past and the future. The Whitney disk $W_{ij} \subset \mathbb{R}^3 \times \{0\}$

pairs up two generic intersection points $\sigma_i \cap \sigma_j$ of opposite signs, and W_{ij} in the figure has a generic intersection point with another 2-cell σ_k. The result of the Whitney move in this setting is shown in Figure 3: the two intersection points $\sigma_j \cap \sigma_i$ are eliminated, but two new intersection points $\sigma_j \cap \sigma_k$ are created instead. In fact, the picture is symmetric with respect to the three sheets $\sigma_i, \sigma_j, \sigma_k$: a neighborhood of the Whitney disk W_{ij} in $\mathbb{R}^4$ is a 4-ball D^4, and the intersection of these three sheets with the boundary 3-sphere ∂D^4 forms the Borromean rings, as shown in Figure 9. Thus any two of the sheets can be arranged to be disjoint in this 4-ball, but not all three simultaneously.

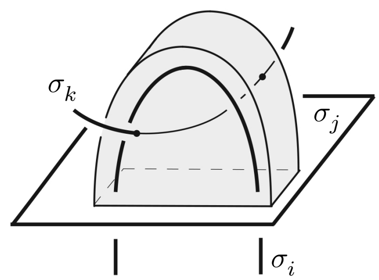

FIGURE 3. The result of the Whitney move

It will be convenient to view these intersections in the context of *capped surfaces* (or more generally *capped gropes* for higher-order intersections) [13, Chapter 2]. This is shown on the right in Figure 2: a tube is added to one of the two sheets, say σ_j as shown in the figure, to eliminate the two intersections $\sigma_i \cap \sigma_j$ at the cost of adding genus to σ_j. The new surface, still denoted σ_j, has two *caps*: disks attached to a symplectic pair of curves on σ_j. One of the caps, C', is obtained from the Whitney disk W_{ij}. The other cap is a disk normal to σ_i and may be thought of as a fiber of the normal bundle to σ_i. A general translation between Whitney towers and capped gropes is discussed in [24]. An advantage of this point of view is the symmetry between the original map of σ_j (intersecting σ_i in two points, as shown on the left in the figure) and the result of the Whitney move where the two intersections $\sigma_i \cap \sigma_j$ are eliminated but σ_j acquires two intersections with σ_k. The first case is obtained by ambient surgery of the capped surface in the figure on the right along the cap C'', and the second case is the surgery along C'. There is an intermediate operation, *symmetric surgery* (also known as *contraction*) [13, Section 2.3] that uses both caps that will be used in the arguments in the next section. The disk obtained by surgery on C' is isotopic to the surgery on C'', and the symmetric surgery may be thought of as the half point of the isotopy.

Consider the following *splitting* operation on Whitney disks. Suppose a Whitney disk W_{ij} pairing up intersections between σ_i, σ_j intersects two other 2-cells, σ_k, σ_l as shown on the left in Figure 4. Consider an arc in W_{ij} (drawn dashed in the figure) which separates the intersections $W_{ij} \cap \sigma_k$, $W_{ij} \cap \sigma_l$ and whose two endpoints are in the interiors of the two Whitney arcs forming the boundary of W_{ij}. Then a finger move on one of the sheets, say σ_i, along the arc introduces two new points of intersection $\sigma_i \cap \sigma_j$ and splits W_{ij} into two Whitney disks W'_{ij}, W''_{ij} as shown in the figure on the right. The advantage of the result is that each Whitney disk intersects only one other 2-cell. In general, if W_{ij} had m intersection points with other 2-cells, an iterated application of splitting yields $m-1$ Whitney disks, each one with a single intersection point in its interior.

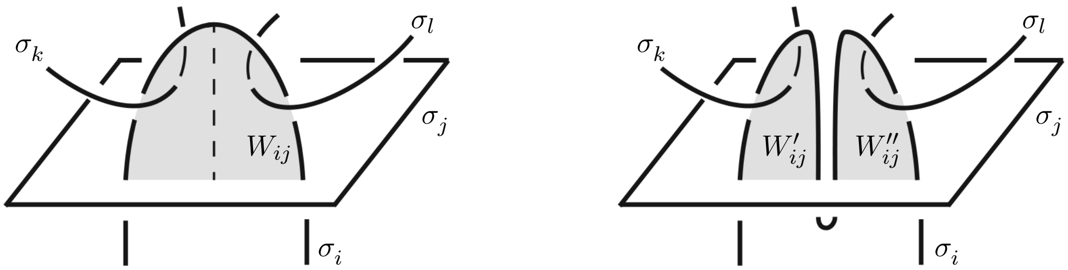

FIGURE 4. Splitting of a Whitney disk

The discussion above referred to the situation where a Whitney disk W_{ij} for $\sigma_i \cap \sigma_j$ intersects 2-cells which are not adjacent to σ_i, σ_j. In general, W_{ij} will have self-intersections as well as intersections with σ_i, σ_j and with 2-cells adjacent to them. Intersections of these types are not considered in the formulation of the obstruction in Section 4.4. (An obstruction involving these more subtle intersections will be explored in a future work. For example, the Arf invariant of a knot in S^3 may be defined using intersections of this type of the disk bounded by the knot in the 4-ball, see Remark 4.6.)

An ingredient in the formulation of higher obstructions in Section 4.4 is a local move on surfaces which replaces an intersection $\sigma_k \cap W_{ij}$ in Figure 2 with an intersection $\sigma_i \cap W_{jk}$ or $\sigma_j \cap W_{ik}$.

To describe this operation in more detail, start with the model situation in Figure 2 where W_{ij} has a single intersection point with σ_k. Perform a finger move on σ_k along an arc from $\sigma_k \cap W_{ij}$ to a point on the Whitney arc in σ_j. The result is shown on the left in Figure 5: now σ_k is disjoint from W_{ij} but there are two new intersections between σ_j and σ_k. The finger move isotopy of σ_k gives rise to a Whitney disk for these two points, denoted W'_{jk} in the figure. Note however that the two Whitney disks W_{ij}, W'_{jk} cannot be both used for Whitney moves since their boundary arcs intersect in σ_j. Resolving this intersection by an isotopy of the Whitney arc in the boundary of W'_{jk} yields a Whitney disk W_{jk} on the right in Figure 5; this Whitney disk has a single intersection point with σ_i. (Note that after this operation the Whitney disk W_{ij} is embedded and disjoint from other 2-cells; a Whitney move along this disk can be used to eliminate the original two intersections $\sigma_i \cap \sigma_j$.)

Therefore to have a well-defined triple intersection number one has to (1) sum over Whitney disks over all pairs of indices, and (2) require that Whitney arcs are disjoint, see Section 4.4.

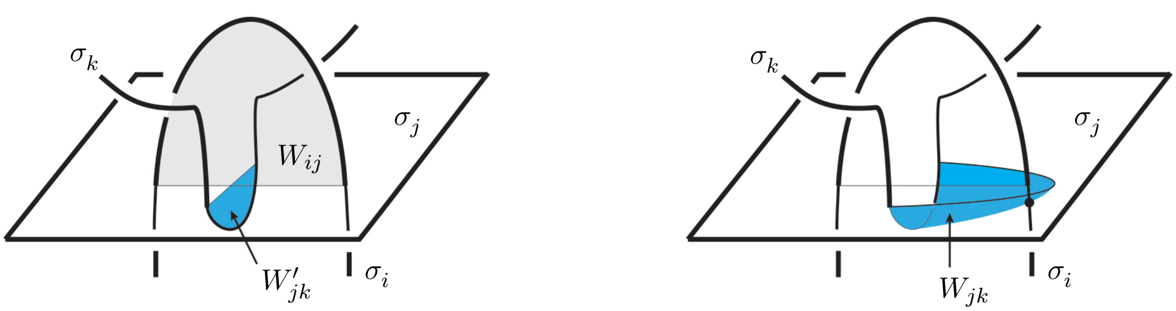

FIGURE 5. From $\sigma_k \cap W_{ij}$ to $\sigma_i \cap W_{jk}$.

4.3. **From Whitney disks to equivariant maps of configuration spaces.** Let K be a 2-complex and suppose the van Kampen obstruction $\mathcal{O}_2^{\mathrm{s}}(K)$ vanishes. Then by Lemma 4.2 there is a map $f\colon K \longrightarrow \mathbb{R}^4$ so that the algebraic intersection number of any two non-adjacent 2-cells in $\mathbb{R}^4$ is zero. As in Section 4.2, pair up the intersections with Whitney disks, so that all Whitney arcs are disjoint in each 2-cell. This condition on the Whitney arcs will be assumed throughout the rest of the paper. The following lemma shows that f together with a choice of Whitney disks W gives rise to a Σ_2-equivariant map $\mathrm{C_s}(K,2) \longrightarrow \mathrm{C}\,(\mathbb{R}^d,2)$. The proof of this lemma explains a basic idea underlying the connection between geometric and homotopy-theoretic approaches to obstruction theory that is established in this paper. A more involved version of this argument will be given in Section 5 to show that there exists a Σ_3-equivariant map of the 5-skeleton of $\mathrm{C_s}(K,3)$ to $\mathrm{C}\,(\mathbb{R}^d,3)$. Recall from Section 4.2 that any given collection of Whitney disks may be *split*, so that any Whitney disk has at most one intersection with a 2-cell of K.

Lemma 4.3. *Let K be a 2-complex and $f\colon K \longrightarrow \mathbb{R}^4$ a general position map such that all intersections of non-adjacent 2-cells are paired up with split Whitney disks W. This data determines a Σ_2-equivariant map $F_{f,W}\colon \mathrm{C_s}(K,2) \longrightarrow \mathrm{C}\,(\mathbb{R}^d,2)$.*

Proof. Given any pair of non-adjacent 2-cells σ_i, σ_j, by assumption all intersections $f(\sigma_i)\cap f(\sigma_j)$ are paired up with Whitney disks W_{ij}, and the Whitney arcs in each 2-cell are disjoint. The self-intersections and intersections of the Whitney disks will not be relevant in the following argument because the simplicial diagonal Δ is missing in the configuration space $\mathrm{C_s}(K,2)$. Since the Whitney disks are split, each W_{ij} intersects a single 2-cell σ_k as in Figure 2. We treat the special case that σ_k is either σ_i or σ_j right away: if W_{ij} intersects σ_i, perform the Whitney move along W_{ij} on σ_i; if it intersects σ_j then perform the Whitney move of σ_j. This results in self-intersections of either $f(\sigma_i)$ or $f(\sigma_j)$ which are irrelevant since we are working with the simplicial configuration space $\mathrm{C_s}(K,2)$, and so the map $F_{f,W}$ does not need to be defined on $\sigma_i\times\sigma_i$, $\sigma_j\times\sigma_j$. Thus the remaining intersections of W_{ij} are with 2-cells σ_k, $k\neq i,j$.

Next we describe the desired map $F_{f,W}\colon \mathrm{C_s}(K,2) \longrightarrow \mathrm{C}\,(\mathbb{R}^4,2)$. By general position the 1-cells and the 2-cells of K are mapped in disjointly by f, so $f\times f$ defines a Σ_2-equivariant map on the 3-skeleton of $\mathrm{C_s}(K,2)$. Thus the goal is to extend it to the 4-skeleton, that is to define $F_{f,W}$ on each product of two non-adjacent 2-cells $\sigma_i\times\sigma_j$. For each such pair σ_i,σ_j we pick an order (i,j); for the other product $\sigma_j\times\sigma_i$ the map $F_{f,W}$ will be defined using Σ_2 equivariance.

In each 2-cell σ_i consider disjoint disk neighborhoods of the Whitney arcs for the intersections of $f(\sigma_i)$ with other 2-cells; the disk neighborhoods corresponding to W_{ij} are denoted D_{ij}, Figure 6. (In general W_{ij} denotes the entire collection of Whitney disks for $f(\sigma_i)\cap f(\sigma_j)$, and D_{ij} denotes the collection of corresponding disk neighborhoods; we illustrate the case of a single component since the argument in general is directly analogous.) If $f(\sigma_i)\cap f(\sigma_j)=\emptyset$, D_{ij} is defined to be empty. Now consider the map $\widetilde{f}_{ij}\colon K \longrightarrow \mathbb{R}^4$ which coincides with f in the complement of the disk D_{ij}. In this disk $\widetilde{f}_{ij}$ is defined to be the result of the Whitney move on $f(\sigma_i)$ along the Whitney disk W_{ij}, making $\widetilde{f}(\sigma_i)$ disjoint from $\widetilde{f}(\sigma_j)$. If W_{ij} intersected another 2-cell σ_k as in Figure 2, as a result of this move $\widetilde{f}_{ij}(\sigma_i)$ intersects $\widetilde{f}_{ij}(\sigma_k)=f(\sigma_k)$.

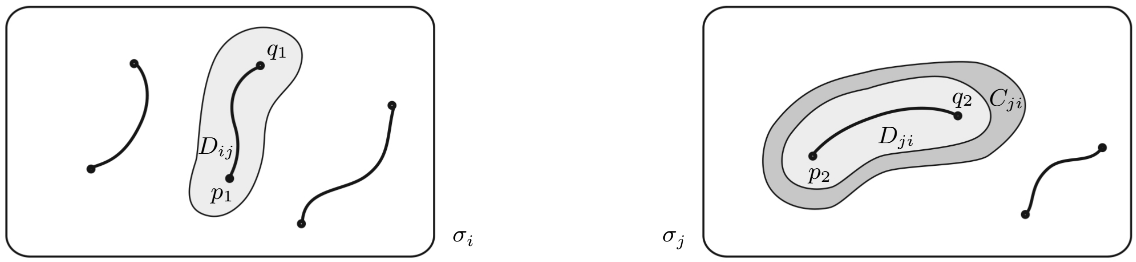

FIGURE 6. Defining the map $\sigma_i \times \sigma_j \longrightarrow \mathrm{C}(\mathbb{R}^4, 2)$. Here $f(p_1) = f(p_2)$ and $f(q_1) = f(q_2)$ are two double points in $f(\sigma_i) \cap f(\sigma_j)$.

