LATTICE COHOMOLOGY AND q-SERIES INVARIANTS OF 3-MANIFOLDS

ROSTISLAV AKHMECHET, PETER K. JOHNSON, AND VYACHESLAV KRUSHKAL

ABSTRACT. An invariant is introduced for negative definite plumbed 3-manifolds equipped with a spin^c-structure. It unifies and extends two theories with rather different origins and structures. One theory is lattice cohomology, motivated by the study of normal surface singularities, known to be isomorphic to the Heegaard Floer homology for certain classes of plumbed 3-manifolds. Another specialization gives BPS q-series which satisfy some remarkable modularity properties and recover SU(2) quantum invariants of 3-manifolds at roots of unity. In particular, our work gives rise to a 2-variable refinement of the $\widehat{Z}$-invariant.

1. INTRODUCTION

The development of low-dimensional topology over the last four decades has been greatly influenced by ideas and methods of gauge theory and quantum topology, dating back to the work of Donaldson [Don83] and Jones [Jon85]. There are many formulations of invariants originating in these theories, but categorically and structurally the two frameworks are quite different: the former is analytic in nature and gives rise to $(3+1)$-dimensional topological quantum field theories, associating to a closed 3-manifold versions of Floer homology (originally defined in the instanton context in [Flo88]). Starting with a quantum group, the latter gives a family of $(2+1)$-dimensional TQFTs, associating to a closed 3-manifold a collection of numerical Witten-Reshetikhin-Turaev invariants [Wit89, RT91] at roots of unity.

Our work builds on two theories that are known to recover, for a certain class of 3-manifolds, Floer homology and SU(2) quantum invariants respectively: lattice cohomology defined by Némethi [Né08] and the $\widehat{Z}$ invariant of Gukov-Pei-Putrov-Vafa [GPPV20]. We show that for negative definite plumbed 3-manifolds, equipped with a spin^c-structure, there is a natural construction giving a common refinement of these two theories. As we discuss below, our construction has novel properties that are not satisfied by either lattice cohomology or the $\widehat{Z}$ invariant. To explain this in more detail, we first summarize the context considered in this paper.

Motivated by the study of normal surface singularities, [Né08] introduced lattice cohomology $\mathbb{H}^*(Y,\mathfrak{s})$ of negative definite plumbed 3-manifolds Y with a spin^c structure $\mathfrak{s}$. For a subclass of negative definite plumbings, $\mathbb{H}^0(Y,\mathfrak{s})$ is isomorphic to Heegaard Floer homology $HF^+(-Y,\mathfrak{s})$ defined by Ozsváth-Szabó [OS04b], as modules over $\mathbb{Z}[U]$, see Section 3 for a more detailed discussion. The generators of $\mathbb{H}^0(Y,\mathfrak{s})$ and the action of U are encoded by the *graded root*, a certain infinite tree associated to $(Y,\mathfrak{s})$, first defined in [Né05].

We construct a refinement, an invariant of $(Y,\mathfrak{s})$ which takes the form of a graded root labelled by a collection of 2-variable Laurent polynomials P_F, see Figure 1 for an example. As in [Né05], the graded root (R,χ) is defined starting from a negative definite plumbing representing Y and a particular representative of the spin^c structure $\mathfrak{s}$. The Laurent polynomials P_F labelling the vertices of the graded root in our construction depend on a choice of

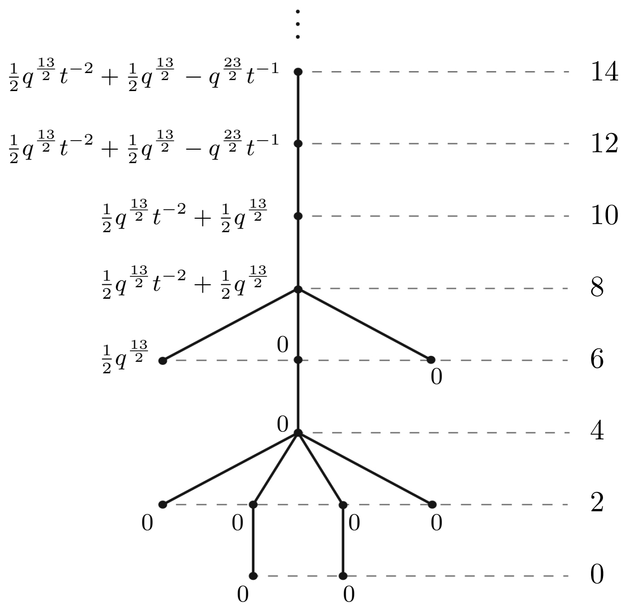

FIGURE 1. The weighted graded root for the Brieskorn sphere $\Sigma(2,7,15)$ corresponding to the admissible family $\widehat{F}$. See grading conventions 3.6 for an explanation of the numbers in the right column. A more detailed discussion of this example is given in Section 8.

admissible functions $F = \{F_n : \mathbb{Z} \to \mathcal{R}\}_{n\geq 0}$ where $\mathcal{R}$ is a commutative ring, see Definition 4.1 for details. We work in the setting of 3-manifolds which are negative definite plumbing trees, as in [Né08]. Our main theorem, proved in Section 5, is that the result is a topological invariant:

Theorem 1.1. *For any admissible family of functions F, the weighted graded root (R, χ, P_F) is an invariant of the 3-manifold Y equipped with the* spin^c *structure $\mathfrak{s}$.*

In Section 6 we show that the sequence of Laurent polynomials P^n_F, obtained by summing the labels over the vertices of the graded root R in grading $\chi = n$, stabilizes to a 2-variable series. Up to an overall normalization, this limit is a Laurent series in q whose coefficients are Laurent polynomials in t. Theorem 6.3 shows that it is an invariant of $(Y, \mathfrak{s})$.

As discussed in Section 4, there is considerable flexibility in the choice of an admissible family of functions F. Each such family gives a weighted graded root and a 2-variable series which are invariants of the 3-manifold with a spin^c structure. A particular choice, denoted $\widehat{F}$ in Section 7, gives rise to a 2-variable series $\widehat{\widehat{Z}}_{Y,\mathfrak{s}}(q,t)$. To state its properties, we recall the context of the GPPV invariant.

Based on the study of BPS states and certain supersymmetric 6-dimensional quantum field theories, [GPV17, GPPV20] formulated a physical definition of homological invariants of 3-manifolds, denoted $\mathcal{H}^{i,j}_{\mathrm{BPS}}(Y, \mathfrak{s})$. When the underlying Lie group is SU(2), the Euler characterisic of this homology theory is expected to be a q-series of the form

$$\widehat{Z}_{Y,\mathfrak{s}}(q) = \sum_{i,j} (-1)^i q^j \operatorname{rk} \mathcal{H}^{i,j}_{\mathrm{BPS}}(Y, \mathfrak{s}) \in 2^{-c} q^{\Delta_{\mathfrak{s}}} \mathbb{Z}[[q]], \tag{1}$$

for some $c \in \mathbb{Z}_+$ and $\Delta_\mathfrak{s} \in \mathbb{Q}$ depending on $(Y, \mathfrak{s})$. A mathematically rigorous definition of $\widehat{Z}_{Y,\mathfrak{s}}(q)$ in general is not yet available. A concrete mathematical formulation for negative definite plumbed 3-manifolds was given in [GPPV20, Appendix A]; also see Section 7 below for a more detailed discussion. An earlier instance of these q-series, motivated by the study of WRT invariants and of modular forms, was considered in the case of the Poincaré homology sphere, and more generally Seifert fibered integer homology spheres with three singular fibers in [LZ99]. For certain classes of negative definite plumbed 3-manifolds, the $\widehat{Z}$ q-series are known to satisfy (quantum) modularity properties, cf. [LZ99, CCF$^+$19, BMM20]. It is not yet known what kinds of modular forms arise as the q-series of other 3-manifolds including more general negative definite plumbings, and other examples such as Dehn surgeries on hyperbolic knots considered in [GM21].

For Seifert fibered integer homology spheres with three singular fibers, $\widehat{Z}_{Y,\mathfrak{s}_0}(q)$ is a holomorphic function in the unit disk $|q| < 1$, and, up to a normalization, radial limits to roots of unity give SU(2) WRT invariants [LZ99, Theorem 3], see also [GM21, Remark 4.5]. More generally, for rational homology spheres it is conjectured [GPV17] that radial limits of a certain linear combination of $\widehat{Z}_{Y,\mathfrak{s}}$ over spin^c structures recovers the WRT invariant of Y; a precise statement is also given in [GM21, Conjecture 3.1].

Our next result, established in Sections 6, 7, relates the 2-variable series that may be read off from the weighted graded root $(R, \chi, P_{\widehat{F}})$, as discussed above, to the $\widehat{Z}$ q-series.

Theorem 1.2. *The* 2*-variable series* $\widehat{\widehat{Z}}_{Y,\mathfrak{s}}(q,t)$ *is an invariant of the* 3*-manifold* Y *with a spin*^{c} *structure* $\mathfrak{s}$*, and its specialization at* $t = 1$ *equals* $\widehat{Z}_{Y,\mathfrak{s}}(q)$*.*

The series $\widehat{\widehat{Z}}_{Y,\mathfrak{s}}(q,t)$ for the Brieskorn sphere $\Sigma(2,7,15)$ in Figure 1 is considered in Example 7.8. It is an interesting question whether there are analogues for the 2-variable series of the properties of $\widehat{Z}$ discussed above, in particular the limiting behavior of $\widehat{\widehat{Z}}(q,t)$ along radial limits of the q variable to roots of unity, as well as modularity of other specializations of $\widehat{\widehat{Z}}(q,t)$.

Some common features of the $\widehat{Z}$ invariant with the gauge theory setting were apparent in [GPV17, GPPV20]; indeed bridging the gap between gauge theoretic and quantum invariants was mentioned as a motivation in [GPPV20]. Crucially, the $\widehat{Z}$ q-series depends not just on a 3-manifold, but also on a spin^c structure. Further, it is shown in [GPP21] that certain numerical gauge-theoretic invariants can be recovered from the $\widehat{Z}$ series, and a physical discussion of a relation with Heegaard Floer homology is given in [GPV17]. Our contribution, as stated in Theorems 1.1, 1.2 is a new structure that is a common refinement of both perspectives; moreover the weighted graded root (R, χ, P_F) has new features that are not present in either of them. Lattice cohomology and the $\widehat{Z}$ q-series are known to be invariant under conjugation of the spin^c structure, see Section 8 for further details. Corollary 8.2 states a more subtle transformation of the 2-variable series $\widehat{\widehat{Z}}$ under this conjugation. Moreover, Example 8.2 gives a plumbing where conjugate spin^c structures have different weighted graded roots. This example also shows that the Laurent polynomial weights of the graded root carry more information than the limiting series.

