

ON THE ASYMPTOTICS OF QUANTUM $SU(2)$ REPRESENTATIONS OF MAPPING CLASS GROUPS.

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Abstract. We investigate the rigidity and asymptotic properties of quantum $SU(2)$ representations of mapping class groups. In the spherical braid group case the trivial representation is not isolated in the family of quantum $SU(2)$ representations. In particular, they may be used to give an explicit check that spherical braid groups and hyperelliptic mapping class groups do not have Kazhdan's property (T). On the other hand, the representations of the mapping class group of the torus do not have almost invariant vectors, in fact they converge to the metaplectic representation of $SL(2, \mathbb{Z})$ on $L^2(\mathbb{R})$. As a consequence we obtain a curious analytic fact about the Fourier transform on \mathbb{R} which may not have been previously observed.

1. INTRODUCTION AND STATEMENT OF RESULTS

The $(2+1)$ -dimensional topological quantum field theories provide a family of interesting projective representations of mapping class groups of surfaces. Some properties of these representations have been established: it is known that the $SU(2)$ quantum representations at prime roots of unity are irreducible [R], and are asymptotically faithful [A], [FWW]. However overall these representations remain relatively unexplored. In this paper we study their asymptotic properties from a different perspective, focusing in particular on rigidity of representations of mapping class groups.

There is a well established analogy between mapping class groups and arithmetic groups, cf [I]. This analogy involves both the algebraic properties such as the Tits alternative and the finiteness of the virtual cohomological dimension and the geometric aspects: the action of the mapping class groups on Teichmüller spaces compared to the action of lattices in semisimple Lie groups on the symmetric spaces. While some of the properties of mapping class groups are similar to those of higher rank lattices, some are analogous to lattices of rank one, cf [FLM].

In this context it is an interesting open question whether mapping class groups have Kazhdan's property (T). In the genus one case the group is $SL(2, \mathbb{Z})$ and doesn't have

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property (T). The affirmative answer for genus greater than one would be similar to the case of higher rank lattices (such as $SL(n, \mathbb{Z})$, $n \geq 2$) which have property (T) [K]. We propose an approach to this question based on quantum representations of mapping class groups. (See section 6.)

Notation. Given a surface S of genus g with b boundary components and n marked points, its mapping class group $\mathcal{M}_{g,n}^b$ is the group of orientation-preserving diffeomorphisms of S fixing ∂S pointwise and preserving the set of marked points, modulo isotopy fixing ∂S and marked points.

Theorem 1. *The quantum $SU(2)$ representations of \mathcal{M}_0^n have almost invariant vectors. In particular, they may be used to explicitly check that the spherical mapping class groups $\mathcal{M}_{0,n}$ and the hyperelliptic mapping class groups do not have Kazhdan's property (T).*

In the case $n = 4$ this implies that \mathcal{M}_2 doesn't have property (T). The fact that the spherical braid groups and the hyperelliptic mapping class groups do not have property (T) may also be proved directly, cf [Ko], using the fact that $SL(2, \mathbb{Z})$ does not have it. A more detailed discussion of property (T) and the proof of theorem 1 are given in section 3.

$PSL(2, \mathbb{Z})$ occurs as a mapping class group in two guises: as $\mathcal{M}_{0,4}$ and as $\mathcal{M}_{1,0}$ modulo center. Therefore the quantum representations give rise to two families of representations of $PSL(2, \mathbb{Z})$. Analyzing the representations of $\mathcal{M}_{0,4}$ (and more generally of $\mathcal{M}_{0,n}$) yields the proof of theorem 1 above. In contrast, the representations of $\mathcal{M}_{1,0}$ exhibit a rather different behavior.

Theorem 2. *The quantum $SU(2)$ representations of the mapping class group of the closed torus (isomorphic to $SL(2, \mathbb{Z})$) converge, as the level $k \rightarrow \infty$, to (an irreducible component of) the metaplectic representation of $SL(2, \mathbb{Z})$ on $L^2(\mathbb{R})$.*

Both the quantum representations and the metaplectic representation are projective, however they may be modified to be genuine linear representations of $SL(2, \mathbb{Z})$ (and actually of $PSL(2, \mathbb{Z})$.) The convergence may be understood in both contexts: for representations of the central extension of $SL(2, \mathbb{Z})$ by \mathbb{Z} , and for modified representations of $SL(2, \mathbb{Z})$. Convergence is in the Fell topology on the space of representations (see below).

Comparing the quantum $SU(2)$ representations of $\mathcal{M}_{1,0}$ with the metaplectic representation over finite fields yields the following result.