Consider a collar $C_{ji} = \partial D_{ji} \times I$ on ∂D_{ji} in $\sigma_j \smallsetminus \mathrm{int}(D_{ji})$, Figure 6. The collars are chosen small enough so that they are disjoint from each other in σ_j for various Whitney arcs. Define

$$F_{f,W}|_{\sigma_i \times (\sigma_j \smallsetminus (D_{ji} \cup C_{ji}))} := (f \times f)|_{\sigma_i \times (\sigma_j \smallsetminus (D_{ji} \cup C_{ji}))}. \tag{4.3}$$

This defines a map into the configuration space $\mathrm{C}(\mathbb{R}^4, 2))$ since $f(\sigma_i)$ is disjoint from $f(\sigma_j \smallsetminus (D_{ji} \cup C_{ji}))$. On $\sigma_i \times D_{ji}$ the map is defined using the result of the Whitney move:

$$F_{f,W}|_{\sigma_i \times D_{ji}} := (\widetilde{f}_{ij} \times \widetilde{f}_{ij})|_{\sigma_i \times D_{ji}} = (\widetilde{f}_{ij} \times f)|_{\sigma_i \times D_{ji}} \tag{4.4}$$

It remains to define $F_{f,W}$ on $\sigma_i \times C_{ji}$ interpolating between the maps (4.3), (4.4). If the Whitney disk W_{ij} was framed and embedded then the original map f and the result of the Whitney move $\widetilde{f}_{ij}$ would be isotopic, with the isotopy supported in the interior of D_{ij}. In general, without these assumptions, these maps are homotopic rather than isotopic. Denote by $f^t_{ij} \colon K \times I \longrightarrow \mathbb{R}^4$ this homotopy $f \simeq \widetilde{f}_{ij}$ given by the Whitney move, and supported in D_{ij}.

Identify $(x, y, t) \in \sigma_i \times \partial D_{ji} \times [0, 1]$ with $(x, y_t) \in \sigma_i \times C_{ji}$ using the product structure on the collar C_{ji}. Using this identification, the following map sends a point (x, y_t) to $(f^t_{ij}(x), f(y_t))$:

$$F_{f,W}|_{\sigma_i \times C_{ji}} := (f^t_{ij} \times f^t_{ij})|_{\sigma_i \times C_{ij}} = (f^t_{ij} \times f)|_{\sigma_i \times C_{ij}}. \tag{4.5}$$

This matches $\widetilde{f}_{ij} \times f$ on $D_i \times \partial D_j$ and $f \times f$ on $D_i \times \partial(D_j \cup C)$. The result is a continuous map $\sigma_i \times \sigma_j \longrightarrow \mathrm{C}(\mathbb{R}^4, 2)$, giving rise to a desired Σ_2-equivariant map $\mathrm{C_s}(K, 2) \longrightarrow \mathrm{C}(\mathbb{R}^d, 2)$. □

A key point in the above proof is that even though the result of the Whitney move $\widetilde{f}_{ij}(\sigma_i)$ intersects $\widetilde{f}_{ij}(\sigma_k) = f(\sigma_k)$, this does not affect the definition of the map $F_{f,W}$ on $\sigma_i \times \sigma_k$. The assumption of Lemma 4.3 is insufficient for producing a map of 3-point configuration spaces, as we make precise in the next subsection.

4.4. **An obstruction from intersections of Whitney disks.** We are now in a position to formulate our geometric embedding obstruction for 2-complexes in $\mathbb{R}^4$ which is defined when the van Kampen obstruction vanishes. Under this assumption, following Lemma 4.2 consider a map $f \colon K \longrightarrow \mathbb{R}^4$ where the intersection number of any two non-adjacent 2-cells $f(\sigma_i) \cap f(\sigma_j)$ in $\mathbb{R}^4$ is zero. As in Section 4.3, consider a collection $W = \{W_{ij}\}$ of Whitney disks for $f(K)$, where W_{ij} denotes the Whitney disks for $f(\sigma_i) \cap f(\sigma_j)$. As above, the Whitney arcs are assumed to be disjoint in each 2-cell σ_i.

The obstruction $\mathcal{W}_3(K)$, defined below, depends on the choice of f and of Whitney disks W. Indeed, in the context of obstruction theory one expects that higher obstructions generally depend on choices of trivializations of lower order obstructions. Recall from Section 2 that the obstruction $\mathcal{O}_3(K)$ to lifting to a Σ_3-equivariant map $\mathrm{C}\,(K,3) \longrightarrow \mathrm{C}\,(\mathbb{R}^4,3)$ depends on the choice of a Σ_2-equivariant map $f_2\colon\ \mathrm{C}\,(K,2) \longrightarrow \mathrm{C}\,(\mathbb{R}^4,2)$. Moreover, by Lemma 4.3 the geometric data – f and W – determine such a map f_2 on the simplicial configuration space $\mathrm{C_s}(K,2)$. The relation between the two theories is extended further in Section 5.

Definition 4.4 (*The obstruction* $\mathcal{W}_3(K)$)**.** Let K, f, W be as above, and endow the 2-cells of K with arbitrary orientations. The orientation of Whitney disks W_{ij}, where (i,j) is an *ordered* pair, is induced from the orientation on its boundary which is oriented from $-$ intersection to $+$ intersection along $f(\sigma_i)$ and from from $+$ to $-$ along $f(\sigma_i)$. Consider the 6-cochain:

$$w_3\colon\ C_6(\mathrm{C_s}(K,3)) \longrightarrow \mathbb{Z}, \tag{4.6}$$

defined as follows. Let $\sigma_i, \sigma_j, \sigma_k$ be 2-cells of K which pairwise have no vertices in common, and define

$$w_3(\sigma_i \times \sigma_j \times \sigma_k) \;=\; W_{ij}\cdot f(\sigma_k) + W_{jk}\cdot f(\sigma_i) + W_{ki}\cdot f(\sigma_j), \tag{4.7}$$

where the algebraic intersection numbers are defined using the orientation convention discussed above. Note that changing the order of i, j reverses the orientation of W_{ij}, so the cochain w_3 in (4.7) is Σ_3 equivariant, where Σ_3 acts on $\mathbb{Z}$ according to the sign representation. This 6-cochain is a cocycle since it is a top-dimensional cochain on $\mathrm{C_s}(K,3)$. The resulting cohomology class is denoted

$$\mathcal{W}_3(K,f,W) \in H^6_{\Sigma_3}(\mathrm{C_s}(K,3);\mathbb{Z}[(-1)]).$$

When f, W are clear from the context, the notation will be abbreviated to $\mathcal{W}_3(K)$.

It is worth noting that the local move in Figure 5 shifts the intersection numbers between the terms of (4.7); it is the sum that gives a meaningful invariant (see also Remark 4.6 below.) Geometrically (4.7) measures intersection numbers that are an obstruction to finding disjoint embedded Whitney disks needed to construct an embedding $K \hookrightarrow \mathbb{R}^4$. The definition depends on various choices: the pairing of $\pm$ intersections of $f(\sigma_i) \cap f(\sigma_j)$, and choices of Whitney arcs and of Whitney disks. By comparing it to the obstruction $\mathcal{O}_3(K)$ in the next section, we show that it really depends only on the homotopy class of the map $F_{f,W}$ constructed in Lemma 4.3, a fact that is not apparent from the geometric framework of the above definition.

In addition to these cell-wise intersection considerations, of course properties of the obstruction $\mathcal{W}_3(K)$ depend on the cohomology of the configuration space $\mathrm{C_s}(K,3)$. This aspect of the obstruction is discussed in Lemma 4.9, and the consequence of its vanishing is the subject of Section 4.5.

Remark 4.5. It is not difficult to see that in the example of [12] there is a map of the 2-complex into $\mathbb{R}^4$ with precisely two 2-cells intersecting in two algebraically canceling points, with a Whitney disk intersecting one other 2-cell as in Figure 2. It follows that the corresponding cochain (4.6) is non-zero on precisely one 6-cell of $\mathrm{C_s}(K,3)$; this example is discussed in detail in Section 6.

Remark 4.6. Our Definition 4.4 extends to the setting of 2-complexes in $\mathbb{R}^4$ the idea of using intersections of Whitney disks with surfaces that has been widely used in 4-manifold topology. The construction of this type in the simplest *relative* case: $K^2 = \coprod^3 D^2$, the disjoint union of three disks whose boundary curves form a given three-component link L in $S^3 = \partial D^4$, is a reformulation of Milnor's $\bar{\mu}$-invariant [21] $\bar{\mu}_{123}(L)$, sometimes referred to as the triple linking number. Such intersections were used to define an obstruction to representing three homotopy classes of maps of 2-spheres into a 4-manifold by maps with disjoint images in [20, 37], and in the non-simply connected setting in [25]. A version considering self-intersections to define the Arf invariant and the Kervaire-Milnor invariant was given in [13, 10.8A], and an extension to non-simply connected 4-manifolds in [25].

The definition of $\mathcal{W}_3(K)$ shares some of the nice features of the geometric definition (4.2) of the van Kampen obstruction. Specifically, we will now describe the higher order analogue ("stabilization") of the finger move homotopy in Figure 1 and of Lemma 4.2.

Definition 4.7 (*Stabilization*)**.** This operation applies to any two 2-cells σ_1, σ_2 and a 1-cell ν of K which are all pairwise non-adjacent, Figure 7a. Perform a finger move introducing two canceling σ_1-σ_2 intersections, and let W'_{12} denote the resulting embedded Whitney disk pairing these two intersection, Figure 7b. Also consider S^2_ν, a small 2-sphere linking $f(\nu)$ in $\mathbb{R}^4$. The final modification applies to the Whitney disk: $\overline{W}_{12}$ is formed as a connected sum of W_{12} and S^2_ν, Figure 7c.

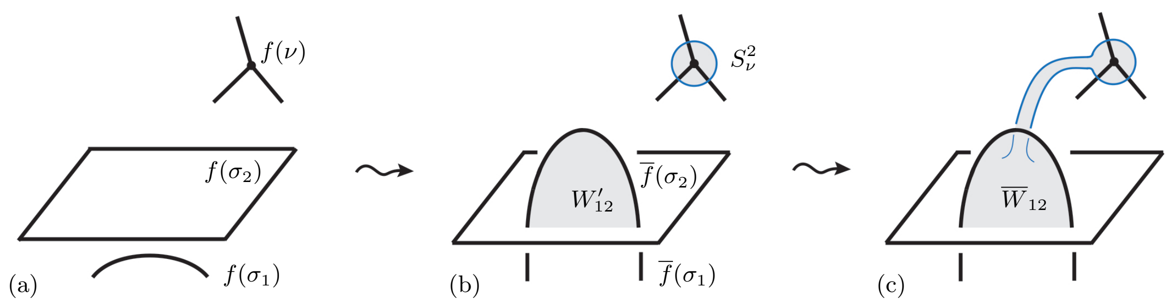

FIGURE 7. Stabilization (modifying the obstruction cocycle by a coboundary)

Proposition 4.8. *Let $(\overline{f}, \overline{W})$ be the result of a stabilization applied to (f, W). Then the Σ_2-equivariant map $F_{\overline{f},\overline{W}}\colon \mathrm{C_s}(K,2) \longrightarrow \mathrm{C}(\mathbb{R}^4, 2)$ associated to $(\overline{f}, \overline{W})$ in Lemma 4.3 is Σ_2-equivariantly homotopic to $F_{f,W}$.*

Proof. The Whitney disk $\overline{W}_{12}$ is used only in the restriction of the map $F_{\overline{f},\overline{W}}$ to $\sigma_1 \times \sigma_2$ (and equivariantly to $\sigma_2 \times \sigma_1$). When $f(\nu)$ and all 2-cells adjacent to it are omitted from the picture, the Whitney disks $\overline{W}_{12}, W'_{12}$ in Figure 7 are isotopic. Thus it is clear from the proof of Lemma 4.3 that the maps of configurations spaces corresponding to these two Whitney disks are homotopic. (Note that the interior of $\overline{W}_{12}$ is disjoint from $f(\sigma_1)$ since σ_1, ν were assumed to be non-adjacent. Thus the result of the Whitney move on $\overline{f}(\sigma_2)$ along $\overline{W}_{12}$ is disjoint from $f(\sigma_1)$.) Moreover, the map f in Figure 7a is isotopic to the result of the Whitney move applied to $\overline{f}$ in Figure 7b, so the induced maps on configuration spaces are again homotopic. □

We are in a position to formulate the analogue of Lemma 4.2 for the new obstruction.

Lemma 4.9. *Any cocycle representative of the cohomology class*

$$\mathcal{W}_3(K,f,W)\in H^6_{\Sigma_3}(\mathrm{C_s}(S,3);\mathbb{Z}[-1])$$

may be realized as the cocycle $w_3(K,f',W')$ associated to some map f' and Whitney disks W'. In particular, if the cohomology class $\mathcal{W}_3(K,f,W)$ is trivial then there exist f',W' whose associated cocycle is identically zero.

Proof. Consider a generator $C_{\sigma_1,\sigma_2,\nu}$ of Σ_3-equivariant 5-cochains on $\mathrm{C_s}(S,3)$, corresponding to non-adjacent 2-cells σ_1,σ_2 and 1-cell ν of K. The stabilization operation $(f,W)\mapsto(\overline{f},\overline{W})$, shown in Figure 7, changes the cocycle $w_3(K,f,W)$ by a coboundary $\pm\delta C_{\sigma_1,\sigma_2,\nu}$, where the sign depends on the orientation of the sphere S^2_ν. Thus changing $w_3(K,f,W)$ by any coboundary may be realized by a suitable sequence of stabilizations. □

As we explain in the next subsection, the vanishing of the cohomology class $\mathcal{W}_3(K,f,W)$ has a geometric consequence: the existence of another layer of Whitney disks, in turn leading to a higher order obstruction.