A version of the theory developed here is likely to have an analogue for knot lattice homology of [OSS14a] and the invariant of plumbed knot complements introduced in [GM21].

This extension is outside the scope of the present paper; we plan to pursue this in future work.

We conclude by recalling the problem of categorifying WRT invariants of 3-manifolds, which remains a central open question in quantum topology. The $\widehat{Z}$ q-series provide a very promising approach to this problem. Indeed, as discussed above there is a physical prediction $\mathcal{H}^{i,j}_{\mathrm{BPS}}(Y,\mathfrak{s})$ for a homology based on the theory of BPS states [GPV17, GPPV20]. It is interesting to note that the 2-variable series $\widehat{\widehat{Z}}_{Y,\mathfrak{s}}$ constructed in this paper is *different* from the expected Poincaré series of the BPS homology, see Section 7.3, thus indicating the possibility of a different (or more refined) categorification.

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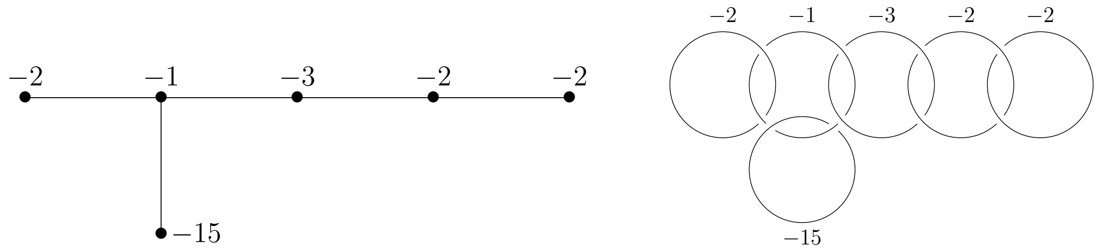


(A) A plumbing tree Γ. (B) The framed link $\mathcal{L}(\Gamma)$.

Figure 2. A negative definite plumbing Γ and its associated framed link $\mathcal{L}(\Gamma)$. The 3-manifold $Y(\Gamma)$ is the Brieskorn sphere $\Sigma(2,7,15)$.

2. Negative definite plumbed 3-manifolds

This section summarizes background material and fixes notational conventions on plumbed 3-manifolds and spin^c structures; the reader is referred to [Neu81, GS99] for more details.

2.1. **Plumbings.** A *plumbing graph* is a finite graph Γ equipped with extra data. For the purposes of this paper, we restrict to plumbing graphs which are trees equipped with a weight function $m : \mathcal{V}(\Gamma) \to \mathbb{Z}$, where $\mathcal{V}(\Gamma)$ is the set of vertices of Γ. Let $s = |\mathcal{V}(\Gamma)|$ be the number of vertices of Γ and let δ_v be the degree of $v \in \mathcal{V}(\Gamma)$. We will often implicitly choose an ordering on $\mathcal{V}(\Gamma)$, so that $\mathcal{V}(\Gamma) = \{v_1, \ldots, v_s\}$, and write quantities associated to $v_i \in \mathcal{V}(\Gamma)$ according to the subscript i. For example, $m_i = m(v_i)$, $\delta_i = \delta_{v_i}$, etc. Denote by $m, \delta \in \mathbb{Z}^s$ the weight and degree vectors, respectively:

$$m = (m_1, \ldots, m_s), \quad \delta = (\delta_1, \ldots, \delta_s).$$

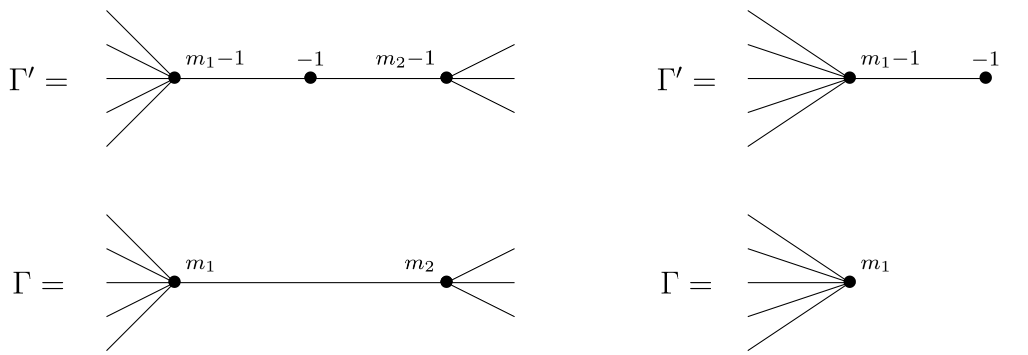
Assign to Γ the symmetric $s \times s$ matrix $M = M(\Gamma)$ with entries:

$$M_{i,j} = \begin{cases} m_i & \text{if } i = j, \\ 1 & \text{if } i \neq j, \text{ and } v_i \text{ and } v_j \text{ are connected by an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

We say Γ is *negative definite* if M is negative definite. From Γ we obtain the following manifolds. Consider the framed link $\mathcal{L}(\Gamma) \subset S^3$ given by taking an unknot at each vertex v with framing $m(v)$, and Hopf linking these unknots when the corresponding vertices are adjacent; see Figure 2 for an example. Let $X = X(\Gamma)$ denote the 4-manifold obtained by attaching 2-handles to D^4 along $\mathcal{L}(\Gamma)$. Equivalently, X is obtained by plumbing disk bundles over S^2 with Euler numbers $m(v)$. Let $Y = Y(\Gamma)$ denote the boundary of X, that is the 3-manifold obtained by Dehn surgery on $\mathcal{L}(\Gamma)$.

A *negative definite plumbed 3-manifold* is a 3-manifold that is homeomorphic to $Y(\Gamma)$ for some negative definite plumbing tree Γ. In [Neu81], the relationship between different representations of a given negative definite plumbed 3-manifold is studied. In particular, from the results in [Neu81], one can deduce the following theorem which is used in [Né08, Proposition 3.4.2] to prove the topological invariance of lattice cohomology:

Theorem 2.1 ([Neu81]). *Two negative definite plumbing trees Γ and Γ' represent the same 3-manifold if and only if they can be related by a finite sequence of type (a) and (b) Neumann moves shown in Figure 3.*



(A) Type (a) move. (B) Type (b) move.

Figure 3. Type (a) and (b) Neumann moves relating negative definite plumbing trees for homeomorphic 3-manifolds.

Notation 2.2. For future reference, we establish notation associated to type (a) and (b) Neumann moves.

- We use primes to distinguish quantities associated with Γ' from those associated with Γ. For example, δ' denotes the degree vector for Γ', whereas δ denotes the degree vector for Γ.
- For a type (a) move, we will always order vertices so that the -1 weighted vertex in Γ' which is blown down is labeled by v_0', and the two adjacent vertices with weights $m_1 - 1$ and $m_2 - 1$ are labeled by v_1' and v_2' respectively. In the Γ graph of a type (a) move, there is no vertex v_0 and the two vertices with weights m_1 and m_2 are labeled by v_1 and v_2 respectively.
- For a type (b) move, the -1 weighted vertex on Γ' is labeled by v_0' and its adjacent vertex is labeled by v_1'. In the Γ graph, there is no v_0 vertex and the vertex with weight m_1 is labeled by v_1.

2.2. **Identification of** spin^c **structures.** Spin^c structures are important ingredients to both lattice cohomology and the $\widehat{Z}$-invariant. The two theories use different identifications of spin^c structures in terms of plumbing data. We recall a translation between the two identifications, following [GM21, Section 4.2].

To begin, we describe the relationships between various (co)homology groups of X and Y. First, note that Γ gives a convenient choice of basis for $H_2(X;\mathbb{Z})$ in the following way. For $v \in \mathcal{V}(\Gamma)$, let $[v] \in H_2(X;\mathbb{Z})$ be the class of the 2-sphere obtained by capping off the core of the 2-handle corresponding to v. Then $H_2(X;\mathbb{Z}) \cong \mathbb{Z}^s$, with a basis given by $\{[v_1], \ldots, [v_s]\}$. With respect to this basis, the intersection form on $H_2(X;\mathbb{Z})$ is the bilinear form associated with M, $\langle x, y\rangle = x^t M y$. We will also write

$$\langle -, - \rangle : \mathbb{Z}^s \times \mathbb{Z}^s \to \mathbb{Z}$$

to denote this bilinear form when $H_2(X;\mathbb{Z})$ is identified with $\mathbb{Z}^s$ as above.

Remark 2.3. In some of the lattice cohomology literature the intersection form is denoted by $(-,-)$. However, in [GM21] the intersection form is denoted using angled brackets $\langle -, - \rangle$, as we do above, and $(-,-)$ instead refers to the usual dot product. To minimize confusion, we will use $\cdot$ for the dot product.

Since X is a 2-handlebody with no 1-handles, $H_1(X;\mathbb{Z}) = 0$, and we can identify $H^2(X;\mathbb{Z})$ with $\mathrm{Hom}_{\mathbb{Z}}(H_2(X;\mathbb{Z}),\mathbb{Z})$. Furthermore, using the above basis of $H_2(X;\mathbb{Z})$, we have a distinguished isomorphism $\mathrm{Hom}_{\mathbb{Z}}(H_2(X;\mathbb{Z}),\mathbb{Z}) \cong \mathbb{Z}^s$, so that a vector $k \in \mathrm{Hom}_{\mathbb{Z}}(H_2(X;\mathbb{Z}),\mathbb{Z})$ is represented as $k = (k([v_1]),\dots,k([v_s])) \in \mathbb{Z}^s$. Combining these two identifications, we get an identification of $H^2(X;\mathbb{Z})$ with $\mathbb{Z}^s$ such that for $k \in H^2(X,\mathbb{Z})$ and $x \in H_2(X;\mathbb{Z})$, we have $k(x) = k \cdot x$. The identifications described above are used throughout the paper.

Definition 2.4. An element $k \in H^2(X;\mathbb{Z})$ is called *characteristic* if $k \cdot x + \langle x, x\rangle \equiv 0 \bmod 2$ for all $x \in H_2(X;\mathbb{Z})$. We denote the set of characteristic vectors of X by $\mathrm{Char}(X)$.

