Topological arbiters

Slava Krushkal (joint work with M. Freedman)

July 4, 2011

Outline:

Given an n-dimensional manifold W, a topological arbiter associates a value 0 or 1 to codimension zero submanifolds of W, subject to natural topological and duality axioms.

Outline:

Given an n-dimensional manifold W, a topological arbiter associates a value 0 or 1 to codimension zero submanifolds of W, subject to natural topological and duality axioms.

- Given a manifold W, there may or may not exist an arbiter on W.
- There is a unique arbiter on $\mathbb{R}P^2$. Homological arbiters on $\mathbb{R}P^{2n}$.

Outline:

Given an n-dimensional manifold W, a topological arbiter associates a value 0 or 1 to codimension zero submanifolds of W, subject to natural topological and duality axioms.

• Given a manifold W, there may or may not exist an arbiter on W.

• There is a unique arbiter on $\mathbb{R}P^2$. Homological arbiters on $\mathbb{R}P^{2n}$.

• There exists an uncountable collection of arbiters in dimension 4. (Proof based on generalized link-slicing problems, Milnor group)

• Arbiters, not induced by homology, also exist in higher dimensions. (Construction is based on nontrivial squares in stable homotopy theory)

Outline:

Given an n-dimensional manifold W, a topological arbiter associates a value 0 or 1 to codimension zero submanifolds of W, subject to natural topological and duality axioms.

• Given a manifold W, there may or may not exist an arbiter on W.

• There is a unique arbiter on $\mathbb{R}P^2$. Homological arbiters on $\mathbb{R}P^{2n}$.

• There exists an uncountable collection of arbiters in dimension 4. (Proof based on generalized link-slicing problems, Milnor group)

• Arbiters, not induced by homology, also exist in higher dimensions. (Construction is based on nontrivial squares in stable homotopy theory)

Motivation: Percolation, 4-dimensional surgery.

Generalization: multi-arbiters

The concept of a topological arbiter is rooted in Poincaré-Lefschetz duality, and it may be thought of as an axiomatization of geometric properties of duality.

Axiomatizing properties of a mathematical structure may lead to a useful notion interesting in its own right.

A well-known example: H. Whitney's generalization of linear independence: matroids.

Given a closed smooth $n-$manifold W, consider the collection $\mathcal{M}_W$ of connected smooth codimension zero submanifolds of W.

Given a closed smooth $n-$manifold W, consider the collection $\mathcal{M}_W$ of connected smooth codimension zero submanifolds of W.

Definition

A *topological arbiter* on W is a function $\mathcal{A}\colon\ \mathcal{M}_W \longrightarrow \{0,1\}$ satisfying axioms (1) – (3):

(1) *"$\mathcal{A}$ is topological"*: If $M, M' \in \mathcal{M}_W$ and M is ambiently isotopic to M' in W then $\mathcal{A}(M) = \mathcal{A}(M')$.

(2) *"Greedy axiom"*: If $M \subset M'$ and $\mathcal{A}(M) = 1$ then $\mathcal{A}(M') = 1$.

(3) *"Duality"*: Suppose $A, B \in \mathcal{M}_W$ are such that $W = A \cup B$, with $A \cap B = \partial A \cap \partial B$. Then $\mathcal{A}(A) + \mathcal{A}(B) = 1$.

(1) If M is ambiently isotopic to M' then $\mathcal{A}(M) = \mathcal{A}(M')$.

(2) If $M \subset M'$ and $\mathcal{A}(M) = 1$ then $\mathcal{A}(M') = 1$.

(3) If $W = A \cup B$, with $A \cap B = \partial A \cap \partial B$. Then $\mathcal{A}(A) + \mathcal{A}(B) = 1$.

(1) If M is ambiently isotopic to M' then $\mathcal{A}(M) = \mathcal{A}(M')$.