4.5. **Higher order obstructions from Whitney towers.** The notion of Whitney towers encodes higher order intersections of surfaces in 4-manifolds, where the vanishing of the intersections inductively enables one to find the next layer of Whitney disks. In a sense Whitney towers approximate an embedded disk as the number of layers increases. A closely related notion of capped gropes [13, Chapter 2] is extensively used in the theory of topological 4-manifolds: they may be found in the context of surgery and of the s-cobordism conjecture where surfaces have duals, cf. Proof of Theorem 5.1A in [13]. We will use the notion of Whitney towers and their intersection theory developed in [26, 27]. Only a brief summary of the relevant definitions is given below; the reader is referred to the above references for details.

In the setting of this paper the ambient 4-manifold is $\mathbb{R}^4$, and the surfaces are the images of non-adjacent 2-cells of a 2-complex K under a general position map $f\colon K\longrightarrow\mathbb{R}^4$. Moreover, we will use the *non-repeating* version of Whitney towers considered in [27].

Whitney towers have a parameter, *order*, and are defined inductively. Whitney towers of order 0 are just surfaces in general position in a 4-manifold. Their intersection numbers may be used to define the van Kampen obstruction, as discussed in Section 4.1. A Whitney tower of order 1 is a collection of surfaces with trivial intersection numbers, together with a collection of Whitney disks pairing up the intersection points. (As in the preceding sections, all Whitney disks are assumed to be framed, and have disjoint boundaries.) This is the setting for the obstruction in Definition 4.4. Note that the Whitney tower incorporates both the map f and the Whitney disks W, so $\mathcal{W}_3(K,f,W)$ may be thought of as being defined in terms of a Whitney tower.

All surface stages and intersection points between them in a general Whitney tower are inductively assigned an order in $\mathbb{Z}_{\geq 0}$ as follows. The base of the construction (*order* 0) is a collection of the original immersed surfaces in $\mathbb{R}^4$. All surfaces of higher order are Whitney disks pairing up intersections of surfaces of lower order. The order of an intersection point of surfaces of orders n_1,n_2 is defined to be n_1+n_2. A Whitney disk pairing up intersection points of order n is said to have order $n+1$.

Finally, a Whitney tower W of order $n+1$ is defined inductively as a Whitney tower of order n together with a collection of Whitney disks pairing up all intersections of order n. For example, a tower of order 2 is illustrated on the left in Figure 8, with the surfaces σ of order 0 and Whitney disks V of order 1 and W of order 2. We say that a map $f\colon K \longrightarrow \mathbb{R}^4$ *admits a Whitney tower* of order n if this condition holds for the images under f of each n-tuple of pairwise non-adjacent 2-cells. Note that given a 2-complex K, an obstruction to the existence of a map f admitting a Whitney tower of order n for any $n \geq 1$ is in particular an obstruction to the existence of an embedding $K \hookrightarrow \mathbb{R}^4$.

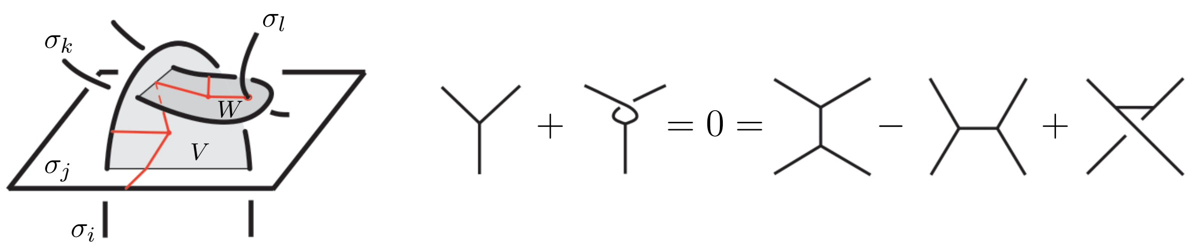

FIGURE 8. Left: a Whitney tower of order 2 and the associated tree. Right: the AS relation and the IHX relation

With this terminology at hand, we are ready to formulate a geometric consequence of Lemma 4.9.

Corollary 4.10. *Let $f\colon K \longrightarrow \mathbb{R}^4$ be an immersion with double points paired up with Whitney disks W, as in Section 4.4. Suppose the cohomology class*

$$\mathcal{W}_3(K, f, W) \in H^6_{\Sigma_3}(\mathrm{C_s}(S, 3); \mathbb{Z}[-1])$$

is trivial. Then there exists a map $\widetilde{f}\colon K \longrightarrow \mathbb{R}^4$ which admits a Whitney tower of order 2.

Indeed, by Lemma 4.9 there exists a map f' and Whitney disks W' such that for each triple of (pairwise non-adjacent) 2-cells, the intersection invariant (4.7) is trivial. By [26, Theorem 2], the map $\widetilde{f}$ is regularly homotopic to f' which admits admits a Whitney tower of order 2, as claimed.

It follows from Lemma 4.2 that if K has trivial van Kampen's obstruction, there exists a map of K into $\mathbb{R}^4$ which admits a Whitney tower of height 1. Corollary 4.10 gives the analogue for the next obstruction: if the class $\mathcal{W}_3(K, f, W) = 0$, there exists a map admitting a Whitney tower of height 2. To define higher obstruction theory, we will now discuss the intersection invariants of Whitney towers.

The obstruction cochain in equation (4.7) was defined using an explicit formula with intersection numbers between Whitney disks and 2-cells. An elegant way of formulating the intersection invariant [26] for a general Whitney tower is in terms of trees, described next.

Each unpaired intersection point p of a Whitney tower determines a trivalent tree t_p: the trivalent vertices correspond to Whitney disks and the leaves are labeled by (distinct) 2-cells of K. The tree embeds in the Whitney tower, as shown on the left in Figure 8, and it inherits a cyclic orientation of each trivalent vertex from this embedding. (Recall that Whitney disks are oriented as in Definition 4.4.)

The relevant obstruction group in our context will be denoted $\mathcal{T}_n$. It is defined as a quotient of the free abelian group generated by trivalent trees with $n+2$ leaves (and thus n trivalent vertices). The leaves are labeled by non-repeating labels $\{1, \ldots, n+2\}$, and the trivalent vertices are cyclically oriented. The quotient is taken with respect to the AS and IHX relations, shown on the right in Figure 8. These relations are well-known in the study of finite type invariants; in the context of Whitney towers the AS (anti-symmetry) relation corresponds to switching orientations of Whitney disks, and the IHX relation reflects choices of Whitney arcs, see [9].

Following [26, Section 2.1], the intersection tree τ_n of an order n Whitney tower W is defined to be

$$\tau_n(W) := \sum_p \epsilon(p)\, t_p \ \in \mathcal{T}_n, \tag{4.8}$$

where the sum is taken over all unpaired (order n) intersections points p, and $\epsilon(p)$ is the sign of the intersection. For example, for order 1 Whitney tower the intersection trees are the Y tree with two possible cyclic orderings of the trivalent vertex; the obstruction group $\mathcal{T}_1$ is isomorphic to $\mathbb{Z}$, and the intersection invariant matches the formula (4.7).

Let $\mathrm{C_s}(K, n)$ denote $K^{\times n}$ minus the simplicial diagonal consisting of all products of simplices $\sigma_1 \times \ldots \times \sigma_n$, where at least two of the simplices σ_i, σ_j have a vertex in common for some $i \neq j$. The symmetric group Σ_n acts in a natural way on the configuration space $\mathrm{C_s}(K, n)$ and also on $\mathcal{T}_{n-2}$. The following definition extends Definition 4.4 to all $n \geq 3$.

Definition 4.11 (*The obstruction* $\mathcal{W}_{\mathrm{n}}(K)$)**.** Let $n \geq 3$ and suppose a map $f\colon K \longrightarrow \mathbb{R}^4$ admits a Whitney tower W of order $n-2$. Endow the 2-cells of K with arbitrary orientations; the orientation of all Whitney disks in W are then determined as in Definition 4.4. Consider the Σ_n-equivariant $2n$-cochain:

$$w_n\colon\ C_{2n}(\mathrm{C_s}(K, n)) \longrightarrow \mathcal{T}_{n-2}, \tag{4.9}$$

whose value on the $2n$-cell $\sigma_1 \times \ldots \times \sigma_n$ is given by the intersection invariant (4.8) of the Whitney tower on the 2-cells $f(\sigma_1), \ldots, f(s_n)$. It is a cocycle since it is a top-dimensional cochain on $\mathrm{C_s}(K, 2n)$. The resulting cohomology class is denoted

$$\mathcal{W}_{\mathrm{n}}(K, W) \in H^{2n}_{\Sigma_n}(\mathrm{C_s}(K, n); \mathcal{T}_{n-2}).$$

Thus $\mathcal{W}_{\mathrm{n}}(K, W)$ is an obstruction to increasing the order of a given Whitney tower W to $n-1$; in particular it is an obstruction to using the data of the Whitney tower W to find an embedding of K.

Remark 4.12. Note that $\mathcal{T}_{n-2}$ is isomotrphic to $\mathbb{Z}^{(n-2)!}$, cf. [27, Lemma 19]; compare this with the coefficients of the cohomology group in Theorem 7.11.

We note that there is an analogue of stabilization in Definition 4.7 for higher trees generating $\mathcal{T}_n$, and an analogue of Corollary 4.10 for higher obstructions $\mathcal{W}_n$. Thus there is an obstruction theory for 2-complexes in $\mathbb{R}^4$ formulated entirely within the context of intersections of Whitney towers. As we mentioned previously, the focus of this paper is on the first new obstruction, $\mathcal{W}_3$; we plan to study higher obstructions in more detail in a future work. $\mathcal{O}_3(K)$ and $\mathcal{W}_3(K)$ are related in the next section; a conjectural relation between $\mathcal{O}_{\mathrm{n}}(K)$ and $\mathcal{W}_{\mathrm{n}}(K)$ for $n > 3$ is stated in Section 8.

5. The obstructions $\mathcal{O}_3(K)$ and $\mathcal{W}_3(K)$ are equal

Here we will relate the obstruction $\mathcal{O}_3(K)$ defined in Section 2 and $\mathcal{W}_3(K)$ from Section 4; the main result of this section is Theorem 5.1. Before we state the result, a brief digression is needed to compare the settings of the two obstructions.[1] As discussed in Section 4.1, the two versions of the van Kampen obstruction are related by $\mathcal{O}_2^{\mathrm{s}}(K) = i^*\,\mathcal{O}_2(K)$, where i is the inclusion map $\mathrm{C_s}(K,2) \subset \mathrm{C}\,(K,2)$. The assumption in the theorem below is that $\mathcal{O}_2(K)$ is trivial; it follows that $\mathcal{O}_2^{\mathrm{s}}(K)$ vanishes as well, and therefore there exists a map $f\colon\; K \longrightarrow \mathbb{R}^4$ and a collection of Whitney disks for intersections of non-adjacent simplices. Then Lemma 4.3 gives a Σ_2-equivariant map $F_{f,W}\colon\; \mathrm{C_s}(K,2) \longrightarrow \mathrm{C}\,(R^4,2)$. However the starting point for the obstruction $\mathcal{O}_3(K)$ is a Σ_2-equivariant map $\mathrm{C}\,(K,2) \longrightarrow \mathrm{C}\,(R^4,2)$. To relate the two contexts, for a given simplicial 2-complex we will take a subdivision fine enough to ensure that the inclusion $\mathrm{C_s}(K,2) \hookrightarrow \mathrm{C}\,(K,2)$ is a homotopy equivalence. Then $F_{f,W}$ induces a map (well defined up to equivariant homotopy) $\mathrm{C}\,(K,2) \longrightarrow \mathrm{C}\,(R^4,2)$, which is needed to define $\mathcal{O}_3(K)$.[2]

Without loss of generality we will assume that the Whitney disks are split as discussed in Section 4.2.

Theorem 5.1. *Given a 2-complex K with trivial van Kampen's obstruction $\mathcal{O}_2(K)$, let W be a collection of split Whitney disks for double points of a map $f\colon\; K \longrightarrow \mathbb{R}^4$. Let $F_{f,W}\colon\; \mathrm{C_s}(K,2) \longrightarrow \mathrm{C}\,(R^4,2)$ be the Σ_2-equivariant map determined by f, W in Lemma 4.3. Then*

$$\mathcal{W}_3(K,f,W) = i^*\,\mathcal{O}_3(K) \in H^6_{\Sigma_3}(\mathrm{C_s}(K,3);\mathbb{Z}[(-1)]), \tag{5.1}$$

where $i\colon\; \mathrm{C_s}(K,3) \longrightarrow \mathrm{C}\,(K,3)$ is the inclusion map.

Proof. The pullback $i^*\,\mathcal{O}_3(K, F_{f,W})$ is the obstruction to the existence of a Σ_3-equivariant dashed map making the following diagram commute up to homotopy.

$$\begin{array}{ccc} \mathrm{C_s}(K,3) & \dashrightarrow & \mathrm{C}\,(\mathbb{R}^4,3) \\ \downarrow{\scriptstyle p_K} & & \downarrow{\scriptstyle p_{\mathbb{R}^4}} \\ \mathrm{C_s}(K,2)^3 & \xrightarrow{(F_{f,W})^3} & \mathrm{C}\,(\mathbb{R}^4,2)^3 \end{array} \tag{5.2}$$

The first step of the proof is to use subdivision to reduce to a model situation where precisely one of the following holds for the image under f of each 2-cell σ of K:

(1) σ is mapped in disjointly from all other non-adjacent 2-cells,
(2) σ intersects exactly one other non-adjacent 2-cell in two points, or
(3) σ has a single intersection point with one of the Whitney disks.