In terms of our identification of $H^2(X;\mathbb{Z})$ with $\mathbb{Z}^s$, it follows that:

$$\mathrm{Char}(X) = m + 2\,\mathbb{Z}^s\,.$$

It is a standard fact in 4-manifold topology that for simply connected X the map which takes a spin^c structure on X to the first Chern class of its determinant line bundle is a bijection from $\mathrm{spin}^c(X)$ to $\mathrm{Char}(X)$, cf. [GS99, Proposition 2.4.16]. Moreover, by restricting spin^c structures to the boundary 3-manifold Y, we get the following identification:

$$\mathrm{spin}^c(Y) \cong \frac{m + 2\,\mathbb{Z}^s}{2M\,\mathbb{Z}^s}. \tag{2}$$

The right side of the above identification is to be interpreted as the set $m + 2\,\mathbb{Z}^s$ up to the equivalence relation defined by $k \sim k'$ if $k - k' \in 2M\,\mathbb{Z}^s$. If $k \in m + 2\,\mathbb{Z}^s$, we let $[k] \in \mathrm{spin}^c(Y)$ denote the equivalence class containing k.

The identification of spin^c structures just described is the one used in the lattice cohomology and Heegaard Floer homology literature (see for example [Né08, Section 2.2.2] and [OS03, Section 1]). The identification used in the $\widehat{Z}$ literature is given as follows:

$$\mathrm{spin}^c(Y) \cong \frac{\delta + 2\,\mathbb{Z}^s}{2M\,\mathbb{Z}^s}. \tag{3}$$

As in the first identification, for $a \in \delta + 2\,\mathbb{Z}^s$, we let $[a] \in \mathrm{spin}^c(Y)$ denote the equivalence class containing a. To avoid confusion between the two spin^c identifications, throughout the paper we will use the letter k for vectors in $m + 2\,\mathbb{Z}^s$ and the letter a for vectors in $\delta + 2\,\mathbb{Z}^s$.

The lattice cohomology and $\widehat{Z}$ identifications of spin^c structures are related to each other in the following way. For a fixed plumbing graph Γ, there is a bijection:

$$\frac{m + 2\,\mathbb{Z}^s}{2M\,\mathbb{Z}^s} \xrightarrow{\sim} \frac{\delta + 2\,\mathbb{Z}^s}{2M\,\mathbb{Z}^s}, \qquad [k] \longmapsto [k - (m + \delta)]. \tag{4}$$

Remark 2.5. If we let $u = (1, 1, \dots, 1) \in \mathbb{Z}^s$, then $Mu = m + \delta$ and we can alternatively express the above bijection as $[k] \mapsto [k - Mu]$.

The above bijection is natural with respect to type (a) and type (b) Neumann moves in the sense that if Γ and Γ' are two plumbing graphs related by either one of these two moves, we get the following commutative diagram:

$$\begin{array}{ccc} \frac{m+2\,\mathbb{Z}^s}{2M\,\mathbb{Z}^s} & \longrightarrow & \frac{\delta+2\,\mathbb{Z}^s}{2M\,\mathbb{Z}^s} \\ \Big\downarrow \alpha & & \Big\downarrow \beta \\ \frac{m'+2\,\mathbb{Z}^{s+1}}{2M'\,\mathbb{Z}^{s+1}} & \longrightarrow & \frac{\delta'+2\,\mathbb{Z}^{s+1}}{2M'\,\mathbb{Z}^{s+1}} \end{array}$$

Here α, β are also bijections, which at the level of representatives are defined as follows.

Type (a) move:

$$\alpha(k) = k' := (0, k) + (1, -1, -1, 0, \ldots, 0), \quad \beta(a) = a' := (0, a). \tag{5}$$

Type (b) move:

$$\alpha(k) = k' := (0, k) + (-1, 1, 0, \ldots, 0), \quad \beta(a) = a' := (0, a) + (-1, 1, 0, \ldots, 0). \tag{6}$$

3. Lattice cohomology

Lattice cohomology was introduced by Némethi in [Né08], building on earlier work of Ozsváth-Szabó in [OS03]. It is a theory which assigns to a given plumbing graph Γ and spin^c structure $[k]$, a $\mathbb{Z}[U]$-module:

$$\mathbb{H}^*(\Gamma, [k]) = \bigoplus_{i=0}^{\infty} \mathbb{H}^i(\Gamma, [k])$$

Each $\mathbb{H}^i(\Gamma, [k])$ is a $(2\,\mathbb{Z})$-graded $\mathbb{Z}[U]$-module. Hence, $\mathbb{H}^*$ is bigraded; it carries a homological grading given by the superscript i as well as an internal $(2\,\mathbb{Z})$-grading.

It was shown in [Né08] that for negative definite plumbings $\mathbb{H}^*$ is invariant under Neumann moves and therefore is a topological invariant. Furthermore, extending results from [OS03], it was shown that for a subset of negative definite plumbed 3-manifolds, namely *almost rational* plumbings, there exists an isomorphism between lattice cohomology and Heegaard Floer homology:

Theorem 3.1 ([Né08, Theorems 4.3.3 and 5.2.2])**.** *If Γ is almost rational, then as graded $\mathbb{Z}[U]$-modules,*

$$\mathbb{H}^i(\Gamma, [k]) \left[- \max_{k' \in [k]} \frac{(k')^2 + s}{4} \right] \cong \begin{cases} HF^+(-Y(\Gamma), [k]) & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Here $-Y$ is Y with the opposite orientation, $(k')^2 := (k')^t M^{-1} k'$, and the left side of the equation is $\mathbb{H}^i(\Gamma, [k])$ with its internal grading shifted up by $-\max_{k' \in [k]} \frac{(k')^2+s}{4}$.

Remark 3.2. The quantity $(k')^2$ has the following geometric meaning: it is the square of the first Chern class of the spin^c structure on $X(\Gamma)$ corresponding to the characteristic vector k'. Even if a vector x is not characteristic, we will still define $x^2 := x^t M^{-1} x \in \mathbb{Q}$.

Remark 3.3. The minimal internal grading of $\mathbb{H}^0(\Gamma, [k])$ is always equal to zero. Hence, after the grading shift, the minimal grading of the left side of equation (7) is equal to $-\max_{k' \in [k]} \frac{(k')^2+s}{4}$. In particular, Theorem 3.1 implies the d-invariant of $-Y$ at the spin^c structure $[k]$ is equal to $-\max_{k' \in [k]} \frac{(k')^2+s}{4}$, or, equivalently, the d-invariant of Y at $[k]$ is $\max_{k' \in [k]} \frac{(k')^2+s}{4}$.

In light of Theorem 3.1, we focus our attention on $\mathbb{H}^0$ rather than recalling the full definition of lattice cohomology. However, it is worth noting that there do exist negative definite plumbings with non-trivial $\mathbb{H}^i$, $i \geq 1$ (see for example [Né08, Example 4.4.1]) and there also exist other classes of plumbed manifolds in which the lattice cohomology - Heegaard Floer homology isomorphism holds, cf. [OSS14b].

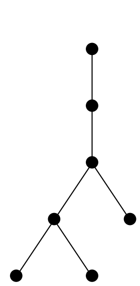


FIGURE 4. An example of a graded root.

3.1. **Graded roots.** The $(2\mathbb{Z})$-graded $\mathbb{Z}[U]$-module $\mathbb{H}^0(\Gamma, [k])$ has the nice feature that it can be encoded by a graph, called a *graded root*, proven in [Né05, Proposition 4.6] to itself be a topological invariant. We now recall the notion of a graded root (R, χ) as an abstract object and show how to associate to it a $(2\mathbb{Z})$-graded $\mathbb{Z}[U]$-module $\mathbb{H}(R, \chi)$. In the next subsection, we show how to obtain a graded root $(R_{[k]}, \chi_{[k]})$ from a pair $(\Gamma, [k])$ and we define $\mathbb{H}^0(\Gamma, [k]) = \mathbb{H}(R_{[k]}, \chi_{[k]})$. For complete details, see [Né05, Section 3].

Definition 3.4. A *graded root* is an infinite tree R, with vertices and edges denoted $\mathcal{V}$ and $\mathcal{E}$ respectively, together with a grading function $\chi : \mathcal{V} \to \mathbb{Z}$ satisfying the properties listed below. We write an edge with endpoints u and v as $[u, v] \in \mathcal{E}$.

(1) $\chi(u) - \chi(v) = \pm 1$ for any $[u, v] \in \mathcal{E}$.
(2) $\chi(u) > \min\{\chi(v), \chi(w)\}$ for any $[u, v], [u, w] \in \mathcal{E}$ with $v \neq w$.
(3) χ is bounded below, each preimage $\chi^{-1}(n)$ is finite, and $|\chi^{-1}(n)| = 1$ for sufficiently large n.

An isomorphism of graded roots is an isomorphism of the underlying graphs which commutes with the grading functions. For $r \in \mathbb{Z}$, let $(R, \chi)[r] = (R, \chi[r])$ denote the graded root with the same underlying tree and the grading shifted up by r, so that $\chi[r](v) = \chi(v) + r$.

We visualize a graded root by embedding it into the plane such that vertices of the same grading are placed at the same horizontal level, see Figure 4 for an example.

The $(2\mathbb{Z})$-graded $\mathbb{Z}[U]$-module $\mathbb{H}(R, \chi)$ associated to a graded root (R, χ) is defined as follows:

- To each $v \in \mathcal{V}$, we associate a copy of $\mathbb{Z}$, which we denote $\mathbb{Z}_v$. By an abuse of notation, we let v also denote a distinguished generator of $\mathbb{Z}_v$.
- As a graded $\mathbb{Z}$-module, we let $\mathbb{H}(R, \chi) := \bigoplus_{v \in \mathcal{V}} \mathbb{Z}_v$, where $\mathbb{Z}_v$ has grading equal to $2\chi(v)$.
- For each generator v, we let $Uv = v_1 + \cdots + v_n$ where

  $$\{v_1, \ldots, v_n\} = \{w \in \mathcal{V} \mid \chi(w) = \chi(v) - 1 \text{ and } w \text{ is connected to } v \text{ by an edge}\}.$$

  In particular, if the above set is empty, then $Uv = 0$.
- We then extend the U-action by $\mathbb{Z}$-linearity. Note U decreases grading by 2.