(2) If $M \subset M'$ and $\mathcal{A}(M) = 1$ then $\mathcal{A}(M') = 1$.

(3) If $W = A \cup B$, with $A \cap B = \partial A \cap \partial B$. Then $\mathcal{A}(A) + \mathcal{A}(B) = 1$.

It is easy to see that for any n, the n-sphere S^n does not admit a topological arbiter.

(1) If M is ambiently isotopic to M' then $\mathcal{A}(M) = \mathcal{A}(M')$.
(2) If $M \subset M'$ and $\mathcal{A}(M) = 1$ then $\mathcal{A}(M') = 1$.
(3) If $W = A \cup B$, with $A \cap B = \partial A \cap \partial B$. Then $\mathcal{A}(A) + \mathcal{A}(B) = 1$.

It is easy to see that for any n, the n-sphere S^n does not admit a topological arbiter.

More generally, any manifold W admitting an open book decomposition does not support a topological arbiter. (However W may admit a *multi*-arbiter, discussed later.)

(Odd-dimensional manifolds admit open book decompositions, simply-connected even-dimensional manifolds (of dimension ≥ 6) are open books, provided that the signature is zero)

(1) If M is ambiently isotopic to M' then $\mathcal{A}(M) = \mathcal{A}(M')$.
(2) If $M \subset M'$ and $\mathcal{A}(M) = 1$ then $\mathcal{A}(M') = 1$.
(3) If $W = A \cup B$, with $A \cap B = \partial A \cap \partial B$. Then $\mathcal{A}(A) + \mathcal{A}(B) = 1$.

A prototypical example: the homological arbiter $\mathcal{A}_h$ on $\mathbb{R}P^2$.

Given $M \subset \mathbb{R}P^2$ define $\mathcal{A}_h(M) = 1$ if M carries the non-trivial first homology class of $\mathbb{R}P^2$, i.e. if

$$H_1(M; \mathbb{Z}/2) \longrightarrow H_1(\mathbb{R}P^2; \mathbb{Z}/2) \cong \mathbb{Z}/2$$

is onto. Set $\mathcal{A}_h(M) = 0$ otherwise.

The first two axioms are immediate, and (3) easily follows from Poincaré duality.

(1) If M is ambiently isotopic to M' then $\mathcal{A}(M) = \mathcal{A}(M')$.
(2) If $M \subset M'$ and $\mathcal{A}(M) = 1$ then $\mathcal{A}(M') = 1$.
(3) If $W = A \cup B$, with $A \cap B = \partial A \cap \partial B$. Then $\mathcal{A}(A) + \mathcal{A}(B) = 1$.

In fact, $\mathcal{A}_h$ is the *unique* topological arbiter on the projective plane:

Suppose there is a different arbiter $\mathcal{A}$ on $\mathbb{R}P^2$, so there is $A \subset \mathbb{R}P^2$ such that $\mathcal{A}(A) = 1$ but $H_1(A; \mathbb{Z}/2) \longrightarrow H_1(\mathbb{R}P^2; \mathbb{Z}/2)$ is the trivial map.

Then by Poincaré-Lefschetz duality the complement $B = \mathbb{R}P^2 \setminus A$ carries the first homology of $\mathbb{R}P^2$. Since A lies in the complement of a non-trivial cycle, it is contained in a 2-cell $D^2 \subset \mathbb{R}P^2$.

It follows from axioms (2), (3) that $\mathcal{A}(D^2) = 0$, so by (2) $\mathcal{A}(A) = 0$, a contradiction proving that $\mathcal{A}_h$ is the unique arbiter on the projective plane.