Lemma 3. *For prime p the quantum $SU(2)$ representation ρ_p of $\mathcal{M}_{1,0} \cong SL(2, \mathbb{Z})$ factors through $SL(2, \mathbb{Z}/p)$.*

The proof shows that ρ_p equals the metaplectic representation over $\mathbb{Z}/p\mathbb{Z}$ composed with the mod p reduction $SL(2, \mathbb{Z}) \rightarrow SL(2, \mathbb{Z}/p\mathbb{Z})$. This fact has been independently observed by Larsen and Wang [LW]. In particular this gives a new proof of the result [G] that the image of the representations ρ_p is finite.

A corollary, to be contrasted with theorem 1, is that the quantum $SU(2)$ representations ρ_k of $\mathcal{M}_{1,0}$ do not have almost invariant vectors.

Corollary 4. *The trivial representation of $SL(2, \mathbb{Z})$ is isolated in the family of quantum $SU(2)$ representations of $\mathcal{M}_{1,0}$.*

Another application concerns Fourier analysis: we show that there exists a subset of the real line of constant density such that no set similar to it supports functions almost invariant under the Fourier transform. The proofs of this result (stated as theorem 5 in section 5) and of corollary 4 follow from property τ (a close relative of Kazhdan's property T) for $SL(2, \mathbb{Z})$. This discussion and the proof of lemma 3 are contained in section 5.

The rest of the paper is organized as follows. The aspects of quantum $SU(2)$ representations of mapping class groups, relevant to our applications, are summarized in section 2. Section 3 recalls the definition and basic properties of Kazhdan's property (T) and Fell topology on the space of representations, and gives the proof of theorem 1. The background on metaplectic representation and the proof of theorem 2 are given in section 4. The paper concludes with a few comments and open questions.

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2. QUANTUM $SU(2)$ REPRESENTATIONS

The purpose of this section is to introduce notations and to describe basic properties of quantum $SU(2)$ representations necessary for our applications. We refer the reader to [BHMV], [Tu], [Wa] for a detailed introduction to TQFT's.

The $SU(2)$ level k representation is denoted ρ_k . Set $r = k + 2$ and define $A = e^{2\pi i/4r}$. Let Σ_g be a closed surface of genus g and let H be a handlebody with $\partial H = \Sigma_g$. Fix a trivalent spine of H , then the basis vectors are given by edge labellings of the spine, where each label j satisfies $0 \leq j \leq r - 2$, with the admissibility conditions on labels a, b, c at each vertex:

- (a) the labels a , b and c satisfy the three triangle inequalities,
- (b) $a + b + c$ is even,
- (c) the “quantum cut off” $a + b + c \leq 2r - 4$.

If Σ is not closed choose a labelling of the boundary components by integers $0 \leq l_\partial \leq r - 2$. Cap off each boundary component of Σ by a disk, denoting the resulting closed surface by $\widehat{\Sigma}$, and let H be a handlebody bounded by $\widehat{\Sigma}$. The basis vectors for Σ are defined analogously to the closed case above with the additional requirement that the spine of H meets each cap of $\widehat{\Sigma}$ once and the corresponding labellings agree with the fixed boundary labels l_∂ .

Let α be a curve in Σ_g which bounds in H . Assume the disk bounded by α intersects the spine in a point, and the label of the corresponding edge is l . Then the basis vectors are eigenvectors for the action of the Dehn twist τ_α with the corresponding eigenvalue equal to

$$A^{-l(l+2)} = e^{-\pi i l(l+2)/2r}. \quad (1)$$

In the genus one case $\mathcal{M}_1 \cong SL(2, \mathbb{Z})$ is generated by the Dehn twists τ_α, τ_β where α, β form a standard symplectic basis of curves on the torus. Consider the solid torus such that α bounds and take the spine to be the core of the solid torus, so there is just one label l , $0 \leq l \leq r - 2$. The Dehn twist τ_α acts according to the formula (1) above. The Dehn twist τ_β is the conjugate of τ_α by the S -matrix corresponding to the homeomorphism of the torus exchanging α and β : $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The S -matrix equals the imaginary part of the discrete Fourier transform [Wi]:

$$S_{l,m} = \sqrt{2/r} \sin(\pi(l+1)(m+1)/r). \quad (2)$$

While (1) and (2) explicitly give the action of a set of generators for $g = 1$ ($\partial = \emptyset$), the action in the higher genus case involves the S -matrix for the punctured torus and $6j$ -symbols. More specifically, $6j$ -symbols (see [KL, Chapter 7]) enter as the coefficients for the change of basis corresponding to different labelled spines, see figure 1 below.