3.2. **Graded roots for negative definite plumbings.** We now recall from [Né05, Section 4] the graded root (R_k, χ_k) associated to a negative definite plumbing Γ and a spin^c representative $k \in m + 2\,\mathbb{Z}^s$. We then show how to obtain a graded root $(R_{[k]}, \chi_{[k]})$ corresponding to $(\Gamma, [k])$, independent of the choice of spin^c representative k.

Define a function $\chi_k : \mathbb{Z}^s \to \mathbb{Z}$ by

$$\chi_k(x) = -(k \cdot x + \langle x, x\rangle)/2. \tag{8}$$

Note that $\chi_k(x) \in \mathbb{Z}$ since k is characteristic.

Consider the standard cubical complex structure on $\mathbb{R}^s$, with 0-dimensional cells located at the lattice points $\mathbb{Z}^s \subset \mathbb{R}^s$. We can extend χ_k to a function on closed cells $\square$ (of any dimension), by defining

$$\chi_k(\square) = \max\{\chi_k(v) \mid v \text{ is a 0-cell of } \square\}$$

Let $S_j \subset \mathbb{R}^s$ denote the subcomplex consisting of cells $\square$ such that $\chi_k(\square) \leq j$. We call S_j a *sublevel set*. Note that each S_j is compact since the intersection form $\langle -, -\rangle$ is assumed to be negative definite. More precisely, if one considers χ_k as a function from $\mathbb{R}^s \to \mathbb{R}$, then it is bounded below and its level sets are $(s-1)$-dimensional ellipsoids.

Write each sublevel set as a disjoint union over its connected components,

$$S_j = C_{j,1} \sqcup \cdots \sqcup C_{j,n_j}$$

The vertices of R_k consist of connected components among all the S_j,

$$\mathcal{V} := \{C_{j,\ell} \mid j \in \mathbb{Z}, 1 \leq \ell \leq n_j\},$$

and the grading is given by $\chi_k(C_{j,\ell}) = j$, where as in [Né05] we use χ_k to denote both a function on closed cells of our cellular decomposition as well as a grading on the vertices of R_k. Edges of R_k correspond to inclusions of connected components: there is an edge connecting $C_{j,\ell}$ and $C_{j+1,\ell'}$ if $C_{j,\ell} \subseteq C_{j+1,\ell'}$. By [Né05, Proposition 4.3], (R_k, χ_k) is a graded root.

Let us now recall from [Né05] how (R_k, χ_k) depends on the choice of a spin^c representative. Let $k \in m + 2\,\mathbb{Z}^s$ and let $k' = k + 2My$ be another representative for $[k] \in \text{spin}^c(Y)$. One readily checks that:

$$\chi_{k'}(x) = \chi_k(x + y) - \chi_k(y) \tag{9}$$

for all $x \in \mathbb{Z}^s$. As stated in [Né05, Proposition 4.4], there is an isomorphism of graded roots:

$$(R_{k'}, \chi_{k'}) \cong (R_k, \chi_k)[-\chi_k(y)], \tag{10}$$

given by applying the translation $x \mapsto x+y$, to each connected component C in each sublevel set of $\chi_{k'}$. Since the collection of graded roots $\{(R_{k'}, \chi_{k'})\}_{k' \in [k]}$ are all isomorphic up to an overall grading shift, we normalize gradings in the following way to obtain a graded root independent of the choice of spin^c representative. This is the same normalization as [Né05, Section 4.5.1].

Definition 3.5. Let $[k] \in \text{spin}^c$ and define $(R_{[k]}, \chi_{[k]})$ by taking any representative $k' \in [k]$ and shifting the $\chi_{k'}$-grading on $(R_{k'}, \chi_{k'})$ so that its minimal grading is zero.

Grading conventions 3.6. When drawing the graded root $(R_{[k]}, \chi_{[k]})$ associated to a plumbing Γ and spin^c structure $[k]$, the numbers we list in the vertical column to the right are the gradings of the corresponding generators of $\mathbb{H}^0(\Gamma,[k])\left[-\max_{k'\in[k]}\frac{(k')^2+s}{4}\right]$. The reason we do this is so that when the isomorphism (7) holds, the gradings one sees are the HF^+ gradings. See for example Figure 1, where $d(-\Sigma(2,7,15))=0$.

4. Admissible functions and weighted graded roots

This section illustrates the main construction of the paper in a preliminary context. Let Γ be a negative definite plumbing and $k\in m+2\,\mathbb{Z}^s$ a spin^c representative. Given a function

$$F_{\Gamma,k}:\mathbb{Z}^s\to\mathcal{R}$$

valued in some ring $\mathcal{R}$, each vertex v in the graded root (R_k,χ_k) can be given a *weight* by taking the sum of $F_{\Gamma,k}$ over lattice points in the connected component representing v. Precisely, for a connected component C in some sublevel set of χ_k, let $L(C)=C\cap\mathbb{Z}^s$ denote its lattice points, and define its weight to be

$$F_{\Gamma,k}(C):=\sum_{x\in L(C)}F_{\Gamma,k}(x). \tag{11}$$

In this section we explain a way to obtain $F_{\Gamma,k}$ from an *admissible* family of functions $F=\{F_n\}_{n\geq 0}$. Theorem 4.3 shows that the graded root (R_k,χ_k) with these weights is an invariant of $(Y(\Gamma),[k])$. This result follows from the more general Theorem 5.10.

Definition 4.1. Fix a commutative ring $\mathcal{R}$. A family of functions $F=\{F_n:\mathbb{Z}\to\mathcal{R}\}_{n\geq0}$ is *admissible* if

(A1) $F_2(0)=1$ and $F_2(r)=0$ for all $r\neq 0$.
(A2) For all $n\geq 1$ and $r\in\mathbb{Z}$,

$$F_n(r+1)-F_n(r-1)=F_{n-1}(r).$$

For an admissible family $F=\{F_n\}_{n\geq0}$, define $F_{\Gamma,k}:\mathbb{Z}^s\to\mathcal{R}$ by

$$F_{\Gamma,k}(x)=\prod_{v\in\mathcal{V}(\Gamma)}F_{\delta_v}\left((2Mx+k-Mu)_v\right), \tag{12}$$

where $u=(1,1,\dots,1)\in\mathbb{Z}^s$ and $(-)_v$ denotes the component corresponding to v.

Remark 4.2. We stress that F_n and $F_{\Gamma,k}$ are only set-theoretic functions rather than homomorphisms. The definition (12) of $F_{\Gamma,k}(x)$ is motivated by the $\widehat{Z}$-invariant, see Section 7 and in particular Proposition 7.4.

Note that if $k'=k+2My$ is another representative for $[k]\in\text{spin}^c(Y(\Gamma))$, then

$$F_{\Gamma,k'}(x)=F_{\Gamma,k}(x+y), \tag{13}$$

so the weights in equation (11) are compatible with the isomorphism $(R_{k'},\chi_{k'})\cong(R_k,\chi_k)[-\chi_k(y)]$ from Section 3.2. Denote by $(R_{[k]},\chi_{[k]},F_{[k]})$ the graded root $(R_{[k]},\chi_{[k]})$ equipped with these weights.

Theorem 4.3. *For any admissible family of functions F, the weighted graded root $(R_{[k]},\chi_{[k]},F_{[k]})$ is an invariant of the 3-manifold $Y(\Gamma)$ endowed with the* spin^c *structure $[k]$.*

The proof of this result follows from Theorem 5.10 upon specializing $q = t = 1$.

We end this section with some remarks about admissible families of functions. Explicit $\mathbb{Z}$- and $\mathbb{Z}[\frac{1}{2}]$-valued examples motivated by the $\widehat{Z}$ invariant are given in Definitions 7.1 and 7.2. Note that not only F_2, but also F_0 and F_1 are uniquely determined by conditions (A1) and (A2):

$$F_1(r) = \begin{cases} 1 & \text{if } r = -1, \\ -1 & \text{if } r = 1, \\ 0 & \text{otherwise.} \end{cases} \qquad F_0(r) = \begin{cases} 1 & \text{if } r = \pm 2, \\ -2 & \text{if } r = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

We can characterize admissible families in the following way. Let $\mathrm{Adm}(\mathcal{R})$ denote the set of all admissible $\mathcal{R}$-valued families of functions, and let $(\mathcal{R} \times \mathcal{R})^{\mathbb{N}}$ be the set of all sequences with entries in $\mathcal{R} \times \mathcal{R}$.

Proposition 4.4. *There is a bijection* $\mathrm{Adm}(\mathcal{R}) \cong (\mathcal{R} \times \mathcal{R})^{\mathbb{N}}$.

Proof. We show that $\Psi : \mathrm{Adm}(\mathcal{R}) \to (\mathcal{R} \times \mathcal{R})^{\mathbb{N}}$ defined by $\Psi(F) = (F_{n+2}(0), F_{n+2}(1))_{n \geq 1}$ is a bijection. Suppose $F = \{F_n\}_{n \geq 0}$ is an admissible family. Fix $n \geq 1$. By applying the recursive relation (A2) inductively, we see that $F_n(r)$ is determined by $F_{n-1}, F_n(0)$ if r is even and $F_{n-1}, F_n(1)$ if r is odd, so Ψ is injective. Likewise, given $(a_n, b_n)_{n \geq 1} \in (\mathcal{R} \times \mathcal{R})^{\mathbb{N}}$, we set $(F_{n+2}(0), F_{n+2}(1)) = (a_n, b_n)$ and use (A2) to inductively construct an admissible family F such that $\Psi(F) = (a_n, b_n)_{n \geq 1}$. □

5. The invariant

This section introduces the main construction of this paper, a refinement of the weights from equation (11) in the form of a collection of two-variable Laurent polynomials. Section 5.2 shows that the resulting weighted graded root is a 3-manifold invariant.

5.1. Refined weights.

We start by establishing the following notation.

Notation 5.1. For $k, x \in \mathbb{Z}^s$, define

$$\Delta_k = -\frac{(k - Mu)^2 + 3s + \sum_v m_v}{4}, \qquad \varepsilon_k(x) = \Delta_k + 2\chi_k(x) + \langle x, u \rangle$$

where the notation $(-)^2$ is the same as in Remark 3.2. Note, the term Δ_k is an overall normalization used to eliminate dependence on the choice of spin^c representative and is similar in form to the d-invariant from Heegaard Floer homology (see Remark 3.3). Also, recall that $Mu = m + \delta$, so $\langle x, u \rangle = x \cdot (m + \delta)$.