(Moreover, the Whitney disks are already assumed to be split, so each one intersects at most one 2-cell as in Figure 2.) To begin with, each 2-cell σ of K has a finite number of disjoint Whitney arcs, as shown in Figure 6, and a finite number of intersection points with Whitney disks. The conditions (1)-(3) above are achieved by subdividing so that each 2-cell

[1] The second author would like to thank Pedro Boavida de Brito for motivating questions.

[2] There are also other ways of relating the two settings; for example one may define a "simplicial" version of $\mathcal{O}_3(K)$ as the homotopy-lifting obstruction in (1.5) where $\mathrm{C_s}(K,3) \longrightarrow \mathrm{C_s}(K,2)^{\times 3}$ is used instead.

contains at most one Whitney arc or intersection point with a Whitney disk. For each pair on intersections of 2-cells σ_i, σ_j as in case (3) we will choose a particular ordering of i, j that will determine which sheet is pushed by the Whitney move.

Let K' denote the 2-complex obtained as the result of the subdivision and let $f'\colon K' \longrightarrow \mathbb{R}^4$ be the resulting map. The map $F_{f,W}$ in Lemma 4.3 was defined by local modifications of f in disk neighborhoods of the Whitney arcs; $F_{f',W'}$ may be assumed to be defined with respect to the same disk neighborhoods (which are now located in distinct 2-cells of K'). It follows that $F_{f,W}$ is the composition

$$\mathrm{C_s}(K,2) \longrightarrow \mathrm{C_s}(K',2) \longrightarrow \mathrm{C}\,(R^4,2)$$

of the inclusion and $F_{f',W'}$. Moreover, the cochain (4.6) defining $\mathcal{W}_3(K)$ is natural with respect to subdivisions, so $\mathcal{W}_3(K)$ is the pullback of $\mathcal{W}_3(K')$ under the inclusion $\mathrm{C_s}(K,3) \longrightarrow \mathrm{C_s}(K',3)$. Thus it suffices to prove Theorem 5.1 for K'. For the rest of the proof we will revert to the notation K for the 2-complex, assuming it is subdivided to satisfy conditions (1)-(3).

Since the homotopy fiber of the map $p_{\mathbb{R}^4}\colon\ \mathrm{C}\,(\mathbb{R}^4,3) \longrightarrow \mathrm{C}\,(\mathbb{R}^4,2)^3$ is 4-connected, there is a lift in (5.2) on the 5-skeleton $\mathrm{Sk}^5\,\mathrm{C_s}(K,3)$.

Construction 5.2. *The construction described below defines a particular Σ_3-equivariant map of the 5-skeleton, $F\colon\ \mathrm{Sk}^5\,\mathrm{C_s}(K,3) \longrightarrow \mathrm{C}\,(\mathbb{R}^4,3)$, lifting up to homotopy the Σ_3-equivariant map $\mathrm{Sk}^5\,\mathrm{C_s}(K,3) \longrightarrow \mathrm{C}\,(\mathbb{R}^4,2)^3$. Its specific geometric form will be used for identifying the point preimages of the map to $S^3 \vee S^3$ in diagram (5.3). The construction relies on the capped surface description of the Whitney move (Figure 2), and is an extension of Lemma 4.3.*

Consider the map on the 4-skeleton induced by f: given any pairwise non adjacent 2-cell σ and 1-cells ν, τ, by general position $f(\sigma)$, $f(\nu)$ and $f(\tau)$ are pairwise disjoint; F is defined on $\sigma \times \nu \times \tau$ (and its orbit under the Σ_3 action) by the Cartesian product $f^{\times 3}$.

The main part of the construction concerns the extension of this map to the 5-cells. We will define F on the boundary of each 6-cell $\partial(\sigma_1 \times \sigma_2 \times \sigma_3)$, where σ_i, $i = 1, 2, 3$ are 2-cells of K, so that the definition is consistent on the overlap of the boundaries of 6-cells. The map will be defined for a particular ordering $\sigma_1, \sigma_2, \sigma_3$ and extended to triple products corresponding to other orderings using Σ_3 equivariance.

There are three cases:

(i) the images of $\sigma_i, i = 1, 2, 3$, are pairwise disjoint,
(ii) two of them, say σ_1, σ_3 intersect, and $W_{13} \cap \sigma_2 = \emptyset$,
(iii) two of them, say σ_1, σ_3 intersect, and $W_{13} \cap \sigma_2$ is a point.

In case (i) the map F is defined on $\partial(\sigma_1 \times \sigma_2 \times \sigma_3)$ as the Cartesian cube $f^{\times 3}$. Consider case (ii). The boundary of the product $\partial(\sigma_1 \times \sigma_2 \times \sigma_3)$ naturally decomposes as the union of three parts. The definition of F on two of the parts is again $f^{\times 3}$. The definition of F on $\sigma_1 \times \partial\sigma_2 \times \sigma_3$ is an analogue of the proof of Lemma 4.3. It is defined on $D_1 \times \partial\sigma_2 \times \sigma_3$ as $f \times f \times \widetilde{f}$, where $\widetilde{f}$ is the result of the Whitney move on σ_3, and D_1 is a disk neighborhood of the Whitney arc in σ_1. As in the proof of that lemma an isotopy in a collar C on the boundary of D_1 is used, so that on $\partial\sigma_1 \times \partial\sigma_2 \times \sigma_3$ the map F equals $f^{\times 3}$.

Now consider the most interesting case (iii), shown in Figure 2. As in the previous case consider a smaller disk neighborhood D_1 of the Whitney arc in σ_1. It will be convenient to use the capped torus interpretation of the Whitney move, discussed in Section 4.2. A neighborhood of the Whitney disk W_{13} in $\mathbb{R}^4$ is a 4-ball D^4, and the intersection of $\sigma_i, i = 1,2,3$ with ∂D^4 is a 3-component link, the Borromean rings, cf. [13, Chapter 12]. An illustration is given in Figure 9; the disk σ_3 may be converted into a punctured torus as in Figure 2.

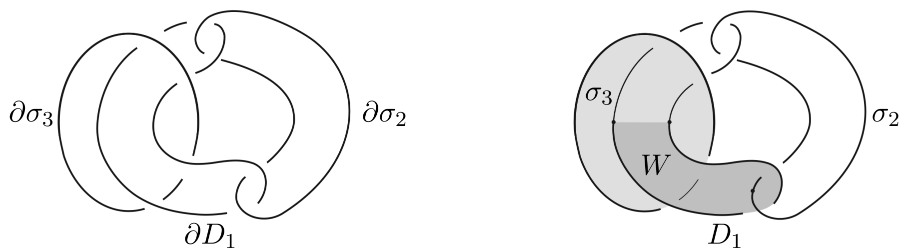

FIGURE 9. Left: the Borromean rings in ∂D^4. Right: The Whitney disk W_{13} intersects σ_2 in a single point

It will be convenient to represent disks in D^4 as movies in $D^3 \times [-1,1]$ with time $-1 \leq t \leq 1$, where most of the activity takes place at time $t = 0$. The remaining figures in this section illustrate $D^3 \times \{0\}$. Figure 10 shows the capped torus (referred to above) bounded by $\partial\sigma_3$ in this representation. The punctured torus consists of two plumbed bands, with caps C' (intersecting σ_2) and C'' (intersecting D_1). The intersections of D_1 and σ_2 with the slice $D^3 \times \{0\}$ are arcs; they extend as (arc$\times I$) into the past and the future.

FIGURE 10. Left: the capped torus bounded by $\partial\sigma_3$ with caps C', C''. Right: the map f defining F_{12}.

The disks bounded by σ_3 in Figures 11, 12 are the surgeries along the two caps and the symmetric surgery, and they will be entirely in the present. The original map f is recovered by the surgery along the cap C'' (Figure 11, left), and the result of the Whitney move $\widetilde{f}$ is the surgery on C' (Figure 11, right).

We will now proceed to define F on the three parts of the boundary $\partial(D_1 \times \sigma_2 \times \sigma_3)$. The map $F_{12}\colon D_1 \times \sigma_2 \times \partial\sigma_3 \longrightarrow \mathrm{C}(\mathbb{R}^4, 3)$ is defined as the Cartesian product $f^{\times 3}$ where f is the original map $K \longrightarrow \mathbb{R}^4$; it is an embedding when restricted to $D_1 \coprod \sigma_2 \coprod \partial\sigma_3 \hookrightarrow \mathbb{R}^4$, Figure 10 (right).

The maps $F_{23}\colon \partial D_1 \times \sigma_2 \times \sigma_3 \longrightarrow \mathrm{C}(\mathbb{R}^4, 3)$, $F_{13}\colon D_1 \times \partial\sigma_2 \times \sigma_3 \longrightarrow \mathrm{C}(\mathbb{R}^4, 3)$ are defined respectively as $f^{\times 3}$, $(\widetilde{f})^{\times 3} = f \times f \times \widetilde{f}$ where f is again the original map which restricts to an embedding $f\colon \partial D_1 \coprod \sigma_2 \coprod \sigma_3 \hookrightarrow \mathbb{R}^4$, and $\widetilde{f}\colon D_1 \coprod \partial\sigma_2 \coprod \sigma_3 \hookrightarrow \mathbb{R}^4$ is the result of the Whitney move on σ_3, Figure 11.

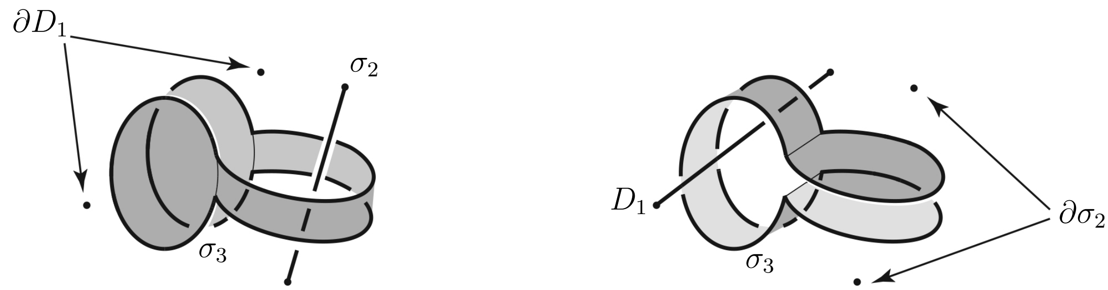

FIGURE 11. The map f defining F_{23} (left) and $\widetilde{f}$ defining F_{13} (right).

The only part of the definition where the map differs from $f^{\times 3}$ is $D_1 \times \partial\sigma_2 \times \sigma_3$, where F is defined as $(\widetilde{f})^{\times 3} = f \times f \times \widetilde{f}$. As in case (ii) and in the proof of Lemma 4.3, consider a collar C on ∂D_1 in σ_1 and extend F to $C \times \partial\sigma_2 \times \sigma_3$ using an isotopy from $\widetilde{f}$ to f. The half point of the isotopy, the symmetric surgery discussed above, is shown in Figure 12. Finally, the map is set to be $f^{\times 3}$ on $(\sigma_1 \smallsetminus (C \cup D_1)) \times \partial\sigma_2 \times \sigma_3$.

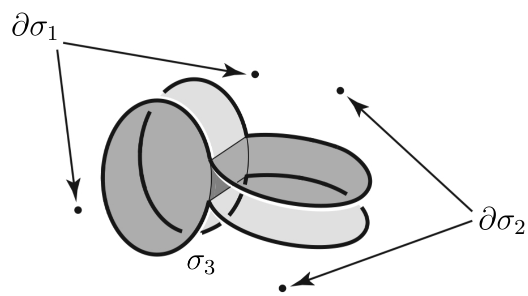

FIGURE 12. The symmetric surgery on the capped torus.

The map F is well-defined on the 5-skeleton: consider an overlap $\partial\sigma_2 \cap \partial\sigma_2'$, where σ_2 intersects W_{13} as in case (iii) and σ_2' is disjoint from W_{13}, as in case (ii). The definition in the two cases above assigns the same map to $\sigma_1 \times (\partial\sigma_2 \cap \partial\sigma_2') \times \partial\sigma_3$.

The constructed map $F\colon \operatorname{Sk}^5 \mathrm{C_s}(K,3) \longrightarrow \mathrm{C}\,(\mathbb{R}^4,3)$ lifts $\operatorname{Sk}^5 \mathrm{C_s}(K,3) \longrightarrow \mathrm{C}\,(\mathbb{R}^4,2)^3$ up to homotopy because the surgeries on the two caps, defining F, are isotopic. This concludes the description of the map F in Construction 5.2.