Higher-dimensional examples: the homological arbiter on even-dimensional real and complex projective spaces:

Lemma

(1) Let $M \subset \mathbb{R}P^{2n}$*. Define* $\mathcal{A}_h(M) = 0$ *if* $H_n(M; \mathbb{Z}/2) \longrightarrow H_n(\mathbb{R}P^{2n}; \mathbb{Z}/2) \cong \mathbb{Z}/2$ *is the trivial map and* $\mathcal{A}_h(M) = 1$ *otherwise.*

(2) Fix a field F *and let* $M \subset \mathbb{C}P^n$*, where* n *is even. Define* $\mathcal{A}_F(M) = 0$ *if* $H_n(M; F) \longrightarrow H_n(\mathbb{C}P^n; F) \cong F$ *is the trivial map and* $\mathcal{A}_F(M) = 1$ *otherwise.*

Then $\mathcal{A}_h$ *is a topological arbiter on* $\mathbb{R}P^{2n}$*, and for any* F*,* $\mathcal{A}_F$ *is a topological arbiter on* $\mathbb{C}P^n$*.*

Proof. The first two axioms are immediate. To prove axiom (3), let $W = \mathbb{R}P^{2n}$ or $\mathbb{C}P^n$, let the coefficients $F = \mathbb{Z}/2$ in the first case and an arbitrary field in the second case. Consider the long exact sequences

$$\begin{array}{ccccc} H_n(A) & \longrightarrow & H_n(W) & \longrightarrow & H_n(W,A) \\ \downarrow & & \downarrow & & \downarrow \\ H^n(W,B) & \longrightarrow & H^n(W) & \longrightarrow & H^n(B) \end{array} \tag{1}$$

where the vertical maps are isomorphisms given by Poincaré-Lefschetz duality. Since $H_n(W;F) \cong F$ it is clear that precisely one of the two maps $H_n(A;F) \longrightarrow H_n(W;F)$, $H_n(B;F) \longrightarrow H_n(W;F)$ is non-trivial. □

Consider $\mathcal{M} = \{(M, \gamma) | M$ is a codimension zero, smooth, compact submanifold of D^{2n}, and $M \cap \partial D^{2n}$ is a tubular neighborhood of an unknotted sphere $\gamma \colon S^{n-1} \subset S^{2n-1}\}$.

A **local topological arbiter** is an invariant $\mathcal{A} \colon \mathcal{M} \longrightarrow \{0, 1\}$ satisfying axioms (1) – (3):

(1) If (M, γ) is ambiently isotopic to (M', γ') in D^{2n} then $\mathcal{A}(M, \gamma) = \mathcal{A}(M', \gamma')$.

(2) If $(M, \gamma) \subset (M', \gamma)$ and $\mathcal{A}(M, \gamma) = 1$ then $\mathcal{A}(M', \gamma') = 1$.

(3) Let $D^{2n} = A \cup B$ be a decomposition of D^{2n} such that the distinguished spheres α, β of A, B form the Hopf link in $S^{2n-1} = \partial D^{2n}$. Then $\mathcal{A}(A, \alpha) + \mathcal{A}(B, \beta) = 1$.

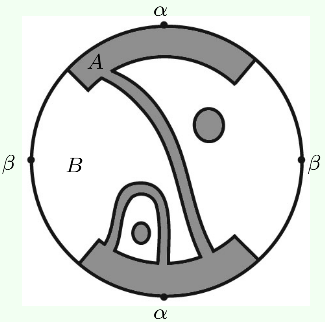
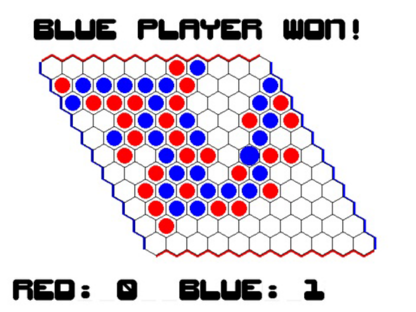
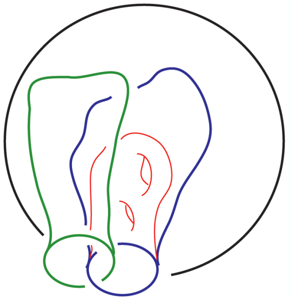


Figure : A $2-$dimensional decomposition, $D^2 = A \cup B$.