Note that ρ_k are *projective* representations of $SL(2, \mathbb{Z})$. It is well known [At] that they may be renormalized to linear representations of $SL(2, \mathbb{Z})$ (and in fact of $PSL(2, \mathbb{Z})$.) More specifically, consider the presentation

$$SL(2, \mathbb{Z}) \cong \langle S, T \mid S^4 = 1, S^2 = (ST)^3 \rangle.$$

The relations holds, under the image of ρ_k , only up to a phase. It is clear from the presentation that the phases of $\rho_k(S), \rho_k(T)$ may be adjusted so that both relations actually hold in the image. A detailed analysis of the phase factors [MR] shows that they may be chosen to have a limit as $k \rightarrow \infty$.

3. KAZHDAN'S PROPERTY T

We begin with the definition and a brief summary of implications of property (T) (cf [L] for a more detailed discussion); the section concludes with the proof of theorem 1. A discrete group G has Kazhdan's property (T) if given any (necessarily infinite dimensional) unitary representation ρ of G , whenever ρ has almost invariant vectors it also has a non-zero invariant vector. Recall that a representation ρ of a group G has *almost invariant vectors* if for any finite subset $S \subset G$ and for any $\epsilon > 0$ there is a unit vector v such that $|\rho(s)v - v| < \epsilon$ for each $s \in S$.

Note that it suffices to check this condition for a fixed finite set of generators of G : if it is true for a, b , then $|\rho(ab)v - v| \leq |\rho(a)(\rho(b)v - v)| + |\rho(a)v - v| = |\rho(b)v - v| + |\rho(a)v - v| < 2\epsilon$.

Property (T) may be expressed in terms of *Fell topology* on the set \tilde{G} of unitary representations of G (and similarly on the set \hat{G} of irreducible unitary representations.) Let H be a Hilbert space, and $\rho: G \rightarrow \mathcal{U}(H)$ a continuous unitary representation of G . Fell topology on \tilde{G} is determined by open neighborhoods: let S be a finite subset of G , $\epsilon > 0$ and $v \in H$ be a unit vector. Then the set of representations $\sigma: G \rightarrow \mathcal{U}(H')$ for which there exists a unit vector $v' \in H'$ such that $|(v, \rho(g)v) - (v', \sigma(g)v')| < \epsilon$ for each g in S is an open neighborhood of (H, ρ) in \tilde{G} . In these terms, G has property (T) if and only if the trivial one-dimensional representation ρ_0 is isolated in \hat{G} with Fell topology, or equivalently if no $\rho \neq \rho_0 \in \hat{G}$ *weakly contains* ρ_0 .

Extending the meaning of the term *almost invariant* in the first paragraph we call a sequence of vectors $\{v_k\}$ belonging to $\{\rho_k\}$ *almost invariant* if for a generating set S , $|\rho_k(s)v_k - v_k| \rightarrow 0$ as $k \rightarrow \infty$. In our examples each ρ_k is finite dimensional.

A short argument proves that infinite amenable groups don't have property (T): suppose G is amenable, and let $\{F_k\}$ be a sequence of Følner sets. Considering the characteristic functions of these sets, it immediately follows that the left regular representation of G on $l^2(G)$ has almost invariant vectors (and since the group is infinite it doesn't have a non-trivial fixed vector.)

It follows that infinite abelian groups don't have property (T). If G has property (T) then so does any quotient group Q . A consequence of these two statements is that any group whose abelianization is infinite is not (T). Another basic property is that a subgroup H of finite index in G has (T) if and only if G does. In particular, $SL(2, \mathbb{Z})$ and the braid groups do not have property (T) since they contain free non-abelian subgroups of finite index.

Finding a subgroup H of finite index in a group G with $H^1(H; \mathbb{Z})$ infinite is a common way of proving that G is not (T). The question of existence of such subgroups

in mapping class groups is open in general. It is known that no such subgroups H exist in genus $g \geq 3$ with the additional assumption that H contains the Torelli group [Mc], on the other hand such subgroups exist in \mathcal{M}_2 [T], [Mc]. This proves, in particular, that \mathcal{M}_2 doesn't have property (T).

Proof of theorem 1. As above, given a surface S of genus g with b boundary components and n marked points, we denote its mapping class group by $\mathcal{M}_{g,n}^b$. We have a central extension

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \mathcal{M}_0^n \longrightarrow \mathcal{M}_{0,n} \longrightarrow 1$$

where the kernel is generated by Dehn twists along boundary parallel curves.