In the remainder of the proof of Theorem 5.1 we will show that the cohomology classes $\mathcal{W}_3(K)$, $i^*\,\mathcal{O}_3(K)$ coincide on the cochain level. The value of the cocycle w_3 in (4.7) is zero on the 6-cell $D^6 := \sigma_1 \times \sigma_2 \times \sigma_3$ in cases (i), (ii) above, and it equals ± 1 in case (iii). Recall that $i^*\,\mathcal{O}_3(K)$ is the obstruction to lifting in the diagram (5.2). The value of the obstruction cochain on the 6-cell D^6 is the element represented by $F(\partial D^6)$ in π_5 of the homotopy fiber of the map $p_{\mathbb{R}^4}\colon\ \mathrm{C}\,(\mathbb{R}^4,3) \longrightarrow \mathrm{C}\,(\mathbb{R}^4,2)^3$; we will focus on the non-trivial case (iii) to match it with the value of w_3. This homotopy group is isomorphic to $\mathbb{Z}$, generated by the Whitehead product of generators of π_3 linking any two of the three diagonals in $\mathrm{C}\,(\mathbb{R}^4,3)$. The value of

$F(\partial D^6)$ will be determined as follows. Consider the fibration [10], $p\colon\ \mathrm{C}\,(\mathbb{R}^4, 3) \longrightarrow \mathrm{C}\,(\mathbb{R}^4, 2)$, $p(x_1, x_2, x_3) = (x_1, x_2)$:

$$
\begin{array}{ccccc}
 & & \mathbb{R}^4 \smallsetminus 2\,\text{points} & \xrightarrow{\simeq} & S^3_{13} \vee S^3_{23} \\
 & & \downarrow & & \\
\partial(\sigma_1 \times \sigma_2 \times \sigma_3) & \xrightarrow{F} & \mathrm{C}\,(\mathbb{R}^4, 3) & & \\
 & & \downarrow p & \searrow^{p_{12}} & \\
 & & \mathrm{C}\,(\mathbb{R}^4, 2) & \xrightarrow{\simeq} & S^3_{12}
\end{array} \tag{5.3}
$$

The composition $p \circ F$ is null-homotopic, where the map $F\colon\ \partial(\sigma_1 \times \sigma_2 \times \sigma_3) \longrightarrow \mathrm{C}\,(\mathbb{R}^4, 3)$ is the result of Construction 5.2. In fact, it is clear from Figure 10 that $p_{12} \circ F$ is not surjective: its image is contained in a ball $D^3 \subset S^3$. Trivializing the fibration over D^3, the map F lifts to the fiber, yielding a map $\widetilde{F}\colon\ S^5 = \partial(\sigma_1 \times \sigma_2 \times \sigma_3) \longrightarrow S^3_{13} \vee S^3_{23}$. The remainder of the proof of Theorem 5.1 amounts to checking that the homotopy class of this map in $\pi_5(S^3_{13} \vee S^3_{23})$ represents the Whitehead product of the two wedge summands.

The compositions of the map $\widetilde{F}$ with the projections of $S^3_{13} \vee S^3_{23}$ onto the wedge summands are homotopic to $p_{13} \circ F$, $p_{23} \circ F$ in the diagram (5.4). In both diagrams, the map $p_{ij}\colon\ \mathrm{C}\,(\mathbb{R}^4, 3) \longrightarrow S^3_{ij}$ is given by $p_{ij}(x_1, x_2, x_3) = (x_i, x_j)/|x_i - x_j|$, $i \neq j \in \{1, 2, 3\}$.

$$
\begin{array}{ccccc}
 & & & & S^3_{13} \\
 & & & \nearrow^{p_{13}} & \\
\partial(\sigma_1 \times \sigma_2 \times \sigma_3) & \xrightarrow{F} & \mathrm{C}\,(\mathbb{R}^4, 3) & & \\
 & & p_{12} \downarrow & \searrow_{p_{23}} & \\
 & & S^3_{12} & & S^3_{23}
\end{array} \tag{5.4}
$$

Using the Potryagin construction, the homotopy class of $\widetilde{F}$ in $\pi_5(S^3_{13} \vee S^3_{23})$ can be determined by the linking number of point preimages of $p_{13} \circ F$, $p_{23} \circ F$.

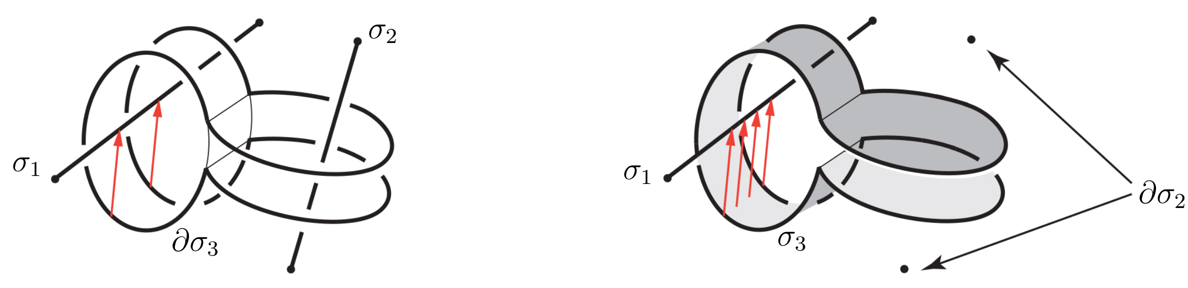

Figure 13.

A transverse point preimage of $p_{13} \circ F$ is shown in Figure 13, where a point in S^3_{13} is represented as a vector in $\mathbb{R}^4$ (colored red online). The preimage of $p_{13} \circ F_{23}$ (defined on the left in Figure 11) is empty. The preimage of $p_{13} \circ F_{12}$ is shown on the left of Figure 13 and consists of two disks. The preimage of $p_{13} \circ F_{13}$ is shown on the right of Figure 13 and consists of an annulus. The entire point preimage of $p_{13} \circ F$ is a 2-sphere.

Similarly, the point preimage of $p_{23} \circ F$ is analyzed in Figure 14. The two points preimages are seen to be the two 2-spheres $\partial D^3 \times \{*\}$, $\{*\} \times \partial D^3 \subset \partial D^3 \times D^3 = \partial(\sigma_1 \times \sigma_2 \times \sigma_3)$. This concludes the proof of Theorem 5.1. $\square$

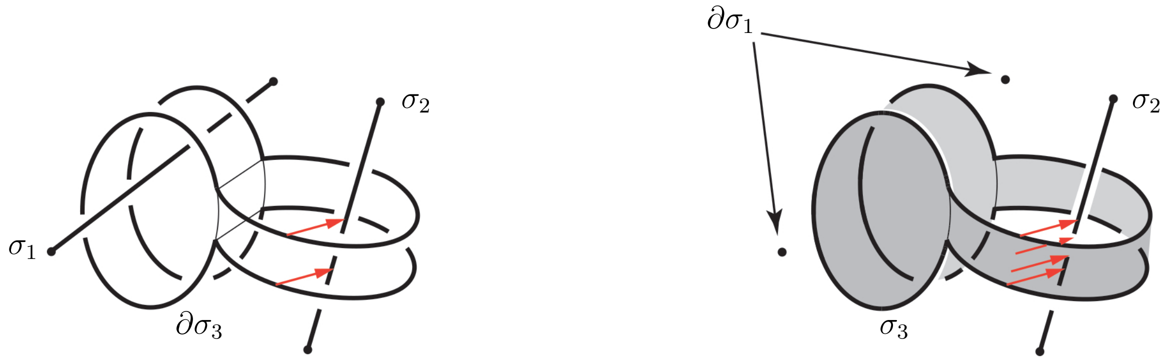

FIGURE 14.

Remark 5.3. Link-homotopy invariants using Whitehead products in configuration spaces were defined and studied in [17]. The context of the above proof is similar, but the actual method and details of the proof are independent of the results of [17].

6. Cohomological obstructions and Examples

An explicit 2-complex K which does not embed into $\mathbb{R}^4$, but has a vanishing van Kampen obstruction was constructed in [12]. In this section we reprove the non-embeddability of K by showing that our obstruction is realized in this example.

Let us begin by reviewing the construction of the complex K in [12]. Let Δ^6 (respectively $\Delta^{6\prime}$) be the six-dimensional simplex with vertex set $v_1, \ldots, v_7$ (respectively $v'_1, \ldots, v'_7$). Denote the triangle on vertices v_a, v_b, v_c by Δ_{abc} and similarly the triangle on vertices v'_a, v'_b, v'_c by Δ'_{abc}.

Let $\mathrm{sk}^n\Delta^6$ denote the n-skeleton of Δ^6. Let G_7 (respectively G'_7) be the 2-skeleton of Δ^6 minus the 2-cell associated with the triangle Δ_{123} (respectively the analogous subcomplex of $\Delta^{6\prime}$).

Let $K_0 = G_7 \vee G'_7$ be the wedge sum obtained by identifying v_1 and v'_1 (in [12] the authors add an edge $v_1v'_1$, but this difference does not matter). Finally, let K be the complex obtained by attaching to K_0 a 2-cell along the commutator of the loops $v_1v_2v_3v_1$ and $v'_1v'_2v'_3v'_1$. The closure of this 2-cell is a torus embedded in K. We denote this torus simply by $\Delta_{123} \times \Delta'_{123}$.

Remark 6.1. This example admits an immediate generalization to a family of examples, where instead of two copies of G_7 and a basic commutator of two loops as above, one takes n copies of the 2-complex G_7 and an element of the mod 2 commutator subgroup of the free group F_n on n generators. The analysis below also goes through for such commutators which are not in the next (second, in the convention of [12, Lemma 7]) term of the mod 2 lower central series of F_n; for simplicity of notation we focus on the basic example described above. We expect that the examples corresponding to higher commutators are detected by our higher obstructions $\mathcal{O}_{\rm n}(K), \mathcal{W}_{\rm n}(K)$; see Section 8.

As explained in [12], van Kampen showed that $\mathrm{sk}^2\Delta^6$ can not be embedded in $\mathbb{R}^4$, but G_7 can. It follows that the complex K_0 can be embedded in $\mathbb{R}^4$.

Let $S \subset G_7$ be the sphere that is the union of the four 2-cells that are disjoint from the triangle Δ_{123}, namely the cells corresponding to $\Delta_{456}, \Delta_{457}, \Delta_{467}$ and Δ_{567}. S is the dual tetrahedron to the triangle Δ_{123} in the 6-simplex. Dually, let $S' \subset G'_7$ be the dual sphere to the triangle Δ'_{123}.

The following key result about embeddings of K_0 into $\mathbb{R}^4$ is proved in [12] (we do not reprove it).

Proposition 6.2 ([12], Lemma 6)**.** *For any PL embedding of K_0 into $\mathbb{R}^4$, the linking numbers of S, S' and $\Delta_{123}, \Delta'_{123}$ satisfy the following (see figure 15 for a schematic illustration):*

$$\mathrm{link}(S, \Delta_{123}) \equiv \mathrm{link}(S', \Delta'_{123}) \equiv 1 (\mathrm{mod} 2).$$

$$\mathrm{link}(S, \Delta'_{123}) = \mathrm{link}(S', \Delta_{123}) = 0.$$

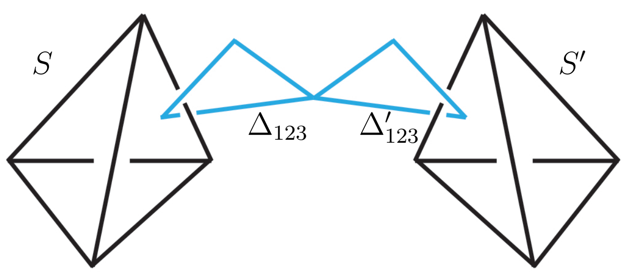

FIGURE 15. The 2-complex K is obtained by attaching a 2-cell along the commutator of Δ_{123} and Δ'_{123}.

It is also shown in [12] that the van Kampen obstruction vanishes on K. Now we can state the main result of this section. Of course it is also proved in [12], using fundamental group instead of cohomology.

Proposition 6.3. *Suppose $f_2\colon \mathrm{C}(K,2) \to \mathrm{C}(\mathbb{R}^4,2)$ is a Σ_2-equivariant map, such that the restriction of f_2 to $\mathrm{C}(K_0,2)$ is induced by some embedding $f\colon K_0 \hookrightarrow \mathbb{R}^4$. Then the following composition map*

$$(6.1)\quad \mathrm{C}(K,3) \to \mathrm{C}(K,\{1,2\}) \times \mathrm{C}(K,\{2,3\}) \times \mathrm{C}(K,\{3,1\}) \to \\ \to \mathrm{C}(\mathbb{R}^4,\{1,2\}) \times \mathrm{C}(\mathbb{R}^4,\{2,3\}) \times \mathrm{C}(\mathbb{R}^4,\{3,1\})$$

does not lift to a map

$$\mathrm{C}(K,3) \to \mathrm{C}(\mathbb{R}^4,3).$$

It follows in particular that no embedding $K_0 \hookrightarrow \mathbb{R}^4$ can be extended to an embedding $K \hookrightarrow \mathbb{R}^4$.