Definition 5.2. Let $k \in m + 2\,\mathbb{Z}^s$ be a spin^c representative and let $F = \{F_n\}_{n \geq 0}$ be an admissible family valued in a commutative ring $\mathcal{R}$. To each vertex of the graded root (R_k, χ_k) we assign a weight valued in $q^{\Delta_k} \cdot \mathcal{R}[q^{\pm 1}, t^{\pm 1}]$ as follows. For a vertex represented by a connected component C in some sublevel set, let $L(C) = C \cap \mathbb{Z}^s$ denote its lattice points. Set

$$P_{F,k}(C) = \sum_{x \in L(C)} F_{\Gamma,k}(x) q^{\varepsilon_k(x)} t^{\langle x, u \rangle}, \tag{15}$$

and let $(R_k, \chi_k, P_{F,k})$ denote the graded root (R_k, χ_k) with these weights. We will often omit the reference to F by writing P_k instead of $P_{F,k}$. Note that specializing $q = t = 1$ recovers the weights in equation (11).

The above weights can be interpreted geometrically as follows. For $n \in \mathbb{Z}$, the coefficient of t^n in $P_k(C)$ is given by summing $F_{\Gamma,k}(x)q^{\Delta_k+2\chi_k(x)+n}$ over all $x \in \mathbb{Z}^s$ which lie on the intersection of C with the hyperplane $\{y \in \mathbb{R}^s \mid \langle y, u\rangle = n\}$.

Let us verify that the weights $P_k(C)$ are compatible with the isomorphisms (10) relating graded roots for different representatives of $[k]$.

Lemma 5.3. *Let $k, k' = k + 2My \in m + 2\,\mathbb{Z}^s$ be two representatives for $[k] \in \mathrm{spin}^c(Y)$. Then $\varepsilon_{k'}(x) = \varepsilon_k(x + y)$ for all $x \in \mathbb{Z}^s$.*

Proof. First note that

$$(k' - Mu)^2 = (k - Mu)^2 + 4y \cdot k - 4\langle y, u\rangle + 4\langle y, y\rangle,$$

which implies

$$\begin{aligned}\Delta_{k'} &= \Delta_k - y \cdot k + \langle y, u\rangle - \langle y, y\rangle \\ &= \Delta_k + 2\chi_k(y) + \langle y, u\rangle.\end{aligned}$$

The desired equality now follows from the above equality and equation (9). □

For $n \in \mathbb{Z}$, let $(R_k, \chi_k, P_k)\{n\}$ denote the weighted graded root with each Laurent polynomial weight multiplied by t^n.

Proposition 5.4. *Let $k \in m + 2\,\mathbb{Z}^s$ and let $k' = k + 2My$ for some $y \in \mathbb{Z}^s$. Then*

$$(R_{k'}, \chi_{k'}, P_{k'}) \cong (R_k, \chi_k, P_k)[-\chi_k(y)]\{-\langle y, u\rangle\}.$$

Proof. Recall that the isomorphism $(R_k, \chi_k) \cong (R_{k'}, \chi_{k'})[-\chi_k(y)]$ of graded roots in equation (10) is induced by the translation $T(x) = x + y$. A lattice point $x \in L(C)$ contributes the summand

$$F_{\Gamma,k'}(x)q^{\varepsilon_{k'}(x)}t^{\langle x,u\rangle}$$

to $P_{F,k'}(C)$, while $T(x)$ contributes

$$t^{\langle y,u\rangle} \cdot F_{\Gamma,k}(x + y)q^{\varepsilon_k(x+y)}t^{\langle x,u\rangle}$$

to $P_{F,k}(T(C))$. Lemma 5.3 and equation (13) imply that

$$F_{\Gamma,k'}(x)q^{\varepsilon_{k'}(x)}t^{\langle x,u\rangle} = F_{\Gamma,k}(x + y)q^{\varepsilon_k(x+y)}t^{\langle x,u\rangle},$$

which completes the proof. □

In light of the above proposition, we consider weighted graded roots up to simultaneous multiplication of all Laurent polynomial weights by some overall power of t; that is, (R_k, χ_k, P_k) is equivalent to $(R_k, \chi_k, P_k)\{n\}$ for all $n \in \mathbb{Z}$.

Definition 5.5. Set $(R_{[k]}, \chi_{[k]}, P_{[k]})$ to be $(R_{[k]}, \chi_{[k]})$, as in Definition 3.5, equipped with the weights P_k for some $k \in [k]$.

Lemma 5.3 and Proposition 5.4 guarantee that, up to the above equivalence, $(R_{[k]}, \chi_{[k]}, P_{[k]})$ does not depend on $k \in [k]$.

Remark 5.6. The t-ambiguity could be removed by fixing the following normalization, similar to the normalization in Definition 3.5. First, if for some $k \in [k]$ all P_k weights are zero then, by Proposition 5.4, for any other $k' \in [k]$ all $P_{k'}$ weights are zero. Hence, there is no ambiguity. Otherwise, let $n' \in \mathbb{Z}$ be the minimal t-power among all nonzero P_k weights of the vertices in χ_k-grading n, and set

$$(R_{[k]}, \chi_{[k]}, P_{[k]}) = (R_{[k]}, \chi_{[k]}, P_k)\{-n'\}.$$

Proposition 5.4 implies $(R_{[k]}, \chi_{[k]}, P_{[k]})$, as defined in the previous equation, is independent of $k \in [k]$.

5.2. **Invariance.** In this section we prove invariance of $(R_{[k]}, \chi_{[k]}, P_{[k]})$ under the two Neumann moves shown in Figure 3. This establishes Theorem 1.1, which is restated as Theorem 5.10 below using a more detailed notation. In what follows, Γ is a negative definite plumbing tree with s vertices, and Γ' is a plumbing tree with $s' = s + 1$ vertices obtained from Γ by one of the type (a) or (b) moves. We will use the conventions established in Notation 2.2, as well as the following additional notation for the two moves.

Type (a): The intersection form for Γ' is given by $M' = \widetilde{M} + A$ where

$$\widetilde{M} = \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix} \qquad A = \begin{pmatrix} -1 & 1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & -1 & 0 & \cdots & 0 \\ 1 & -1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix} \tag{16}$$

As in [Né08, Proposition 3.4.2], define the projection $\pi_* : \mathbb{Z}^{s+1} \to \mathbb{Z}^s$ by

$$\pi_*(x_0, x_1, \ldots, x_s) = (x_1, \ldots, x_s) \tag{17}$$

and the inclusion $\pi^* : \mathbb{Z}^s \to \mathbb{Z}^{s+1}$ by

$$\pi^*(x_1, x_2, \ldots, x_s) = (x_1 + x_2, x_1, x_2, \ldots, x_s). \tag{18}$$

Type (b): The intersection form for Γ' is given by $M' = \widetilde{M} + A$ where

$$\widetilde{M} = \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix} \qquad A = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \tag{19}$$

As in the type (a) case, let $\pi_* : \mathbb{Z}^{s+1} \to \mathbb{Z}^s$ denote the projection $\pi_*(x_0, x_1, \ldots, x_s) = (x_1, x_2, \ldots, x_s)$, and define two inclusions $\pi^*, \rho^* : \mathbb{Z}^s \to \mathbb{Z}^{s+1}$ by

$$\pi^*(x_1, \ldots, x_s) = (x_1, x_1, \ldots, x_s), \tag{20}$$

$$\rho^*(x_1, \ldots, x_s) = (x_1 - 1, x_1, \ldots, x_s). \tag{21}$$

With the notation in place, we now record several results regarding various contributions to the Laurent polynomial weights.

Lemma 5.7. *Let Γ, Γ' be negative definite plumbings related by a Neumann move as above. Let $k \in m + 2\,\mathbb{Z}^s$ be a* spin^c *representative, and let $k' \in m' + 2\,\mathbb{Z}^{s+1}$ denote the associated representative, as in equations* (5) *and* (6)*. Then $\Delta_k = \Delta_{k'}$.*

Proof. We first address the type (a) move. Let $u' = (1, u)$. Observe that $3s + \sum_v m_v = 3s' + \sum_v m'_v$, so it remains to verify that

$$(k' - M'u')^2 = (k - Mu)^2.$$

Expressions for M' and k' are given in equations (16) and (5). Note that

$$k' - M'u' = (0, k - Mu).$$

Let $y = (y_1, y_2, \dots, y_s) \in \mathbb{Q}^s$ be such that $k - Mu = My$. Then $M'(y_1 + y_2, y) = (0, k - Mu)$, and it follows that $(k' - M'u')^2 = (k - Mu)^2$ which completes the proof of the type (a) case.

For the type (b) move, M' and k' are given in equations (19), (6). Note

$$k' - M'u' = (-1, 1, 0, \dots, 0) + (0, k - Mu).$$

Denote these two summands by $w := (-1, 1, 0, \dots, 0)$ and $\widetilde{k} := (0, k - Mu)$. We claim that

$$(\widetilde{k})^t (M')^{-1} \widetilde{k} = (k - Mu)^t M^{-1} (k - Mu). \tag{22}$$

To prove the claim, let $y = M^{-1}(k - Mu)$ or equivalently $k - Mu = My$. Then one checks that $\widetilde{k} = M'(y_1, y)$, where y_1 is the first coordinate of y. Thus the left-hand side of (22) equals $(0, k - Mu)^t (y_1, y)$ which equals the right-hand side of (22), verifying the claim. Expanding linearly, consider

$$(k' - M'u')^t (M')^{-1} (k' - M'u') - (k - Mu)^t M^{-1} (k - Mu) =$$

$$w^t (M')^{-1} w + 2(\widetilde{k})^t (M')^{-1} w. \tag{23}$$

It follows from equations (19) that $(M')^{-1} w = (1, 0, \dots, 0)$, thus the first term in (23) equals -1 and the second term is zero. Under the (b) move, we have $3s + \sum_v m_v = 3s' + \sum_v m'_v - 1$, precisely offsetting the change in $(k - Mu)^2$ computed above.

□

For the type (a) move in the following lemma, recall the function π^* from equation (18). For the type (b) move, recall the functions π^*, and ρ^* from equations (20) and (21).