Consider the hyperelliptic involution J on the closed surface Σ_g of genus g , inducing a double cover $\Sigma_g \rightarrow S^2$ branched over $2g+2$ points. Let \mathcal{M}_g^J denote the hyperelliptic mapping class group of Σ_g : the subgroup consisting of elements of \mathcal{M}_g which commute with J . Then we have a short exact sequence [BH]

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathcal{M}_g^J \longrightarrow \mathcal{M}_{0,2g+2} \longrightarrow 1$$

where the generator of $\mathbb{Z}/2\mathbb{Z}$ is mapped to the involution J . In particular, it suffices to show that $\mathcal{M}_{0,2g+2}$ doesn't have property (T): then the result for \mathcal{M}_g^J is immediate. Note that $\mathcal{M}_1^J = \mathcal{M}_1 \cong SL(2, \mathbb{Z})$ and $\mathcal{M}_2^J = \mathcal{M}_2$ are known not to have property (T), therefore our argument below specialized to $g = 1, 2$ gives another proof for these groups.

Consider the family ρ_k of projective representations of \mathcal{M}_0^n , introduced in section 2. Fixing a spine, a basis for the representations is indexed by labellings of its edges. The strategy is to find almost invariant vectors $\{v_k\}$ for $\{\rho_k\}$ with *fixed boundary labels* and then to consider $\rho_k \otimes \bar{\rho}_k$ (the bar denotes the complex-conjugate representation.) Then $v_k \otimes \bar{v}_k$ are almost invariant under $\rho_k \otimes \bar{\rho}_k$, and since the extension $\mathcal{M}_0^n \rightarrow \mathcal{M}_{0,n}$ is central, $\rho_k \otimes \bar{\rho}_k$ descends to a linear representation of $\mathcal{M}_{0,n}$ (this follows from formula (1) for the action of the Dehn twists along the boundary parallel curves.) Note that this also eliminates the projective ambiguity.

Consider the case of the 4-punctured sphere ($n = 4$), the general case follows easily from this. The spine is H-shaped: there are four "boundary" labels a_i and one "interior" label b , see figure 1. (Note that this is an *orthogonal* basis.) Assume $r = k + 2$ is large and fixed. Take all labels a_i equal to $a = f(r)$ where $f(r)$ is any function of r with $f(r) \rightarrow \infty$ and $f(r)/\sqrt{r} \rightarrow 0$ as $r \rightarrow \infty$. Then it follows from the admissibility of the labels that b ranges from 0 to $2a$ – denote the corresponding basis vector by $e_{a,b}$. (Here we omit reference to the level k to simplify the notation.)

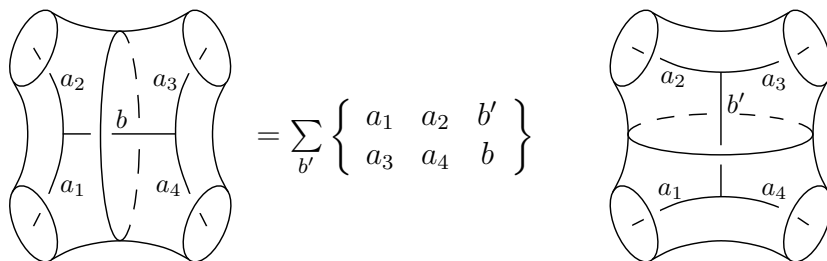


FIGURE 1. Change of basis: the coefficients are given by $6j$ symbols. The summation is over admissible labels b' : in the context of the proof all labels a_i equal $a = f(r)$ and $0 \leq b' \leq 2a$.

Fix a as above and consider the vector

$$v_k = \sum_{b=0}^{2a} \frac{e_{a,b}}{\|e_{a,b}\|}.$$

$\mathcal{M}_{0,4}$ is generated by the Dehn twists τ_1, τ_2 along the curves pictured in figure 1: one bounds a disk intersecting the bar of the H-spine in a point, the other one corresponds to the dual I-spine. It follows from the choice of the function $f(r)$ and formula (1) in section 2 that v is almost invariant under $\rho_k(\tau_1)$. The action of $\rho_k(\tau_2)$ is easily understood in the I -basis. Rewriting v in this basis is hard since this requires calculating the $6j$ -symbols (coefficients for the change of basis), however due to the choice of the boundary labels and the admissibility (quantum cut off) of labels, we know that the new interior label still satisfies $0 \leq b' \leq 2a$, making v also almost invariant under $\rho_k(\tau_2)$.

It remains to check that the unit vectors $w_k := v_k \otimes \bar{v}_k / \|v_k \otimes \bar{v}_k\|$ do not converge to the fixed subspaces of $\rho_k \otimes \bar{\rho}_k$ as $k \rightarrow \infty$. According to formula (1), the basis vectors $e_{a,b} \otimes \bar{e}_{a,c}$ are eigenvectors for $\rho_k(\tau_1)$ with the corresponding eigenvalue

$$e^{2\pi i[b(b+2)-c(c+2)]/r} = e^{2\pi i(b-c)(b+c+2)/r}.$$

Therefore for prime r the fixed vectors of $\rho_k \otimes \bar{\rho}_k$ have to lie in the subspace spanned by $\{e_{a,b} \otimes \bar{e}_{a,b}, 0 \leq b \leq r-2\}$ and by $\{e_{a,b} \otimes \bar{e}_{a,r-2-b}, 0 \leq b \leq r-2\}$.