Analogously to the case of $\mathbb{R}P^2$, there is a unique local arbiter in dimension 2.


BLUE PLAYER WON!
RED: 0 BLUE: 1

Dimension 4: Given a decomposition $D^4 = A \cup B$ extending the standard genus 1 Heegaard decomposition of $S^3 = \partial D^4$, by Alexander duality either (a multiple of) α bounds in A, or (a multiple of) β bounds in B.



Local homological arbiters on D^4 (or analogously on D^{2n} for any $n > 1$):

Fix a field F and let $(M, \gamma) \subset (D^4, \partial D^4)$. Define $\mathcal{A}_F(M, \gamma) = 1$ if $\gamma = 0 \in H_1(M; F)$ and $\mathcal{A}_F(M) = 1$ otherwise.

It follows from the universal coefficient theorem that the arbiter $\mathcal{A}_F$ on D^{2n} depends only on the characteristic of the field F, so $\{\mathcal{A}_F\}$ is a countable collection.

All arbiters discussed so far are induced by homology (with various coefficients). Focusing on D^4, we construct a collection of arbiters different from the homological ones:

Theorem (Freedman - K.)

There are uncountably many topological arbiters on D^4.

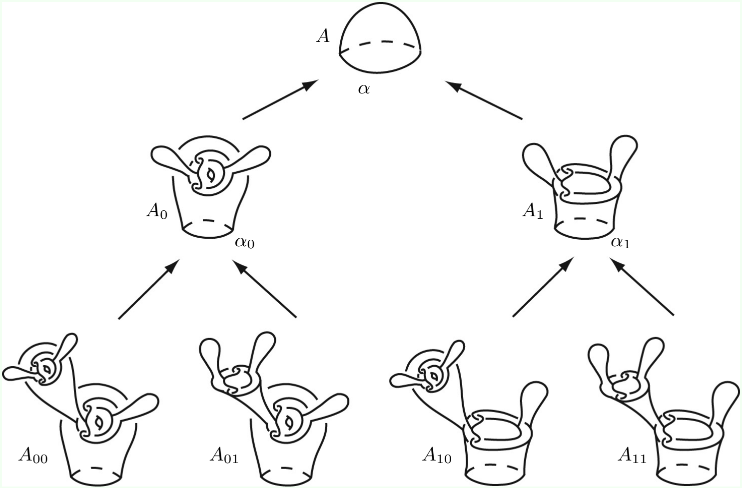
Theorem (Freedman - K.)

There are uncountably many topological arbiters on D^4.

Tools used in the proof: nilpotent embedding obstructions in dimension 4 (measuring the failure of the Whitney trick), reflected in particular by the Milnor group.

A general problem: *Given two submanifolds $(M, \gamma), (M', \gamma') \subset (D^4, \partial D^4)$, what are obstruction to disjoint embedding M, M' into D^4 so that γ, γ' form the Hopf link in ∂D^4?*

Outline of the proof of the theorem: construction of a tree of submanifolds with pairwise non-embedding properties:



Since the Whitney trick is valid in dimensions > 4, our construction of (uncountably many) arbiters on D^4 does not have an immediate analogue in higher dimensions.

Theorem

To each non-trivial square in the stable homotopy ring of spheres there is associated a local topological arbiter not induced by homology on D^{2n} for sufficiently large n. In particular, there exist local arbiters not induced by homology on D^{2n} for each $n > 2$.

Main ingredient: a homotopy-theoretic obstruction to a generalized link-slicing problem:

For each $n > 2$ we show that there exist submanifolds $(M, S^{n-1}) \subset (D^{2n}, S^{2n-1})$, where S^{n-1} is a distinguished "attaching" sphere in the boundary of M, such that

(1) for any coefficient ring R the attaching $(n-1)$-sphere of M is non-trivial in $H_{n-1}(M; R)$, and

(2) The components of the Hopf link $S^{n-1} \cup S^{n-1} \subset S^{2n-1}$ do not bound disjoint embeddings of two copies of M into D^{2n}.