Lemma 5.8. *Let $\Gamma, \Gamma', k,$ and k' be as in the statement of Lemma 5.7.*

(1) In the type (a) case, for any $x \in \mathbb{Z}^s$,

$$\langle x, u \rangle = \langle \pi^*(x), u' \rangle.$$

(2) In the type (b) case, for any $x \in \mathbb{Z}^s$,

$$\langle x, u \rangle = \langle \pi^*(x), u' \rangle = \langle \rho^*(x), u' \rangle.$$

Proof. Recall that $\langle x, u \rangle = x \cdot (m + \delta)$. For item (1), $m' + \delta' = (1, -1, -1, 0, \dots, 0) + (0, m + \delta)$. Then

$$\begin{aligned} \pi^*(x) \cdot (m' + \delta') &= (x_1 + x_2, x_1, x_2, \dots, x_s) \cdot [(1, -1, -1, 0, \dots, 0) + (0, m + \delta)] \\ &= x \cdot (m + \delta). \end{aligned}$$

For item (2), $m' + \delta' = (0, m + \delta)$, and the desired equality follows. □

Lemma 5.9. *Let $\Gamma, \Gamma', k,$ and k' be as in the statement of Lemma 5.7.*

(1) For the type (a) move, $F_{\Gamma,k}(x) = F_{\Gamma',k'}(\pi^*(x))$ for all $x \in \mathbb{Z}^s$,

(2) For the type (b) move, $F_{\Gamma,k}(x) = F_{\Gamma',k'}(\pi^*(x)) + F_{\Gamma',k'}(\rho^*(x))$ for all $x \in \mathbb{Z}^s$.

Proof. To show item (1), first note that

$$2M'\pi^*(x) + k' - M'u' = (0, 2Mx + k - Mu),$$

so, using property (A1), $F_{\Gamma',k'}(\pi^*(x)) = F_2(0)\prod_{i=1}^{s} F_{\delta_i}((2Mx+k-Mu)_{v_i}) = F_{\Gamma,k}(x)$.

We now verify item (2). Observe that

$$2M'\pi^*(x) + k' - M'u' = (0, 2Mx + k - Mu) + (-1, 1, 0, \ldots, 0),$$
$$2M'\rho^*(x) + k' - M'u' = (0, 2Mx + k - Mu) + (1, -1, 0, \ldots, 0).$$

Introduce the notation

$$r = (2Mx + k - Mu)_{v_1}, \qquad \overline{r} = \prod_{i=2}^{s} F_{\delta_i}(2Mx + k - Mu)_{v_i},$$

and recall from equation (14) that $F_1(\pm 1) = \mp 1$. Then we have

$$F_{\Gamma',k'}(\pi^*(x)) = F_{\delta_1+1}(r+1)\cdot \overline{r},$$
$$F_{\Gamma',k'}(\rho^*(x)) = -F_{\delta_1+1}(r-1)\cdot \overline{r},$$
$$F_{\Gamma,k}(x) = F_{\delta_1}(r)\cdot \overline{r},$$

and the desired equality follows from property (A2). □

We are in a position to prove our main result:

Theorem 5.10. *For any admissible family of functions F, the weighted graded root $(R_{[k]}, \chi_{[k]}, P_{[k]})$ is an invariant of the 3-manifold $Y(\Gamma)$ equipped with the* spin^c *structure $[k]$.*

Proof. We will demonstrate an isomorphism $(R_k, \chi_k, P_k) \cong (R_{k'}, \chi_{k'}, P_{k'})$ of weighted graded roots when Γ, Γ', k, and k' are as in the statement of Lemma 5.7. Note, here the symbol $\cong$ means isomorphism with no t-ambiguity; the t-ambiguity only becomes relevant when one changes spin^c representative.

For each of the two moves we first give an explicit isomorphism of graded roots $(R_k, \chi_k) \cong (R_{k'}, \chi_{k'})$, following the proofs of [Né05, Proposition 4.6] and [Né08, Proposition 3.4.2]. We then show that this isomorphism respects our Laurent polynomial weights.

We begin with the type (a) move. Recall the functions π_* and π^* from (17) and (18), and that $M' = \widetilde{M} + A$, as in equation (16).

For $x' = (x_0, x_1, x_2, \ldots, x_s) \in \mathbb{Z}^{s+1}$, we have

$$(x')^t A x' = -(x_0 - x_1 - x_2)^2,$$

so $(x')^t M' x' = \pi_*(x')^t M \pi_*(x') - (x_0 - x_1 - x_2)^2$. It is then straightforward to verify that

$$\chi_{k'}(x') = \chi_k(\pi_*(x')) + \frac{1}{2}[(x_0 - x_1 - x_2)(x_0 - x_1 - x_2 - 1)].$$

In particular, substituting $x' = \pi^*(x)$ for $x \in \mathbb{Z}^s$, this implies

$$\chi_{k'} \circ \pi^* = \chi_k, \tag{24}$$

so π^* induces an inclusion $\chi_k^{-1}((-\infty, j]) \hookrightarrow \chi_{k'}^{-1}((-\infty, j])$ of sublevel sets. As in the proof of [Né08, Proposition 3.4.2], π^* also induces a bijection, denoted $\tilde{\pi}^*$, on connected components in these sublevel sets. The isomorphism $(R_k, \chi_k) \cong (R_{k'}, \chi_{k'})$ of graded roots is given by $\tilde{\pi}^*$, sending a connected component C to the connected component $\tilde{\pi}^*(C)$ that contains $\pi^*(C)$.

Fix a connected component C in some sublevel set of χ_k. We will now show that

$$P_k(C) = P_{k'}(\tilde{\pi}^*(C)).$$

The term on the right-hand side above is a sum over contributions from lattice points in the component $\tilde{\pi}^*(C)$, which contains all the lattice points in $\pi^*(C)$, but is in general strictly bigger. As we shall now see, only lattice points in $\pi^*(C)$ contribute. We have

$$(2M'x' + k' - M'u')_{v'_0} = -2(x_0 - x_1 - x_2).$$

Since $\delta_{v'_0} = 2$, property (A1) implies that $F_{\Gamma',k'}(x') = 0$ unless $x_0 = x_1 + x_2$, so

$$P_{k'}(\tilde{\pi}^*(C)) = \sum_{x' \in \pi^*(L(C))} F_{\Gamma',k'}(x') q^{\varepsilon_{k'}(x')} t^{\langle x', u' \rangle}.$$

Therefore, it suffices to show

$$F_{\Gamma,k}(x) q^{\varepsilon_k(x)} t^{\langle x, u \rangle} = F_{\Gamma',k'}(\pi^*(x)) q^{\varepsilon_{k'}(\pi^*(x))} t^{\langle \pi^*(x), u' \rangle}.$$

for all $x \in C$. Equation (24), Lemma 5.7, and item (1) of Lemma 5.8 guarantee that the powers of q and t are equal, and $F_{\Gamma,k}(x) = F_{\Gamma',k'}(\pi^*(x))$ by Lemma 5.9 (1). This concludes the proof of the type (a) move.

We now address the type (b) move. Recall the functions π_*, π^*, and ρ^* from equations (17), (20), (21), and that $M' = \widetilde{M} + A$ as in equation (19). For $x' \in \mathbb{Z}^{s+1}$, we have

$$(x')^t A x' = -(x_0 - x_1)^2,$$

so $(x')^t M' x' = \pi_*(x')^t M \pi_*(x') - (x_0 - x_1)^2$. It is then easy to see that

$$\chi_{k'}(x') = \chi_k(\pi_*(x')) + \frac{1}{2}(x_0 - x_1)(x_0 - x_1 + 1),$$

which implies

$$\chi_{k'} \circ \pi^* = \chi_k = \chi_{k'} \circ \rho^*. \tag{25}$$

Thus both π^* and ρ^* induce inclusions $\chi_k^{-1}((-\infty, j]) \hookrightarrow \chi_{k'}^{-1}((-\infty, j])$ of sublevel sets. As in the type (a) case above, π^* also induces a bijection $\tilde{\pi}^*$ between connected components of each sublevel set, and the isomorphism of graded roots $(R_k, \chi_k) \cong (R_{k'}, \chi_{k'})$ is given by $\tilde{\pi}^*$.

To complete the proof we check that

$$P_k(C) = P_{k'}(\tilde{\pi}^*(C))$$

for every connected component C of each sublevel set of χ_k. As in the type (a) case, we will now see that only a particular subset of lattice points in $\tilde{\pi}^*(C)$ contribute to $P_{k'}(\tilde{\pi}^*(C))$. To begin, note

$$(2M'x' + k' - M'u')_{v'_0} = -2(x_0 - x_1) - 1.$$

Since $\delta_{v'_0} = 1$, the formula for F_1 from equation (14) implies that $F_{\Gamma',k'}(x') = 0$ unless $-2(x_0 - x_1) - 1 = \pm 1$, or, equivalently, unless $x' = \pi^*(x)$ or $x' = \rho^*(x)$ for some $x \in \mathbb{Z}^s$. Observe that $\pi^*(x) - \rho^*(x) = (1, 0, \ldots, 0)$, so $\pi^*(x)$ and $\rho^*(x)$ are in the same component of $\chi_{k'}^{-1}(j)$. It follows that

$$P_{k'}(\tilde{\pi}^*(C)) = \sum_{x' \in \pi^*(L(C))} F_{\Gamma',k'}(x') q^{\varepsilon_{k'}(x')} t^{\langle x', u' \rangle} + \sum_{x' \in \rho^*(L(C))} F_{\Gamma',k'}(x') q^{\varepsilon_{k'}(x')} t^{\langle x', u' \rangle}.$$

To complete the proof, we have

$$F_{\Gamma,k}(x)q^{\varepsilon_k(x)}t^{\langle x,u\rangle} = F_{\Gamma',k'}(\pi^*(x))q^{\varepsilon_{k'}(\pi^*(x))}t^{\langle \pi^*(x),u'\rangle} + F_{\Gamma',k'}(\rho^*(x))q^{\varepsilon_{k'}(\rho^*(x))}t^{\langle \rho^*(x),u'\rangle}$$

by combining Lemma 5.7, equation (25), item (2) of Lemma 5.8, and item (2) of Lemma 5.9. □

6. The two-variable series

In this section we extract a two-variable series from $(R_{[k]}, \chi_{[k]}, P_{[k]})$ by taking a limit (in an appropriate sense) of the weights $P_k(C)$. Theorem 6.3 shows that this limiting procedure yields a well-defined invariant of $(Y(\Gamma), [k])$. Throughout this section some $\mathcal{R}$-valued admissible family F will be fixed, and references to it will be omitted for brevity of notation.