The component of the unit vector w_k along the “diagonal”: the span of $\{e_{a,b} \otimes \bar{e}_{a,b}\}$ tends to zero as $k \rightarrow \infty$, and since $b < 2a \ll r$ the component along $\{e_{a,b} \otimes \bar{e}_{a,r-2-b}\}$ is trivial. This concludes the proof for $\mathcal{M}_{0,4}$.

For $\mathcal{M}_{0,n}$, $n > 4$, again choose a spine and fix all boundary labels to be $a = f(r)$ as above. The tree structure of the spine (and in fact of *all* spines) yields the required bound $b_j \ll \sqrt{r}$ for all interior labels b_j provided r is large enough. \square

A similar argument allows one to find almost invariant vectors, with a fixed boundary label, for representations ρ_k of the mapping class group \mathcal{M}_1^1 . Again $\rho_k \otimes \bar{\rho}_k$ descends to a representation of $\mathcal{M}_1 \cong SL(2, \mathbb{Z})$ with almost invariant vectors. The situation is different for the *closed* torus: see section 5 and the discussion in section 6.

4. THE METAPLECTIC (SEGAL-SHALE-WEIL) REPRESENTATION AND THE MODULAR GROUP

We first give a brief overview of the metaplectic representation, defined on the double cover of the symplectic group $Sp(2n, \mathbb{R})$ (cf [Fo] for a detailed exposition.) The main discussion will concern the case $n = 1$, where restricting to the integers we have a projective representation of $Sp(2, \mathbb{Z}) \cong SL(2, \mathbb{Z})$ which as mentioned in section 2 can be adjusted to be a linear representation of $SL(2, \mathbb{Z})$. We give an abstract definition in terms of intertwining operators and then present an explicit formula in the $n = 1$ case. The section concludes with the proof of theorem 2.

The Heisenberg group \mathbf{H}_n is defined as \mathbb{R}^{2n+1} with the coordinates $(p, q, t) = (p_1, \dots, p_n, q_1, \dots, q_n, t)$ and with the multiplication rule

$$(p, q, t)(p', q', t') = (p + p', q + q', t + t' + \frac{1}{2}(pq' - qp')).$$

(This group is isomorphic to the group of $(n+2) \times (n+2)$ matrices $M(p, q, t)$ with 1's on the diagonal and with $M_{1,i} = p_{i-1}$, $M_{i,n+2} = q_{i+1}$, $i = 2, \dots, n+1$, and $M_{n+2,n+2} = t$.)

Exponentiating the classical quantum-mechanical representation of the Heisenberg algebra – P_j acts by $(1/2\pi i) \partial/\partial x_j$, Q_j acts by multiplication by x_j – one gets the Schrödinger unitary representation ρ of \mathbf{H}_n on $L^2(\mathbb{R}^n)$:

$$\rho(p, q, t)f(x) = e^{2\pi i t + 2\pi i q x + \pi i p q} f(x + p).$$

The symplectic group $Sp(2n, \mathbb{R})$ acts on \mathbf{H}_n by automorphisms:

$$(p, q, t) \longmapsto (S(p, q), t) \text{ for } S \in Sp(2n, \mathbb{R}),$$

in particular for any such S we have a representation $\rho \circ S$ of \mathbf{H}_n on $L^2(\mathbb{R}^n)$.

By the Stone-von Neumann theorem these representations are equivalent: there exists a unitary operator $\mu(S)$ on $L^2(\mathbb{R}^n)$ such that $\rho \circ S = \mu(S) \rho \mu(S)^{-1}$. By Schur's lemma these intertwining operators define a projective representation of the symplectic group: $\mu(TS) = C_{T,S} \mu(T) \mu(S)$ for a scalar $C_{T,S}$ with $|C_{T,S}| = 1$; moreover it can be shown that it lifts to a linear representation μ of the non-trivial double cover $Mp(2n, \mathbb{R})$ of $Sp(2n, \mathbb{R})$.

The spaces $L^2_{\text{even}}(\mathbb{R}^n)$ and $L^2_{\text{odd}}(\mathbb{R}^n)$ of even and odd functions are invariant subspaces of the metaplectic representation μ , and it can be proved that the subrepresentations of μ on L^2_{even} and L^2_{odd} are irreducible (and inequivalent).