To prove the proposition, we give a cohomological interpretation of our obstruction $\mathcal{O}_3(K)$ in terms of the Arnold class, which may be of independent interest. Consider, once again, the problem of constructing a Σ_3-equivariant lift in a diagram of the following form

$$\begin{array}{ccc} & & \mathrm{C}(\mathbb{R}^4,\{1,2,3\}) \\ & \nearrow & \downarrow p \\ \mathrm{C}(K,3) & \xrightarrow{h} & \mathrm{C}(\mathbb{R}^4,\{1,2\})\times\mathrm{C}(\mathbb{R}^4,\{2,3\})\times\mathrm{C}(\mathbb{R}^4,\{3,1\}) \end{array}$$

Recall that for any two points i,j, $u_{ij}\in H^3(\mathrm{C}(\mathbb{R}^4,\{i,j\}))$ denotes a generator that corresponds to u_{12} under the canonical homeomorphism. Then we have the Arnold class

$$u_{12}\otimes u_{23}\otimes 1+1\otimes u_{23}\otimes u_{31}-u_{12}\otimes 1\otimes u_{31}\in H^6(\mathrm{C}(\mathbb{R}^4,\{1,2\})\times\mathrm{C}(\mathbb{R}^4,\{2,3\})\times\mathrm{C}(\mathbb{R}^4,\{3,1\})).$$

By Lemma 2.4, this class generates the kernel of p in H^6. We get the following easy sufficient condition for our obstruction to be non-zero

Lemma 6.4. *Referring to the diagram above, suppose $h^*(u_{12}\otimes u_{23}\otimes 1+1\otimes u_{23}\otimes u_{31}-u_{12}\otimes 1\otimes u_{31})\neq 0$. Then a lift does not exists and $\mathcal{O}_3(K)\neq 0$.*

One can make the connection between $\mathcal{O}_3(K)$ and the Arnold class a little more precise. By definition, $\mathcal{O}_3(K)$ is an element in the Σ_3-equivariant cohomology group $H^6_{\Sigma_3}(\mathrm{C}(K,3);\mathbb{Z}^{\pm})$. There is a natural homomorphism

$$H^6_{\Sigma_3}(\mathrm{C}(K,3);\mathbb{Z}^{\pm})\to H^6(\mathrm{C}(K,3);\mathbb{Z}^{\pm})^{\Sigma_3}\subset H^6(\mathrm{C}(K,3))$$

Lemma 6.5. *The image of $\mathcal{O}_3(K)$ in $H^6(\mathrm{C}(K,3))$ under this homomorphism is (the image of) the Arnold class under the map $p_k\circ f_2^3\colon \mathrm{C}(K,3)\to\mathrm{C}(\mathbb{R}^4,2)^3$.*

Proof. We saw in Section 2 that $\mathcal{O}_3(K)$ is represented by a map

$$\mathrm{C}(\mathbb{R}^n,2)^3\to K(\mathbb{Z},2n-2)$$

with the property that the sequence

$$\mathrm{C}(\mathbb{R}^n,3)\to\mathrm{C}(\mathbb{R}^n,2)^3\to K(\mathbb{Z},2n-2)$$

induces a split short exact sequence in H_{2n-2} and in H^{2n-2}. It follows that the map $\mathrm{C}(\mathbb{R}^n,2)^3\to K(\mathbb{Z},2n-2)$ representing our obstruction sends a generator of $H^{2n-2}(K(\mathbb{Z},2n-2))$ to a generator of $\ker(H^{2n-2}(\mathrm{C}(\mathbb{R}^n,2)^3)\to H^{2n-2}(\mathrm{C}(\mathbb{R}^n,3))$, which is precisely the Arnold class (up to sign, which we can adjust). □

Now let us prove the main result of this section.

Proof of Proposition 6.3. The map (6.1) induces a homomorphism in cohomology

$$H^6(\mathrm{C}(\mathbb{R}^4,\{1,2\})\times\mathrm{C}(\mathbb{R}^4,\{2,3\})\times\mathrm{C}(\mathbb{R}^4,\{3,1\}))\to H^6(\mathrm{C}(K,3)).$$

By Lemma 6.4, it is enough to show that this homomorphism does not send the element $u_{12}\otimes u_{23}\otimes 1+1\otimes u_{23}\otimes u_{31}-u_{12}\otimes 1\otimes u_{31}$ to zero.

Inside K there are three disjoint subspaces: the spheres S and S', and the torus $\Delta_{123}\times\Delta'_{123}$. Since these subspaces are disjoint, the obvious inclusion

$$S\times S'\times(\Delta_{123}\times\Delta'_{123})\hookrightarrow K\times K\times K$$

factors through an inclusion

$$S \times S' \times (\Delta_{123} \times \Delta'_{123}) \hookrightarrow \mathrm{C}\,(K, 3).$$

We will want to give names to elements in the cohomology of $S \times S' \times (\Delta_{123} \times \Delta'_{123})$. For this purpose, let u, u', τ_{123}, and τ'_{123} be generators of $H^2(S), H^2(S'), H^1(\Delta_{123}), H^1(\Delta'_{123})$ respectively.

Consider the composition

$$(6.2)\quad S \times S' \times (\Delta_{123} \times \Delta'_{123}) \to \mathrm{C}\,(K, 3) \to \mathrm{C}\,(K, \{1, 2\}) \times \mathrm{C}\,(K, \{2, 3\}) \times \mathrm{C}\,(K, \{3, 1\}) \to \\ \to \mathrm{C}\,(\mathbb{R}^4, \{1, 2\}) \times \mathrm{C}\,(\mathbb{R}^4, \{2, 3\}) \times \mathrm{C}\,(\mathbb{R}^4, \{3, 1\}).$$

We want to analyze the effect of this map on cohomology. So let us consider the three projections of this map. The map $S \times S' \times (\Delta_{123} \times \Delta'_{123}) \to \mathrm{C}\,(\mathbb{R}^4, \{1, 2\})$ factors through the projection $S \times S' \times (\Delta_{123} \times \Delta'_{123}) \to S \times S'$. The map $S \times S' \to \mathrm{C}\,(\mathbb{R}^4, \{1, 2\})$ is zero on reduced cohomology for the obvious reason that the target only has non-trivial cohomology in degree 3 and the source has trivial cohomology in degree 3. It follows that the terms of the Arnold class that involve u_{12} are sent to zero by this map.

It remains to see what happens to the term $1 \otimes u_{23} \otimes u_{31}$. Let us consider the map $S \times S' \times (\Delta_{123} \times \Delta'_{123}) \to \mathrm{C}\,(\mathbb{R}^4, \{3, 1\})$. This map factors as a composition

$$S \times S' \times (\Delta_{123} \times \Delta'_{123}) \to S \times (\Delta_{123} \times \Delta'_{123}) \to \mathrm{C}\,(\mathbb{R}^4, \{3, 1\}).$$

It follows from Proposition 6.2 that this composite map sends the generator u_{31} of $H^3(\mathrm{C}\,(\mathbb{R}^4, \{3, 1\})$ to an odd multiple of $u \otimes \tau_{123}$. Similarly, the map $S \times S' \times (\Delta_{123} \times \Delta'_{123}) \to \mathrm{C}\,(\mathbb{R}^4, \{2, 3\})$ sends the generator u_{23} to an odd multiple of $u' \otimes \tau'_{123}$.

It follows that the map (6.2) in cohomology sends the Arnold class to an odd multiple of $u \otimes u' \otimes \tau_{123} \otimes \tau'_{123}$. In particular, to a non-zero element of $H^6(\mathrm{C}\,(K, 3))$. □

Remark 6.6. In the discussion above we focused on the case of 2-complexes in $\mathbb{R}^4$, but a similar calculation shows that the obstruction $\mathcal{O}_3(K)$ detects non-embeddability of examples (with vanishing obstruction $\mathcal{O}_2(K)$) in all dimensions outside the metastable range, $2d < 3(m + 1)$, such that $d \geq \max(4, m)$. Such examples of m-dimensional complexes were constructed and shown to not admit an embedding in $\mathbb{R}^d$ in [29, 28]. The construction involves the Whitehead product of meridional spheres $S^l, l = d - m - 1$, linking two m-spheres S, S', rather than the commutator of loops in the construction above. Still, there are three disjoint subspaces in the complex: the spheres S, S', and a $2l$-torus, and the calculation of the Arnold class analogous to the above shows that it is non-trivial. This gives a unified proof of non-embeddability of the examples in [12] and in [29, 28], while the arguments in these original references are quite different, both from each other and from the new perspective in this paper.

7. The tower

In this section we show how the obstruction $\mathcal{O}_2(K)$ and $\mathcal{O}_3(K)$ can be extended to a sequence of obstructions $\mathcal{O}_{\mathrm{n}}(K)$, using a primitive version of the Goodwillie-Weiss tower. We will then give a conjectural description of a framed cobordism refinement of $\mathcal{O}_{\mathrm{n}}(K)$.

Definition 7.1. Let $\mathbb{I}$ be the category of finite sets and injective functions between them, and $\mathbb{I}_n \subset \mathbb{I}$ be the full subcategory consisting of sets of cardinality at most n.

As before, let $\mathrm{Emb}(K, \mathbb{R}^d)$ denote the space of topological embeddings of K into $\mathbb{R}^d$. In the case when i is a finite set and X is any space, $\mathrm{Emb}(i, X)$ is the configuration space of ordered i-tuples of pairwise distinct points of X. We also denote this space by $\mathrm{C}\,(X, i) := \mathrm{Emb}(i, X)$.

Given a small category $\mathcal{C}$ and functors $F, G \colon \mathcal{C} \to \mathrm{Top}$, we let $\mathrm{Nat}_{\mathcal{C}}(F, G)$ denote the space of natural transformations from F to G, and let $\mathrm{hNat}_{\mathcal{C}}(F, G)$ denote the space of derived natural transformations from F to G. In other words, $\mathrm{hNat}_{\mathcal{C}}(F, G)$ is the space of natural transformations from a cofibrant replacement of F to a fibrant replacement of G. The (co)fibrant replacements can be taken in any Quillen model structure on the functor category $[\mathcal{C}, \mathrm{Top}]$, where the weak equivalences are defined levelwise. We will use the projective model structure, in which every functor is fibrant.

Remark 7.2. To save notation, if F, G are functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Top}$, we will use the notation $\mathrm{hNat}_{\mathcal{C}}(F, G)$ rather than $\mathrm{hNat}_{\mathcal{C}^{\mathrm{op}}}(F, G)$.

A topological space K determines a functor $\mathrm{C}\,(K, -) \colon \mathbb{I}^{\mathrm{op}} \to \mathrm{Top}$ that sends a set i to $\mathrm{C}\,(K, i) = \mathrm{Emb}(i, K)$. A topological embedding $f \colon K \hookrightarrow \mathbb{R}^d$ gives rise to a natural transformation $\mathrm{C}\,(K, -) \to \mathrm{C}\,(\mathbb{R}^d, -)$, which sends an embedding $\alpha \colon i \hookrightarrow K$ to the embedding $f \circ \alpha \colon i \hookrightarrow \mathbb{R}^d$. This gives rise to natural maps.

$$\mathrm{Emb}(K, \mathbb{R}^d) \to \mathrm{Nat}_{\mathbb{I}}(\mathrm{C}\,(K, -), \mathrm{C}\,(\mathbb{R}^d, -)) \to \mathrm{hNat}_{\mathbb{I}}(\mathrm{C}\,(K, -), \mathrm{C}\,(\mathbb{R}^d, -)). \tag{7.1}$$

One useful feature of the space $\mathrm{hNat}_{\mathbb{I}}(\mathrm{C}\,(K, -), \mathrm{C}\,(\mathbb{R}^d, -))$ is that it admits a natural tower of approximations.

Definition 7.3. For each $n \geq 1$ define

$$T_n \, \mathrm{Emb}(K, \mathbb{R}^n) = \mathrm{hNat}_{\mathbb{I}_n}(\mathrm{C}\,(K, -), \mathrm{C}\,(\mathbb{R}^d, -)) \tag{7.2}$$

The inclusions of categories $\cdots \mathbb{I}_{n-1} \subset \mathbb{I}_n \subset \cdots \subset \mathbb{I}$ give rise to a tower whose homotopy inverse limit is equivalent to $\mathrm{hNat}_{\mathbb{I}}(\mathrm{C}\,(K, -), \mathrm{C}\,(\mathbb{R}^d, -))$

$$\mathrm{hNat}_{\mathbb{I}}(\mathrm{C}\,(K, -), \mathrm{C}\,(\mathbb{R}^d, -)) \to \cdots \to T_n \, \mathrm{Emb}(K, \mathbb{R}^n) \to T_{n-1} \, \mathrm{Emb}(K, \mathbb{R}^d) \to \cdots$$

Remark 7.4. Readers familiar with the embedding calculus of Goodwillie and Weiss will readily recognize $T_n \, \mathrm{Emb}(K, \mathbb{R}^n)$ as a primitive analogue of the n-the Taylor approximation in the Goodwillie tower. Indeed, the Goodwillie-Weiss construction is essentially the same as the one in Definition 7.3, except that instead of the category $\mathbb{I}_n$ of sets with at most n elements, they use the category whose objects are manifolds diffeomorphic to the disjoint union of at most n copies of $\mathbb{R}^m$, and whose morphisms are smooth embeddings. At least this is one way to construct the Goodwillie-Weiss tower. For more information about this approach to the Goodwillie-Weiss calculus see the paper of Boavido and Weiss [5].

The following lemma is an immediate consequence of the existence of the map (7.1).

Lemma 7.5. *If* $\mathrm{hNat}_{\mathbb{I}_n}(\mathrm{C}\,(K, -), \mathrm{C}\,(\mathbb{R}^d, -))$ *is empty for some n, then there does not exists an embedding of K into $\mathbb{R}^d$.*

Our goal is to study obstructions for a path component of $T_{n-1} \, \mathrm{Emb}(K, \mathbb{R}^d)$ to be in the image of $T_n \, \mathrm{Emb}(K, \mathbb{R}^d)$. For this purpose it is useful to have an inductive description of $T_n \, \mathrm{Emb}(K, \mathbb{R}^d)$. Such a description is given by Proposition 7.8 below. The proposition is elementary and no doubt well-known. But for completeness we will give a proof. We need some preparation.

Definition 7.6. Let

$$\mathrm{C}_0\left(\mathbb{R}^d, n\right) = \operatorname*{holim}_{S \subsetneq \{1,\ldots,n\}} \mathrm{C}\left(\mathbb{R}^d, S\right)$$

In words, $\mathrm{C}_0\left(\mathbb{R}^d, n\right)$ is the homotopy limit of all the ordered configuration spaces of proper subsets of $\{1, \ldots, n\}$ into $\mathbb{R}^d$.

Remark 7.7. It is worth noting that $\mathrm{C}_0\left(\mathbb{R}^d, 3\right) \simeq \mathrm{C}\left(\mathbb{R}^d, 2\right)^2$-a space that we encountered in sections 2 and 3. Everything we are doing in this section is a generalization of what we did in those two sections for $n = 2, 3$.