We first establish some preliminary notions. For a commutative ring $\mathcal{R}$, denote by $\mathcal{R}[q^{-1}, q]]$ the ring of Laurent series in q and by $\widetilde{\mathcal{R}}$ the set of Laurent series in q whose coefficients are Laurent polynomials in t,

$$\widetilde{\mathcal{R}} = \left(\mathcal{R}[t^{\pm 1}]\right)[q^{-1}, q]].$$

Given $\Delta \in \mathbb{Q}$, $f \in q^{\Delta} \cdot \widetilde{\mathcal{R}}$, and $i, j \in \mathbb{Z}$, let $[f]_{i,j} \in \mathcal{R}$ be the coefficient of $q^{\Delta+i}t^j$ in f.

Definition 6.1. We say a sequence $f_1, f_2, \ldots \in q^{\Delta} \cdot \mathcal{R}[q^{\pm 1}, t^{\pm 1}]$ *stabilizes* if for all $i, j \in \mathbb{Z}$, the sequence of coefficients $([f_1]_{i,j}, ([f_2]_{i,j}, \ldots)$ is eventually constant. For such a sequence, the *limit* f is the bi-infinite series in q, t defined by setting $[f]_{i,j}$ to be the limit of $[f_n]_{i,j}$ as $n \to \infty$.

As stated in the definition, the limit of a stabilizing sequence in general is a bi-infinite series in q, t. In Theorem 6.3 below, the limit is claimed to be an element of $q^{\Delta} \cdot \widetilde{\mathcal{R}}$. In addition to proving that the sequence (f_n) stabilizes, this will be shown by establishing that

(i) there exists $i_0 \in \mathbb{Z}$ such that $[f_n]_{i,j} = 0$ for all $n \geq 0$, $j \in \mathbb{Z}$, and $i \leq i_0$, and
(ii) for any fixed i, the set of j such that $[f_n]_{i,j} \neq 0$ is bounded.

Returning to weighted graded roots, fix a negative definite plumbing tree Γ and spin^c representative $k \in m + 2\,\mathbb{Z}^s$. Consider the weighted graded root (R_k, χ_k, P_k), as given in Definition 5.2. For $n \in \mathbb{Z}$, let

$$P_k^n := \sum_{C \in \chi_{[k]}^{-1}(n)} P_k(C) \in q^{\Delta_k} \cdot \mathcal{R}[q^{\pm 1}, t^{\pm 1}] \tag{26}$$

denote the sum of the Laurent polynomial weights over vertices C of R_k in χ_k-grading n. Recall that χ_k is bounded below by some $n_0 \in \mathbb{Z}$, and consider the sequence $(P_k^{n_0}, P_k^{n_0+1}, P_k^{n_0+2}, \ldots)$.

Remark 6.2. Note that P_k^n is the sum of $F_{\Gamma,k}(x)q^{\varepsilon_k(x)}t^{\langle x,u\rangle}$ over all lattice points x in the entire n-sublevel set of χ_k. Moreover, since there is only one connected component for large enough n, one may just as well start the sequence at a sufficiently high χ_k-grading, making the sum in (26) be given by a single $P_k(C)$.

Theorem 6.3. *The sequence* $\left(P_k^{n_0}, P_k^{n_0+1}, P_k^{n_0+2}, \ldots\right)$ *stabilizes to an element of* $q^{\Delta_k} \cdot \widetilde{\mathcal{R}}$. *Up to multiplication by a power of* t, *its limit* P_k^{∞} *is an invariant of the* 3*-manifold* $Y(\Gamma)$ *equipped with the* spin^c *structure* $[k]$.

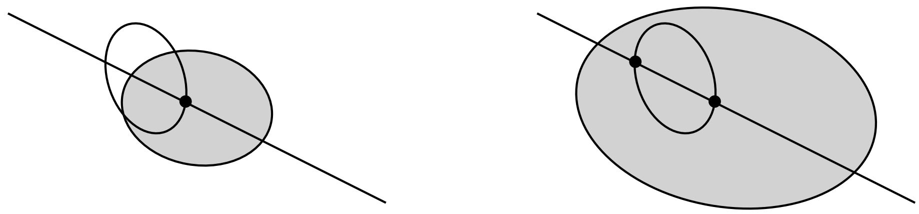


FIGURE 5. A schematic depiction of stabilization. Left: $\mathcal{S}_n$ is represented by the shaded ellipse, $\partial\widetilde{\mathcal{S}}_i$ is the unshaded ellipse, and the hyperplane $\mathcal{A}_j$ is the straight line (the actual sets are discrete subsets of the illustration). The intersection $\mathcal{S}_n \cap \partial\widetilde{\mathcal{S}}_i \cap \mathcal{A}_j$ is marked by a dot. Right: taking $n >> 0$ ensures $\partial\widetilde{\mathcal{S}}_i \subset \mathcal{S}_n$, and the intersection (two dots) is the same for all sufficiently large n.

Proof. For $n \in \mathbb{Z}$, define

$$\mathcal{S}_n = \{x \in \mathbb{Z}^s \mid \chi_k(x) \le n\}, \qquad \widetilde{\mathcal{S}}_n = \{x \in \mathbb{Z}^s \mid 2\chi_k(x) + \langle x, u\rangle \le n\},$$
$$\partial\widetilde{\mathcal{S}}_n = \{x \in \mathbb{Z}^s \mid 2\chi_k(x) + \langle x, u\rangle = n\}, \qquad \mathcal{A}_n = \{x \in \mathbb{Z}^s \mid \langle x, u\rangle = n\}.$$

By definition,

$$P_k^n = \sum_{x \in \mathcal{S}_n} F_{\Gamma,k}(x) q^{\varepsilon_k(x)} t^{\langle x,u\rangle}.$$

It follows from Notation 5.1 that for fixed $i, j \in \mathbb{Z}$, the coefficient of $q^{\Delta_k + i} t^j$ in P_k^n is equal to

$$\sum_x F_{\Gamma,k}(x), \tag{27}$$

where the sum is over $x \in \mathcal{S}_n \cap \partial\widetilde{\mathcal{S}}_i \cap \mathcal{A}_j$. Both $(\mathcal{S}_n)$, $(\widetilde{\mathcal{S}}_n)$ are sequences of nested finite sets whose union is $\mathbb{Z}^s$. Hence for a fixed i there exists N such that $\mathcal{S}_i \subset \mathcal{S}_n$ for all $n \ge N$. Then for $n \ge N$, we have

$$\mathcal{S}_n \cap \partial\widetilde{\mathcal{S}}_i \cap \mathcal{A}_j = \partial\widetilde{\mathcal{S}}_i \cap \mathcal{A}_j,$$

so that the sum in equation (27) is independent of n for n sufficiently large. See Figure 5 for an illustration when $s = 2$. This verifies stabilization of the sequence.

As discussed after Definition 6.1, we will check two conditions (i), (ii) ensuring that the limit is an element of $q^\Delta \cdot \widetilde{\mathcal{R}}$. The condition (i) follows from the fact that the q-powers in the P_k^n are given by ε_k, which is bounded below. To check (ii), observe that for a fixed i, the exponent of t is given by $\langle x, u\rangle$ which is bounded on the set $\widetilde{\mathcal{S}}_i$.

That P_k^∞ is an invariant of $(Y(\Gamma), [k])$ up to multiplication by t follows from Proposition 5.4 and Theorem 5.10. □

For $f, g \in q^\Delta \cdot \widetilde{\mathcal{R}}$, we write $f \doteq g$ if $f = t^n g$ for some $n \in \mathbb{Z}$. As a result of the above theorem, we can now make the following definition.

Definition 6.4. Set $P_{[k]}^\infty = P_k^\infty$ for any choice of $k \in [k]$, with the understanding that $P_{[k]}^\infty$ is well-defined up to multiplication by a power of t. More explicitly, it follows from the definition (15) of the Laurent polynomial weights and from Theorem 6.3 that

$$P^{\infty}_{[k]} \doteq \sum_{x\in\mathbb{Z}^s} F_{\Gamma,k}(x) q^{\varepsilon_k(x)} t^{\langle x,u\rangle} \tag{28}$$

for any choice of $k \in [k]$. To specify the underlying admissible family F, the notation $P^{\infty}_{F,[k]}$ will be used.

Remark 6.5. For any F, setting $t = 1$ in $P^{\infty}_{[k]}$ gives a well-defined Laurent q-series invariant of $(Y(\Gamma,[k]))$. Moreover, if the t-powers in $(R_{[k]}, \chi_{[k]}, P_{[k]})$ are normalized as in Remark 5.6, then the weights stabilize to a well-defined element of $q^{\Delta_k} \cdot \widetilde{\mathcal{R}}$.

7. The $\widehat{Z}_a(q)$ power series

This section starts with a review of the GPPV invariant of negative definite plumbed manifolds, motivating the definition of the admissible family of functions $\widehat{F}$. In fact, three closely related admissible families are discussed in Section 7.1, $\widehat{F}$, $\widehat{F}^{+}$, and $\widehat{F}^{-}$. Section 7.2 reformulates the $\widehat{Z}$ invariant using the lattice cohomology convention for spin^c structures. Theorem 7.6 shows that the GPPV invariant is a specialization of the 2-variable series $P^{\infty}_{\widehat{F},[k]}$ at $t = 1$, thus establishing Theorem 1.2 stated in the introduction. Additionally, Section 7.3 gives calculations in specific examples.

Let $a \in \delta + 2\,\mathbb{Z}^s$ be a representative of a spin^c structure $[a]$ on Y, using the convention (3). Following [GPPV20] (see also [GM21, Section 4.3]), consider

$$\widehat{Z}_a(q) := q^{-\frac{3s+\sum_v m_v}{4}} \cdot v.p. \oint\limits_{|z_v|=1} \prod_{v\in\mathcal{V}(\Gamma)} \frac{dz_v}{2\pi i z_v} \left(z_v - \frac{1}{z_v}\right)^{2-\delta_v} \cdot \Theta^{-M}_a(z), \tag{29}$$

where

$$\Theta^{-M}_a(z) := \sum_{\ell\in a+2M\,\mathbb{Z}^s} q^{-\frac{\ell^t M^{-1}\ell}{4}} \prod_{v\in\mathcal{V}(\Gamma)} z_v^{\ell_v}. \tag{30}$$

In (29), $v.p.$ denotes the principal value, that is the average of the integrals over $|z_v| = 1+\varepsilon$ and over $|z_v| = 1-\varepsilon$, for small $\varepsilon > 0$. Note that, since M is negative definite, for each power of q the expression (30) for $\Theta^{-M}_a(z)$ is a Laurent polynomial in the variables $\{z_v\}_{v\in\mathcal{V}}$, and the exponent $(-\ell^t M^{-1}\ell)/4$ as ℓ varies is bounded below. It is clear from the definition that $\widehat{Z}_a(q)$ is independent of the choice of representative $a \in [a]$.