In the case $n = 1$ this representation has a particularly nice explicit integral expression. Fix the standard generators of $SL(2, \mathbb{Z})$:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then (up to a phase) $\mu(S)$ is the inverse of the Fourier transform and $\mu(T)$ is the multiplication by the imaginary Gaussian:

$$\mu(S)f(x) = \int_{\mathbb{R}} e^{2\pi ixy} f(y) dy, \quad \mu(T)f(x) = e^{-\pi ix^2} f(x).$$

It is easily seen that the subrepresentations μ' , μ'' of μ on $L^2_{\text{odd}}(\mathbb{R})$ and $L^2_{\text{even}}(\mathbb{R})$ are given by

$$\mu'(S)f(x) = \int_{\mathbb{R}} \sin(2\pi xy) f(y) dy \quad \text{and} \quad \mu''(S)f(x) = \int_{\mathbb{R}} \cos(2\pi xy) f(y) dy$$

(in both cases T acts by multiplication by $e^{-\pi ix^2}$.)

Theorem 2 asserts that the representations ρ_k converge in Fell topology to μ' . For the proof, consider the formulas (1) and (2) in section 2 for the action of T and S . It is convenient to rescale the variables: $a = l/\sqrt{2r}$, $b = m/\sqrt{2r}$. We always use the notation $r = k + 2$ and in our estimates we assume r is large. The new parameters range from 0 to $\sim \sqrt{r/2}$ and values are spaced $1/\sqrt{2r}$ apart. Denoting the basis vector for ρ_k corresponding to the label l (and to the rescaled value a) by e_a , the representation is given by

$$\rho_k(T)e_a \simeq e^{-\pi ia^2} e_a, \quad \rho_k(S)e_a \simeq \sum_b \sqrt{2/r} \sin(2\pi ab) e_b$$

According to the definition of Fell topology, given an odd function $f \in L^2(\mathbb{R})$ of norm 1 we need to find a sequence of unit vectors $\{v_k\}$, $v_k \in \rho_k$ such that $\forall \epsilon > 0$

$$|(f, \mu'(S)f)_{L^2} - (v_k, \rho_k(S)v_k)| < \epsilon \quad \text{and} \quad |(f, \mu'(T)f)_{L^2} - (v_k, \rho_k(T)v_k)| < \epsilon$$

for sufficiently large k . We will view vectors in ρ_k as step functions on $[0, \sqrt{r/2}]$, with the step length equal to $1/\sqrt{2r}$. Since step functions are dense in $L^2(\mathbb{R})$, for a sufficiently large k there exists a vector $v_k \in \rho_k$ whose corresponding step function g satisfies $|g - f|_{L^2} < \epsilon/3$. More precisely, note that the given function f is *odd* while the available step functions are supported on the interval $[0, \sqrt{r/2}]$ which is in \mathbb{R}_+ . Therefore the correspondence between step functions g and vectors v_k involves keeping only the $x \geq 0$ half of g and multiplying by $\sqrt{2}$ to keep the norm equal to 1. Increasing k if necessary we may assume also that

$$|(g, \mu'(S)g)_{L^2} - (v_k, \rho_k(S)v_k)| < \epsilon/3.$$

Then

$$\begin{aligned} |(f, \mu'(S)f) - (v_k, \rho_k(S)v_k)| &\leq |(f, \mu'(S)f) - (g, \mu'(S)g)| + \epsilon/3 \leq \\ &|(f - g, \mu'(S)f)| + |(g, \mu'(S)(f - g))| + \epsilon/3 \leq \epsilon. \end{aligned}$$

Here we used the fact that $\mu'(S)$ is an isometry. The estimate for the action of T is proved identically. \square

5. PROPERTY τ , THE METAPLECTIC REPRESENTATION OVER $\mathbb{Z}/p\mathbb{Z}$, AND AN ANALYTIC FACT.

The question of whether the quantum representations ρ_k of the mapping class group of the closed torus – $SL(2, \mathbb{Z})$ – have almost invariant vectors leads to an interesting problem in Fourier analysis. In particular, the sequence of Gaussians $e^{-\pi l^2/r}$ “almost” solves the problem as $r \rightarrow \infty$: they are almost invariant under the Fourier transform, however their support is exactly balanced against the quadratic behavior of the eigenvalues (1) of the Dehn twist T .

In the limit the problem is to find a function in $L^2(\mathbb{R})$ almost invariant under the Fourier transform $\hat{}$ (more precisely, under the *sin*-transform) and supported near the set of points $\{\sqrt{n}, n \in \mathbb{Z}\}$. It follows from the uncertainty principle that a solution cannot be given by a function similar to the Gaussian: there doesn't exist a function supported near zero, say on $(-\epsilon, \epsilon)$ – making it almost invariant under the action of T – whose Fourier transform is also mostly supported on $(-\epsilon, \epsilon)$ for small ϵ .