(1) for any coefficient ring R the attaching $(n-1)$-sphere of M is non-trivial in $H_{n-1}(M;R)$, and

(2) The components of the Hopf link $S^{n-1} \cup S^{n-1} \subset S^{2n-1}$ do not bound disjoint embeddings of two copies of M into D^{2n}.

The proof of (1) and (2) is based on a "secondary" obstruction to disjointness where $n+1$-cells are attached to S^{n-1} via the generator of π_1^S, $n > 3$.

To prove the theorem define a "partial" arbiter, setting it equal to 1 on all submanifolds of D^{2n} containing M and equal to zero on all submanifolds contained in $D^{2n} \smallsetminus M$.

(2) implies that this partial arbiter is consistently defined, and (1) shows that it is different from any homological arbiter.

M.Freedman *"Percolation on the projective plane"* (MRL 1997)

Theorem In the Voronoi model at any level of refinement and at critical phase, $p_c = .5$, the homological observable on $\mathbb{R}P^2$ (the presence of the essential cycle in $H_1(\mathbb{R}P^2; \mathbb{Z}/2)$) is conformally (equivalently metric) invariant. It occurs with probability $q = .5$ independent of the metric on $\mathbb{R}P^2$.

Proposition. A topological arbiter satisfying Axioms (1)-(4) is an obstruction to 4-dimensional topological surgery.

Axiom (4): Suppose $\mathcal{A}(M', \gamma') = 1$ and $\mathcal{A}(M'', \gamma';) = 1$. Then $\mathcal{A}(D(M', M''), \gamma) = 1$ where $D(M', M'')$ is the "Bing double".

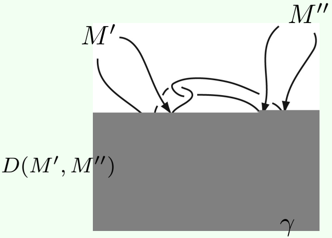


Figure : The Bing double of M', M''.

Definition

(*"Poincaré multiarbiters"*) Let M be a codimension zero submanifold of a d-dimensional manifold W. A *multiarbiter* f associates to M a number $f(M) \in \{0, \ldots, d-1\}$, subject to the axioms:

(1) If M is ambiently isotopic to M' in W then $f(M) = f(M')$.

(2) If $M \subset M'$ then $f(M) \leq f(M')$.

(3) Suppose A, B are codimension zero submanifolds such that $W = A \cup B$, with $A \cap B = \partial A \cap \partial B$. Then $f(A) + f(B) = d - 1$.

(4) $f(A \cap B) \geq f(A) + f(B) - d$.

In each dimension d, the homological multiarbiter on the projective space $\mathbb{R}P^d$ may be defined by setting $f(M) =$maximal k such that $H_k(M;\mathbb{Z}/2)$ maps onto $H_k(\mathbb{R}P^d;\mathbb{Z}/2)$. The axioms (3)-(4) follow from duality and intersection theory.

An example of a non-homological Poincaré multiarbiter on $\mathbb{R}P^3$: Given a codimension zero submanifold $M \subset \mathbb{R}P^3$, set $f(M) = 2$ iff M contains a standard copy of $\mathbb{R}P^2$. Set $f(M) = 0$ iff M can be isotoped into a 3-ball. $f(M)$ is defined to be 1 on all other submanifolds. The duality axiom holds since the complement of $\mathbb{R}P^2$ is a ball, and axiom (4) is a consequence of intersection theory.

Questions:

- Classification of arbiters and multi-arbiters in dimensions $n > 2$?
- Arbiters for graphs and complexes?