There is another, equivalent, construction of the space $\mathrm{C}_0\left(\mathbb{R}^d, n\right)$ that will come up. Let $\mathbb{I}_{n-1} \downarrow n$ be the category whose objects are injective maps of sets $i \hookrightarrow n$, where n is shorthand for $\{1, \ldots, n\}$ and $i \in \mathbb{I}_{n-1}$ denotes a set with strictly fewer elements than n. Morphisms in $\mathbb{I}_{n-1} \downarrow n$ are commuting triangles. There is a functor from $\mathbb{I}_{n-1} \downarrow n$ to the category (poset) of proper subsets of $\{1, \ldots, n\}$ which sends an injective map $i \hookrightarrow n$ to its image. This functor is easily seen to be faithful, full and surjective, so it is an equivalence of categories. Therefore it induces an equivalence

$$\operatorname*{holim}_{S \subsetneq \{1,\ldots,n\}} \mathrm{C}\left(\mathbb{R}^d, S\right) \xrightarrow{\simeq} \operatorname*{holim}_{i \hookrightarrow n \in \mathbb{I}_{n-1} \downarrow n} \mathrm{C}\left(\mathbb{R}^d, i\right). \tag{7.3}$$

Another notion that we will use in the proof of Proposition 7.8 is that of a homotopy right Kan extension. Let us quickly review what this is. Suppose $\mathcal{C}$ is a category and $\mathcal{C}_0$ is a subcategory. Let $F\colon \mathcal{C} \to \mathrm{Top}$ be a functor. We denote the restriction of F to $\mathcal{C}_0$ by $F|_{\mathcal{C}_0}$ (sometimes we may denote the restriction of F simply by F). Next, suppose $G\colon \mathcal{C}_0 \to \mathrm{Top}$ is a functor defined on a subcategory of $\mathcal{C}$. Then let $RG\colon \mathcal{C} \to \mathrm{Top}$ denote the homotopy right Kan extension of G from $\mathcal{C}_0$ to $\mathcal{C}$. Recall that RG can be defined on the objects of $\mathcal{C}$ by the following formula

$$RG(x) = \operatorname*{holim}_{x \to z \in x \downarrow \mathcal{C}_0} G(z).$$

The homotopy right Kan extension is a derived right adjoint to the restriction functor. This means that there is a natural equivalence

$$\mathrm{hNat}_{\mathcal{C}}(F, RG) \simeq \mathrm{hNat}_{\mathcal{C}_0}(F|_{\mathcal{C}_0}, G) \tag{7.4}$$

The adjunction also means that there is a natural transformation of functors $F \to RF|_{\mathcal{C}_0}$. If $\mathcal{C}_0$ is a full subcategory of $\mathcal{C}$ then this natural transformation is an equivalence when evaluated on objects of $\mathcal{C}_0$.

Now we are ready to state and prove the inductive description of $T_n \operatorname{Emb}(K, \mathbb{R}^d)$

Proposition 7.8. *There is a homotopy pullback square, where the right vertical map is induced by the canonical map* $\mathrm{C}\left(\mathbb{R}^d, n\right) \to \mathrm{C}_0\left(\mathbb{R}^d, n\right)$

$$\begin{array}{ccc} T_n \operatorname{Emb}(K, \mathbb{R}^d) & \to & \operatorname{map}(\mathrm{C}\left(K, n\right), \mathrm{C}\left(\mathbb{R}^d, n\right))^{\Sigma_n} \\ \downarrow & & \downarrow \\ T_{n-1} \operatorname{Emb}(K, \mathbb{R}^d) & \to & \operatorname{map}(\mathrm{C}\left(K, n\right), \mathrm{C}_0\left(\mathbb{R}^d, n\right))^{\Sigma_n} \end{array}$$

Proof. Since $T_n \operatorname{Emb}(K, \mathbb{R}^d) = \operatorname{hNat}_{\mathbb{I}_n}(\mathrm{C}\,(K, -), \mathrm{C}\,(\mathbb{R}^d, -))$, our task is to prove that there exists a homotopy pullback diagram of the following form

$$\begin{array}{ccc} \operatorname{hNat}_{\mathbb{I}_n}(\mathrm{C}\,(K,-), \mathrm{C}\,(\mathbb{R}^d,-)) & \longrightarrow & \operatorname{map}(\mathrm{C}\,(K,n), \mathrm{C}\,(\mathbb{R}^d,n))^{\Sigma_n} \\ \downarrow & & \downarrow \\ \operatorname{hNat}_{\mathbb{I}_{n-1}}(\mathrm{C}\,(K,-), \mathrm{C}\,(\mathbb{R}^d,-)) & \longrightarrow & \operatorname{map}(\mathrm{C}\,(K,n), \mathrm{C}_0\,(\mathbb{R}^d,n))^{\Sigma_n} \end{array} \tag{7.5}$$

The strategy is to express all four corners of this square as spaces of homotopy natural transformations between functors defined on $\mathbb{I}_n$ using homotopy right Kan extension.

Let $R^n_{n-1}\,\mathrm{C}\,(\mathbb{R}^d, -)$ be the homotopy right Kan extension of the functor $\mathrm{C}\,(\mathbb{R}^d, -)$ from $\mathbb{I}_{n-1}$ to $\mathbb{I}_n$. By (7.4) we know that restriction from $\mathbb{I}_n$ to $\mathbb{I}_{n-1}$ induces an equivalence

$$\operatorname{hNat}_{\mathbb{I}_n}(\mathrm{C}\,(K,-), R^n_{n-1}\,\mathrm{C}\,(\mathbb{R}^d,-)) \xrightarrow{\simeq} \operatorname{hNat}_{\mathbb{I}_{n-1}}(\mathrm{C}\,(K,-), \mathrm{C}\,(\mathbb{R}^d,-)).$$

Now let us analyse the functor $R^n_{n-1}\,\mathrm{C}\,(\mathbb{R}^d, -)$. There is a natural transformation of (contravariant) functors on $\mathbb{I}_n$

$$\mathrm{C}\,(\mathbb{R}^d,-) \to R^n_{n-1}\,\mathrm{C}\,(\mathbb{R}^d,-).$$

This natural transformation is an equivalence when evaluated on objects of $\mathbb{I}_{n-1}$ because $\mathbb{I}_{n-1}$ is a full subcategory of $\mathbb{I}_n$. On the other hand, we have the following formula for $R^n_{n-1}\,\mathrm{C}\,(\mathbb{R}^d, n)$

$$R^n_{n-1}\,\mathrm{C}\,(\mathbb{R}^d,n) \simeq \operatorname*{holim}_{i \hookrightarrow n \in \mathbb{I}_{n-1}\downarrow n} \mathrm{C}\,(\mathbb{R}^d,i)$$

By (7.3) we have an equivalence

$$\mathrm{C}_0\,(\mathbb{R}^d,n) = \operatorname*{holim}_{S \subsetneq \{1,\dots,n\}} \mathrm{C}\,(\mathbb{R}^d,S) \xrightarrow{\simeq} \operatorname*{holim}_{i \hookrightarrow n \in \mathbb{I}_{n-1}\downarrow n} \mathrm{C}\,(\mathbb{R}^d,i).$$

Therefore there is an equivalence

$$R^n_{n-1}\,\mathrm{C}\,(\mathbb{R}^d,n) \simeq \mathrm{C}_0\,(\mathbb{R}^d,n).$$

And the map $\mathrm{C}\,(\mathbb{R}^d, n) \to R^n_{n-1}\,\mathrm{C}\,(\mathbb{R}^d, n)$ is equivalent to the natural map $\mathrm{C}\,(\mathbb{R}^d, n) \to \mathrm{C}_0\,(\mathbb{R}^d, n)$. Now consider the full subcategory of $\mathbb{I}_n$ consisting of the single object n and its endomorphisms. This category is the symmetric group, and we will denote it by Σ_n. A functor from Σ_n to Top is the same thing as a space with an action of Σ_n. Given a space X_n with an action of Σ_n, we let $R^{\mathbb{I}}_{\Sigma} X_n(-)$ denote the homotopy right Kan extension of this functor from Σ_n to $\mathbb{I}_n$. Since there are no morphisms in $\mathbb{I}_n$ from n to smaller sets, it follows that $R^{\mathbb{I}}_{\Sigma} X_n(n) = X_n$ and $R^{\mathbb{I}}_{\Sigma} X_n(i) \simeq *$ for $i < n$.

From the discussion above we conclude that there is a homotopy pullback square of functors from $\mathbb{I}_n$ to Top

$$\begin{array}{ccc} \mathrm{C}\,(\mathbb{R}^d,-) & \longrightarrow & R^{\mathbb{I}}_{\Sigma}\,\mathrm{C}\,(\mathbb{R}^d,n)(-) \\ \downarrow & & \downarrow \\ R^n_{n-1}\,\mathrm{C}\,(\mathbb{R}^d,-) & \longrightarrow & R^{\mathbb{I}}_{\Sigma}\,\mathrm{C}_0\,(\mathbb{R}^d,n)(-) \end{array} \tag{7.6}$$

Indeed, when evaluated at a set $i < n$, the vertical morphisms in this square are equivalences, and when evaluated at n, the horizontal morphisms are equivalences. So it is a homotopy pullback square of functors.

Applying $\mathrm{hNat}_{\mathbb{I}_n}(\mathrm{C}\,(K,-),-)$ to (7.6), we obtain a homotopy pullback square

$$\begin{array}{ccc} \mathrm{hNat}_{\mathbb{I}_n}(\mathrm{C}\,(K,-),\mathrm{C}\,(\mathbb{R}^d,-)) & \longrightarrow & \mathrm{hNat}_{\mathbb{I}_n}(\mathrm{C}\,(K,-),R^{\mathbb{I}}_{\Sigma}\,\mathrm{C}\,(\mathbb{R}^d,n)(-)) \\ \downarrow & & \downarrow \\ \mathrm{hNat}_{\mathbb{I}_n}(\mathrm{C}\,(K,-),R^n_{n-1}\,\mathrm{C}\,(\mathbb{R}^d,-)) & \longrightarrow & \mathrm{hNat}_{\mathbb{I}_n}(\mathrm{C}\,(K,-),R^{\mathbb{I}}_{\Sigma}\,\mathrm{C}_0\,(\mathbb{R}^d,n)(-)) \end{array}$$

Using the fact that right Kan extension is derived right adjoint to restriction we obtain that this square is equivalent to the desired square (7.5) at the beginning of the proof. So we have proved that a homotopy pullback square of this form exists. □

Remark 7.9. One can interpret the homotopy pullback square (7.6) as an inductive description of the coskeletal filtration on a functor defined on a (generalized) Reedy category. See [4, Section 6]

Proposition 7.8 leads to an inductive procedure for constructing obstructions to the existence of an embedding $K \hookrightarrow \mathbb{R}^d$. Suppose we have a point $g_{n-1}\colon\ \mathrm{hNat}_{\mathbb{I}_{n-1}}(\mathrm{C}\,(K,-),\mathrm{C}\,(\mathbb{R}^d,-))$, and we want to know if (the path component of) g_{n-1} lies in the image of

$$\mathrm{hNat}_{\mathbb{I}_n}(\mathrm{C}\,(K,-),\mathrm{C}\,(\mathbb{R}^d,-)).$$

The bottom map in diagram (7.5) sends g_{n-1} to a Σ_n-equivariant map

$$\tilde{f}_n\colon\ \mathrm{C}\,(K,n) \to \mathrm{C}_0\,(\mathbb{R}^d,n),$$

which really factors as a composite

$$\mathrm{C}\,(K,n) \to \mathrm{C}_0\,(K,n) \to \mathrm{C}_0\,(\mathbb{R}^d,n).$$

The path component of g_{n-1} is in the image of a path component of $\mathrm{hNat}_{\mathbb{I}_n}(\mathrm{C}\,(K,-),\mathrm{C}\,(\mathbb{R}^d,-))$ if and only if $\tilde{f}_n$ lifts up to homotopy to a Σ_n-equivariant map $f_n\colon\ \mathrm{C}\,(K,n) \to \mathrm{C}\,(\mathbb{R}^d,n)$, as per the following diagram

(7.7)
$$\begin{array}{ccc} & & \mathrm{C}\,(\mathbb{R}^d,n) \\ & \overset{f_n}{\nearrow} & \downarrow \\ \mathrm{C}\,(K,n) & \xrightarrow{\tilde{f}_n} & \mathrm{C}_0\,(\mathbb{R}^d,n) \end{array}$$

At this point obstruction theory kicks in. We will assume that $d \geq 3$, so that the spaces $\mathrm{C}\,(\mathbb{R}^d,n)$ and $\mathrm{C}_0\,(\mathbb{R}^d,n)$ are simply connected. The first obstruction to the existence of a lift f_n lies in the equivariant cohomology of $\mathrm{C}\,(K,n)$ with coefficients in the first non-trivial homotopy group of the homotopy fiber of the map $\mathrm{C}\,(\mathbb{R}^d,n) \to \mathrm{C}_0\,(\mathbb{R}^d,n)$. The following proposition is known [14, 16].

Proposition 7.10. *The map* $\mathrm{C}\,(\mathbb{R}^d,n) \to \mathrm{C}_0\,(\mathbb{R}^d,n)$ *is* $(d-2)(n-1)+1$*-connected. Let* F *be the homotopy fiber of this map. The first non-trivial homotopy group of* F *is*

$$\pi_{(d-2)(n-1)+1}(F) \cong \mathbb{Z}^{(n-2)!}.$$

Since the space F is simply-connected, the action of Σ_n on these spaces induces a well-defined action on the first non-trivial homotopy group of F. Thus the group $\mathbb{Z}^{(n-2)!}$ is a representation of Σ_n. Standard obstruction theory implies the following result.

Theorem 7.11. *There is a cohomological obstruction* $\mathcal{O}_{\mathrm{n}}(K)$ *to the existence of a lift* f_n *as above. The class* $\mathcal{O}_{\mathrm{n}}(K)$ *is an element of the equivariant cohomology group.*

$$H^{(d-2)(n-1)+2}_{\Sigma_n}\left(\mathrm{C}\,(K,n),\mathbb{Z}^{(n-2)!}\right).$$

If $\dim(K)\cdot n=(d-2)(n-1)+2$ *then* $\mathcal{O}_{\mathrm{n}}(K)$ *is a complete obstruction to the existence of a lift* f_n*. In particular, this holds when* $\dim(K)=2$ *and* $d=4$*.*

It is easy to see that for $n=2,3$ the general definition of $\mathcal{O}_{\mathrm{n}}(K)$ agrees with the definitions of $\mathcal{O}_2(K)$ and $\mathcal{O}_3(K)$ that we saw earlier.