In fact, as stated in lemma 3 in the introduction, the quantum representations ρ_k (for k prime) factor through $SL(2, \mathbb{Z}/k\mathbb{Z})$ (the proof of this is given below.) It follows

from Selberg's theorem [L] that $SL(2, \mathbb{Z})$ has property τ with respect to the family of congruence subgroups

$$N_k = \ker[SL(2, \mathbb{Z}) \longrightarrow SL(2, \mathbb{Z}/k\mathbb{Z})],$$

in other words the trivial representation is isolated in the family of representations $\{\rho \mid \ker(\rho) \supset N_k \text{ for some } k\}$. This proves corollary 4: our representations ρ_k of $SL(2, \mathbb{Z})$ do not have almost invariant vectors; in particular the problem concerning the Fourier transform on \mathbb{R} above does not have a solution. In particular we can prove:

Theorem 5. *There are sets $X \subset \mathbb{R}$ of constant density so that no set similar to X supports almost invariant functions.*

Notations: We say X has *constant density* if there exist $N, c > 0$ such that

$$\mu(X \cap [r, r + N])/N > c \text{ for all } r \in \mathbb{R}$$

where μ is the Lebesgue measure. We say X supports *almost invariant functions* if there exist L^2 functions $f_i: X \longrightarrow \mathbb{C}$ such that $\|f_i - \hat{f}_i\| \longrightarrow 0$. $\hat{}$ denotes the Fourier transform. We say Y is *similar* to X if there exist $r_1 \in \mathbb{R} \setminus 0$ and $r_2 \in \mathbb{R}$ such that $Y = r_1 X + r_2$.

Proof. Let, for $\epsilon > 0$, $X_\epsilon = \{r \in \mathbb{R} \mid |r - (\pm\sqrt{n})| < \epsilon/\sqrt{n}, n \text{ a positive integer}\}$. Clearly all X_ϵ have constant density. If each supported almost invariant functions, a diagonalization argument and the preceding discussion would contradict property τ for $SL(2, \mathbb{Z})$. If X is replaced by $r_1 X$ rescale $a = l/r_1\sqrt{r}$, $b = m/r_1\sqrt{r}$; translation by r_2 introduces a phase $e^{2\pi i r_2}$ into \hat{f} but since our analysis is on $\rho_k \otimes \bar{\rho}_k$ such phases are irrelevant. \square .

To prove lemma 3 we note that the construction of the metaplectic representation μ , sketched in the previous section, can be carried through for finite fields K in place of \mathbb{R} , see [N] and references therein. As in the case of \mathbb{R} , one considers the action of the symplectic group $Sp(2n, K)$ on the Heisenberg group \mathbf{H}_n (defined in terms of K), and the Stone-von Neumann theorem asserts that the Schrödinger representation is the unique unitary representation \mathbf{H}_n with the given action of the center of \mathbf{H}_n . Again, the irreducible subspaces consist of even and odd functions.

Theorem 4.1 in [N] gives an explicit formula for the metaplectic representation μ over finite fields, in particular taking $K = \mathbb{Z}/p\mathbb{Z}$ one has (for $n = 1$ and the standard generators S, T of $Sp(2, \mathbb{Z}/p\mathbb{Z}) \cong SL(2, \mathbb{Z}/p\mathbb{Z})$)

$$\mu(T)f(x) = e^{-\pi ix^2/2p}f(x), \quad \mu(S)f(x) = \frac{1}{\sqrt{p}} \sum_{y \in K} e^{\pi ixy/p}f(y) \quad (3)$$

Comparing (3) with the formulas (1), (2) in section 2 and restricting to the subspace of odd functions, observe that (for prime p) the quantum representation ρ_p may be viewed as the irreducible component μ' of the metaplectic representation of $SL(2, \mathbb{Z}/p\mathbb{Z})$. \square

6. COMMENTS AND QUESTIONS

1. Property (T) and mapping class groups. We established in section 5 that the family of quantum $SU(2)$ -representations of $SL(2, \mathbb{Z})$ that arise as the mapping class group of the closed torus doesn't have almost invariant vectors. On the other hand, the representations of the mapping class groups of the punctured torus, and of the sphere with 4 (or more) punctures do have almost invariant vectors. It is an interesting question which of the two possibilities holds for closed surfaces of higher genus. (Of course the latter one would imply that the mapping class groups of all orientable surfaces do not have property (T).) An answer to this question would likely require a detailed analysis of the action of a generating set of Dehn twists on explicit vectors under the quantum representations. Checking that a proposed vector v is almost invariant would involve calculations with $6j$ -symbols and with the S -matrix for the punctured torus. The last ingredient is known [Ki] in terms of Macdonald polynomials but the calculations are difficult in practice.

2. Limits of quantum representations. We showed that the representations for the closed torus converge, as the level approaches infinity, to the metaplectic representation of $SL(2, \mathbb{Z})$. It would be interesting to find out whether there is a limiting representation of the mapping class group \mathcal{M}_g in the higher genus case.

Specifically consider the higher-dimensional metaplectic representation ρ . The maps induced on the first homology of the surface by homeomorphisms give a surjection $\pi: \mathcal{M}_g \rightarrow Sp(2g, \mathbb{Z})$. The quantum $SU(2)$ representations faithfully detect the mapping class groups modulo center [A], [FWW] so they cannot factor through the symplectic group. However it is possible that the quantum representations ρ_k converge to $\rho \circ \pi$ in Fell topology. Convergence in this topology is rather weak and proving this would require exhibiting, for any function in $L^2(\mathbb{R}^{2g})$, a sequence of vectors v_k that are transformed by ρ_k applied to a generating set of Dehn twists approximately as by $\rho \circ \pi$ – generalizing our proof in the genus 1 case.

REFERENCES

- [A] J.E. Anderson, *Asymptotic faithfulness of quantum $SU(n)$ representations of mapping class groups*, preprint arXiv:math.QA/0204084.
- [At] M. Atiyah, *On framings of 3-manifolds*, *Topology* 29 (1990), 1-7.
- [BH] J. Birman, H. Hilden, *On isotopies of homeomorphisms of Riemann surfaces*, *Ann. Math.* 97 (1973), 424-439.
- [BHMV] C. Blanchet, N. Habegger, G. Masbaum, P. Vogel, *Topological quantum field theories derived from the Kauffman bracket*, *Topology* 34 (1995), 883-927.
- [FLM] B. Farb, A. Lubotzky, Y. Minsky, *Rank-1 phenomena for mapping class groups*, *Duke Math. J.* 106 (2001), 581-597.
- [Fo] G. Folland, *Harmonic analysis in phase space*, Princeton University Press, 1989.
- [FWW] M. Freedman, K. Walker and Z. Wang, *Quantum $SU(2)$ faithfully detects mapping class groups modulo center*, *Geom. Topol.* 6 (2002), 523-539.
- [G] P. Gilmer, *On the Witten-Reshetikhin-Turaev representations of mapping class groups*, *Proc. AMS* 127 (1999), 2483-2488.
- [I] N. Ivanov, *Mapping class groups*, *Handbook of geometric topology*, 523-633, North-Holland, Amsterdam, 2002.
- [K] D.A.Kazhdan, *Connection of the dual space of a group with the structure of its closed subgroups*, *Functional Anal. Appl.* 1 (1967), 63-65.
- [KL] L.H. Kauffman, S. Lins, *Temperley-Lieb recoupling theory and invariants of 3-manifolds*, Princeton University Press, 1994.
- [Ki] A. Kirillov, *On an inner product in modular tensor categories*, *J. Amer. Math. Soc.* 9 (1996), 1135-1169.
- [Ko] M. Korkmaz, *On cofinite subgroups of mapping class groups*, *Turkish J. Math.* 27 (2003), 115-123.
- [LW] M. Larsen and Z. Wang, *Density of the $SO(3)$ TQFT representation of mapping class groups*, preprint, August 2004 arXiv:math.GT/0408161.
- [L] A. Lubotzky, *Discrete groups, expanding graphs and invariant measures*, Birkhäuser, 1994.
- [MR] G. Masbaum and J. Roberts, *On central extensions of mapping class groups*, *Math. Ann.* 302 (1995), 131-150.
- [Mc] J. McCarthy, *On the first cohomology group of cofinite subgroups in surface mapping class groups*, *Topology* 40 (2001), 401-418.
- [N] M. Neuhauser, *An explicit construction of the metaplectic representation over a finite field*, *J. Lie Theory* 12 (2002), 15-30.
- [R] J. Roberts, *Irreducibility of some quantum representations of mapping class groups*, *Math. Proc. Cam. Phil. Soc.* 115 (1994), 53-77.
- [T] F. Taherkhani, *The Kazhdan property of the mapping class group of closed surfaces and the first cohomology group of its cofinite subgroups*, *Experiment. Math.* 9 (2000), 261-274.

- [Tu] V. Turaev, Quantum invariants of knots and 3-manifolds, de Gruyter Studies in Math. Vol. 18 (1994)
- [Wa] K. Walker, On Witten's 3-manifold invariants, preprint, 1991.
- [Wi] E. Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Phys. 121 (1989), 351-399.