We end this section by describing a conjectural refinement of $\mathcal{O}_{\mathrm{n}}(K)$ to an obstruction $\mathcal{O}^{\mathrm{fr}}_{\mathrm{n}}(K)$ living in equivariant stable cobordism, extending the definitions of $\mathcal{O}^{\mathrm{fr}}_2(K)$ and $\mathcal{O}^{\mathrm{fr}}_3(K)$ that we saw earlier.

The construction of $\mathcal{O}^{\mathrm{fr}}_{\mathrm{n}}(K)$ uses a geometric realization of the group $\pi_{(d-2)(n-1)+1}(F)\cong\mathbb{Z}^{(n-2)!}$ as the cohomology of the *space of non 2-connected graphs*. Recall that a graph G is called 2-*connected* if G connected, and for every vertex x, $G\setminus\{x\}$ is connected.

Definition 7.12. For $n>1$, let Δ^2_n be the poset of non-trivial non 2-connected graphs with vertex set $\{1,\ldots,n\}$. Let T_n be the unreduced suspension of the geometric realization of Δ^2_n.

The space T_n was initially introduced by Vassiliev, and was studied in the paper [3]. For example, $T_2=S^0$, $T_3=S^2$, with the standard (non-trivial) action of Σ_3.

The following is well-known [3]

Theorem 7.13. *There is a homotopy equivalence*

$$T_n\simeq\bigvee_{(n-2)!}S^{2n-4}$$

Conjecture 7.14. *There is a natural* Σ_n*-equivariant map*

$$\mathrm{C}_0\,(\mathbb{R}^d,n)\to map_*(T_n,\Omega^\infty\Sigma^\infty S^{d(n-1)})$$

So that there is an $(d-2)n+1$*-cartesian square*

$$\begin{array}{ccc}\mathrm{C}\,(\mathbb{R}^d,n) & \longrightarrow & \mathrm{C}_0\,(\mathbb{R}^d,n)\\ \downarrow & & \downarrow\\ * & \longrightarrow & \mathrm{map}_*(T_n,\Omega^\infty\Sigma^\infty S^{d(n-1)})\end{array}$$

Assuming the conjecture, we have natural maps

$$T_{n-1}\,\mathrm{Emb}(K,\mathbb{R}^d)\to\mathrm{map}(\mathrm{C}\,(K,n),\mathrm{C}_0\,(\mathbb{R}^d,n))^{\Sigma_n}\to\mathrm{map}_*(\mathrm{C}\,(K,n)_+\wedge T_n,\Omega^\infty\Sigma^\infty S^{d(n-1)})^{\Sigma_n}$$

This map associates to a point in $T_{n-1}\,\mathrm{Emb}(K,\mathbb{R}^d)$ an element of the equivariant stable cohomotopy group of $\mathrm{C}\,(K,n)\wedge T_n$. This element is an obstruction $\mathcal{O}^{\mathrm{fr}}_{\mathrm{n}}(K)$ to the point of $T_{n-1}\,\mathrm{Emb}(K,\mathbb{R}^d)$ being in the image of $T\,\mathrm{Emb}(K,\mathbb{R}^d)$. The obstruction is complete so long as $d\geq\dim(K)+2$.

Remark 7.15. Our reasons to believe Conjecture 7.14 come from Orhogonal calculus. The functor that sends $\mathbb{R}^d$ to the spectrum

$$\mathrm{map}_*(T_n, \Omega^\infty\Sigma^\infty S^{d(n-1)})$$

is the bottom non-trivial layer of the difference between $\mathrm{C}(\mathbb{R}^d, n)$ and $\mathrm{C}_0(\mathbb{R}^d, n)$. In fact, the conjecture is almost a formal consequence of the existence of Orthogonal Calculus and what we know about the derivatives of functors related to $\mathrm{C}(\mathbb{R}^d, n)$. However, it would be interesting to have an explicit map

$$\mathrm{C}_0(\mathbb{R}^d, n) \to \mathrm{map}_*(T_n, \Omega^\infty\Sigma^\infty S^{d(n-1)})$$

with some sort of geometric interpretation. The map that we defined for the case $n = 3$ in Section 3 does not seem to generalize easily to higher values of n.

8. Questions and conjectures

In conclusion we will mention several problems motivated by the results of this paper.

8.1. **Equivalence of higher obstructions.** Given a 2-complex K with trivial van Kampen's obstruction $\mathcal{O}_2(K)$, according to Lemma 4.3 a map $f\colon K \longrightarrow \mathbb{R}^4$ together with Whitney disks W for intersections of non-adjacent 2-cells determine a Σ_2-equivariant map $F_{f,W}\colon \mathrm{C_s}(K,2) \longrightarrow \mathrm{C}(\mathbb{R}^d, 2)$. Theorem 5.1 then shows that the obstruction $\mathcal{W}_3(K)$ equals the pullback of $\mathcal{O}_3(K)$ to $H^6_{\Sigma_3}(\mathrm{C_s}(K,3); \mathbb{Z}[(-1)])$.

We conjecture that the analogous relation holds for higher obstructions as well. More precisely, we conjecture that a map $f\colon K \longrightarrow \mathbb{R}^4$ which admits a Whitney tower of order $n-2$ for any $n \geq 4$ determines a point in $T_{n-1}\,\mathrm{Emb}(K, \mathbb{R}^4)$, and the obstruction $\mathcal{W}_\mathrm{n}(K)$ from Definition 4.11 equals the pullback of $\mathcal{O}_\mathrm{n}(K)$ to $H^{2n}_{\Sigma_n}\left(\mathrm{C_s}(K,n), \mathbb{Z}^{(n-2)!}\right)$.

8.2. **Conjectural higher cohomological obstructions.** Recall the discussion of the Arnold class (Definition 2.3) and its relation with the obstruction $\mathcal{O}_3(K)$ (Lemma 6.5). Here we formulate a certain version of Massey products, defined when the Arnold class vanishes.

For convenience of choosing signs below, we will focus on the case $d = 4$; analogous cohomological classes can be constructed for any d.

We will use the notation $\mathrm{C}(\mathbb{R}^4, \{i,j\})$ defined in (2.5) and the generators $u_{ij} \in H^3(\mathrm{C}(\mathbb{R}^d, \{i,j\}))$ defined in the paragraph following (2.5). Denote by U_{ij} some cocycle representatives of u_{ij}. Assume that there exists $\tilde{f}_4\colon \mathrm{C}(K,4) \longrightarrow \mathrm{C}_0(\mathbb{R}^d, 4)$ as in diagram (7.7). Denote by V_{ij} the 3-cocycles on $\mathrm{C}(K,4)$ obtained as the pull-backs of U_{ij}, $i,j \in \{1,\dots,4\}$. Consider

$$V_{i,j,k} := V_{ij}V_{jk} + V_{jk}V_{ki} + V_{ki}V_{ij}.$$

It follows from the existence of the map $\tilde{f}_4\colon \mathrm{C}(K,4) \longrightarrow \mathrm{C}_0(\mathbb{R}^d, 4)$ and the resulting vanishing of the Arnold class in Lemma 6.5 that for each subset $\{i,j,k\} \subset \{1,\dots,4\}$ the cohomology class of $V_{i,j,k}$ in $H^6(\mathrm{C}(K,4))$ is trivial. Consider 5-cochains X_{ijk} on $C(K,4)$, defined by $\delta X_{ijk} = V_{ijk}$. Consider the 8-cochain on $\mathrm{C}(K,4)$:

$$Y_{(12)(34)} := X_{123}(U_{14} - U_{24}) + X_{234}(U_{31} - U_{41}) + X_{341}(U_{32} - U_{42}) + X_{412}(U_{13} - U_{23}) \tag{8.1}$$

One checks that this is a cocycle; in fact there are two additional cocycles which we denote $Y_{(13)(24)}, Y_{(14)(23)}$; for example

$$Y_{(13)(24)} := X_{123}(U_{34} - U_{14}) + X_{234}(U_{41} - U_{21}) + X_{341}(U_{12} - U_{32}) + X_{412}(U_{23} - U_{43}) \tag{8.2}$$

The sum of these three cocycles is zero. We conjecture that these cohomology classes are obstructions to lifting $\tilde{f}_4$ in diagram (7.7) to a map $f_4\colon \mathrm{C}(K,4) \longrightarrow \mathrm{C}(\mathbb{R}^d,4)$, and that they are related to the obstructions $\mathcal{O}_4(K), \mathcal{W}_4(K) \in H^8_{\Sigma_4}(\mathrm{C}(K,4), \mathbb{Z}^2)$ analogously to Lemma 6.5. Moreover, formulas (8.1), (8.2) suggest that these classes admit a systematic generalization to $C(K,n)$ for larger n as well.

8.3. **4-complexes in $\mathbb{R}^7$.** Recall that the validity of the Whitney trick in higher dimensions implies that an m-complex K admits an embedding into $\mathbb{R}^{2m}$ if and only if van Kampen's obstruction $\mathcal{O}_2(K)$ vanishes, $m > 2$. (And more generally, by Weber's theorem, this holds for m-complexes in $\mathbb{R}^d$ in the metastable range, $2d \geq 3(m+1)$.)

It is an interesting question whether outside of the metastable range, when the dimension and codimension are both sufficiently large, the vanishing of just one additional obstruction is sufficient for embeddability. To be specific, suppose K is a 4-complex such that both $\mathcal{O}_2(K)$ and $\mathcal{O}_3(K)$ are trivial. Does this imply that K admits a PL embedding into $\mathbb{R}^7$?

It seems reasonable to conjecture that the answer is affirmative. A central role in Sections 4, 5 of this paper is played by the study of 2-complexes in $\mathbb{R}^4$ where the Whitney trick fails. The analysis of 4-complexes in $\mathbb{R}^7$ is back in the dimensions where the (suitably generalized) Whitney trick works both for primary and secondary intersections, so one may expect that the vanishing of algebraic obstructions should lead to an embedding.

8.4. **(Weak) convergence of the tower.** To our knowledge the problem of convergence to the tower (7.2) is open in dimensions $2d < 3(m+1)$. (Section 8.3 discussed a particular case, $m = 4, d = 7$.)

In particular, the case of 2-complexes in $\mathbb{R}^4$ is interesting. As mentioned in Remark 6.1, the examples in Section 6 admit a generalization where a 2-cell is attached to an $(n-2)$-fold commutator of $n-1$ circles, $n > 3$; we expect that the obstructions $\mathcal{W}_{\mathrm{n}}(K), \mathcal{O}_{\mathrm{n}}(K)$ detect their non-embeddability.

Question. *Let K be a 2-complex such that all obstruction $\mathcal{O}_{\mathrm{n}}(K)$ are trivial, $n \geq 2$. Does K necessarily admit an embedding in $\mathbb{R}^4$, in either PL or topologically flat category?*

As mentioned in the introduction, in the special *relative* case where K is the disjoint union of disks D^2_i and the embedding problem in the 4-ball has a prescribed boundary condition – a link L formed by the boundaried of the disks ∂D^2_i in $S^3 = \partial D^4$ – our obstructions correspond to the Milnor invariants of L. There are well-known examples (boundary links) which have trivial Milnor's invariants but are not slice. (Further, there are examples [7] of links with vanishing Milnor invaraints which are not concordant to boundary links.) However in our context there is no boundary condition present, and there is considerable flexibility in re-embedding; we are not aware of an example contradicting the possibility of an affirmative answer to the question above.

8.5. **Intrinsic characterization of the obstructions.** Given a 2-complex K with trivial $\mathcal{O}_2(K)$, is there an intrinsic characterization of classes in $H^3(\mathrm{C_s}(K,2))$ that arise (as the pullback of a generator of $H^3(\mathrm{C}(\mathbb{R}^4,2)))$ from maps to $\mathbb{R}^4$ as in Lemma 4.3? The proof of non-embeddability of examples in Section 6 relies on Proposition 6.2. A characterization of such classes $H^3(\mathrm{C_s}(K,2))$ might lead to an obstruction theory (the Arnold class, and higher cohomological operations in Section 8.2) defined without a reference to maps into configurations spaces of $\mathbb{R}^4$.

8.6. **Complexity of embeddings.** There have been recent advances in the subject of complexity of embeddings of complexes into Euclidean spaces, both from algorithmic and geometric perspectives, cf. [19, 11]. In higher dimensions there is an upper bound $O(\exp(N^{4+\epsilon}))$ on the *refinement complexity* (r.c.), i.e. the number of subdivisions needed to PL embed a simplicial m-complex (with trivial $\mathcal{O}_2(K)$) into $\mathbb{R}^{2m}$, $m > 2$, in terms of the number N of simplices of K. For 2-complexes in $\mathbb{R}^4$ the complexity problem is open. The examples in [11] (relying on the van Kampen obstruction) have exponential r.c., and the embedding problem in this dimension is NP-hard [19]. But to the authors' knowledge it is an open question whether r.c. could even be non-recursive (and correspondingly whether the embedding problem is algorithmically undecidable). It is a natural question whether the higher obstruction theory developed in this paper may be used to shed new light on the problem. This question is closely related to the convergence problem discussed in Section 8.4: indeed, if the vanishing of all obstructions implies embeddability, one might be able to get an upper bound on complexity using a translation from the vanishing of cohomological obstructions to geometric moves realizing an embedding.

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Gregory Arone
Department of Mathematics, Stockholm University

Vyacheslav Krushkal
Department of Mathematics, University of Virginia, Charlottesville, VA 22904-4137

Email address: krushkal@virginia.edu