We begin by rewriting (29) as

$$\widehat{Z}_a(q) = q^{-\frac{3s+\sum_v m_v}{4}} \sum_{\ell\in a+2M\,\mathbb{Z}^s} \left[\prod_v v.p. \frac{1}{2\pi i} \oint\limits_{|z_v|=1} \frac{1}{z_v}\left(z_v - \frac{1}{z_v}\right)^{2-\delta_v} z_v^{\ell_v} dz_v\right] q^{-\frac{\ell^t M^{-1}\ell}{4}}. \tag{31}$$

Our analysis of the $\widehat{Z}$ invariant, in particular Proposition 7.4 and Theorem 7.6 below, depends on the properties of the coefficient given by the expression in the square brackets in equation (31). Thus we start by rewriting it in more concrete terms. We note that this preliminary analysis, leading to Definition 7.1, amounts to taking a detailed look at the coefficients denoted $F_{\vec{\ell}}$ in [GM21, equation (43)].

To compute the integral in equation (31), write $(z_v - z_v^{-1})^{2-\delta_v}$ as a Laurent series E_v^- in z_v for the integral over $|z_v| = 1-\varepsilon$, and as a Laurent series E_v^+ in z_v^{-1} for the integral over $|z_v| = 1+\varepsilon$. Then for $\ell_v \in \mathbb{Z}$,

$$v.p. \frac{1}{2\pi i} \oint\limits_{|z_v|=1} \frac{1}{z_v}\left(z_v - \frac{1}{z_v}\right)^{2-\delta_v} z_v^{\ell_v} dz_v = \frac{1}{2}\left[\operatorname{Res}\left(z_v^{\ell_v-1}E_v^-, 0\right) + \operatorname{Res}\left(z_v^{-\ell_v-1}E_v^+(z_v^{-1}), 0\right)\right].$$

Note that $\operatorname{Res}\left(z_v^{\ell_v-1}E_v^-, 0\right)$ and $\operatorname{Res}\left(z_v^{-\ell_v-1}E_v^+(z_v^{-1}), 0\right)$ equal the coefficient of $z^{-\ell_v}$ in E_v^- and E_v^+, respectively. We will now identify the Laurent series E_v^- and E_v^+ more explicitly.

When $\delta_v \leq 2$, the exponent in $(z_v - z_v^{-1})^{2-\delta_v}$ is non-negative and $E_v^+ = E_v^- = (z_v - z_v^{-1})^{2-\delta_v}$ is a Laurent polynomial. In particular, if $\delta_v \leq 2$ for all vertices v, then $\widehat{Z}_a(q)$ is a Laurent polynomial with integer coefficients. More generally, coefficients of $\widehat{Z}_a(q)$ are in $2^{-c} \cdot \mathbb{Z}$ where c is the number of vertices of degree at least 3.

We now describe the Laurent series expansions $E_v^{\pm}$ of $(z_v - z_v^{-1})^{2-\delta_v}$ when $\delta_v \geq 3$. Fix $n \geq 3$. For $|z| < 1$, using the expansion $\left(z - \frac{1}{z}\right)^{-1} = \frac{-z}{1-z^2} = -\sum\limits_{i\geq 0} z^{2i+1}$, we can write

$$\left(z - \frac{1}{z}\right)^{2-n} = \left(-\sum_{i\geq 0} z^{2i+1}\right)^{n-2}. \tag{32}$$

For $|z| > 1$, the expansion $\left(z - \frac{1}{z}\right)^{-1} = \frac{z^{-1}}{1-z^{-2}} = \sum\limits_{i\geq 0} z_v^{-(2i+1)}$ gives

$$\left(z - \frac{1}{z}\right)^{2-n} = \left(\sum_{i\geq 0} z^{-(2i+1)}\right)^{n-2}. \tag{33}$$

Then E_v^- and E_v^+ are given by substituting $z = z_v$, $n = \delta_v$ into the right-hand side of (32) and (33), respectively. We summarize the discussion so far: the expression in square brackets in equation (31) equals the product over $v \in \mathcal{V}(\Gamma)$ of the average of the coefficients of $z^{-\ell_v}$ in (32), (33).

We now define a family of functions $\widehat{F} = \{\widehat{F}_n\}_{n\geq 0}$ which record the coefficients in the average of the two expansions. In Proposition 7.3 we show this family is admissible.

Definition 7.1. Consider the following family of functions $\{\widehat{F}_n : \mathbb{Z} \to \mathbb{Q}\}_{n\in\mathbb{Z}_+}$. For $0 \leq n \leq 2$, set $\widehat{F}_n(r)$ to be the coefficient of z^{-r} in $(z - z^{-1})^{2-n}$. For $n \geq 3$, $\widehat{F}_n(r)$ is defined to be the average of the coefficients of z^{-r} in equations (32) and (33).

Note that $\widehat{F}_n$ takes values in $\mathbb{Z}$ for $0 \leq n \leq 2$ and in $\frac{1}{2}\mathbb{Z}$ for $n \geq 3$. Thus $\widehat{F}_{\Gamma,k}$, defined in equation (12), takes values in $2^{-c}\mathbb{Z}$ where c is the number of vertices of degree at least 3. Although an explicit formula for $\widehat{F}_n$, $n \geq 3$, will not be used in this paper, for the reader's convenience we record it in equation (34).

$$\widehat{F}_n(r) = \begin{cases} \frac{1}{2}\operatorname{sgn}(r)^n \dbinom{\frac{n+|r|}{2} - 2}{n-3} & \text{if } |r| \geq n-2 \text{ and } r \equiv n \bmod 2 \\ 0 & \text{otherwise.} \end{cases} \tag{34}$$

Here $\operatorname{sgn}(r) \in \{-1, 1\}$ denotes the sign of r.

7.1. **Three admissible families.** In this section we introduce families $\widehat{F}^{+}$, $\widehat{F}^{-}$, closely related to $\widehat{F}$, and show that they are all admissible.

Definition 7.2. For $r \in \mathbb{Z}$ and $0 \leq n \leq 2$, set $\widehat{F}_n^{+}(r) = \widehat{F}_n^{-}(r) = \widehat{F}_n(r)$ to be the coefficient of z^{-r} in $(z - z^{-1})^{2-n}$. For $n \geq 3$, $\widehat{F}_n^{-}(r)$ and $\widehat{F}_n^{+}(r)$ are defined to be the coefficient of z^{-r} in (32) and (33), respectively.

The following general observation is used in the proof of the proposition below. If $F^1, F^2, \ldots, F^m$ are admissible families valued in a field of characteristic zero, then the family $\mathrm{av}(F^1, \ldots, F^m)$ given by the average

$$\big(\mathrm{av}(F^1, \ldots, F^m)\big)_n = \frac{1}{m}\sum_{i=1}^{m} F_n^1 + \cdots + F_n^m \tag{35}$$

is again admissible.

Proposition 7.3. *The families $\widehat{F}^{+}$, $\widehat{F}^{-}$, and $\widehat{F}$ are admissible.*

Proof. Property (A1) is straightforward to verify. To show property (A2) for $\widehat{F}^{+}$, note that

$$(z - z^{-1}) \sum_{i \geq 0} z^{-(2i+1)} = 1.$$

Therefore

$$z\left(\sum_{i\geq 0} z^{-(2i+1)}\right)^{n-2} - z^{-1}\left(\sum_{i\geq 0} z^{-(2i+1)}\right)^{n-2} = \left(\sum_{i\geq 0} z^{-(2i+1)}\right)^{n-3},$$

which demonstrates (A2). Alternatively, (A2) may also be seen from a binomial coefficient identity, using an explicit formula for $\widehat{F}^{\pm}$, analogous to (34). The calculation for $\widehat{F}^{-}$ is similar. Finally, note that $\widehat{F}$ is the average of $\widehat{F}^{+}$ and $\widehat{F}^{-}$ and is therefore admissible. $\square$

7.2. **The lattice and $\widehat{Z}$.** In this section we reformulate $\widehat{Z}$ as a sum of contributions of the associated function $\widehat{F}_{\Gamma,k}$ (see equation (12) and Remark 4.2) over lattice points, using the lattice cohomology identification of spin^c structures.

As a first step, we reparameterize definition (29) in the following way. Every $\ell \in a + 2M\,\mathbb{Z}^s$ can be written in the form $\ell = a + 2Mx$ for a unique $x \in \mathbb{Z}^s$. Then

$$\frac{\ell^t M^{-1} \ell}{4} = \frac{a^2}{4} + a \cdot x + \langle x, x\rangle = \frac{a^2}{4} - 2\chi_a(x),$$

using the notation of Remark 3.2 and (8). Compare with [GM21, Equation (46)]. Thus we get

$$\widehat{Z}_a(q) = q^{-\frac{a^2+3s+\sum_v m_v}{4}} v.p. \oint_{|z_v|=1} \prod_v \frac{dz_v}{2\pi i z_v}\left(z_v - \frac{1}{z_v}\right)^{2-\delta_v}\left(\sum_{x\in\mathbb{Z}^s} q^{2\chi_a(x)} \prod_v z_v^{(a+2Mx)_v}\right). \tag{36}$$

We now move on to the main goal of this section. Recall from Section 2.2 that graded roots and lattice cohomology use a different identification of spin^c structures than the $\widehat{Z}$ invariant. The translation between these two identifications is given in equation (4). Given

$k \in m + 2\,\mathbb{Z}^s$, let $a = k - Mu \in \delta + 2\,\mathbb{Z}^s$ denote the corresponding spin^c representative, and set

$$\widehat{Z}_k^{\#}(q) := \widehat{Z}_a(q).$$

Recall Notation 5.1 for Δ_k and $\varepsilon_k(x)$ in the following statement.

Proposition 7.4. *For $k \in m + 2\,\mathbb{Z}^s$, we have*

$$\widehat{Z}_k^{\#}(q) = \sum_{x \in \mathbb{Z}^s} \widehat{F}_{\Gamma,k}(x) q^{\varepsilon_k(x)}.$$

Proof. Note that $2\chi_a(x) = 2\chi_k(x) + \langle x, u\rangle$ for all $x \in \mathbb{Z}^s$, so equation (36) with $a = k - Mu$ can be written as

$$\widehat{Z}_a(q) = q^{\Delta_k} v.p. \oint\limits_{|z_v|=1} \prod_v \frac{dz_v}{2\pi i z_v} \left(z_v - \frac{1}{z_v}\right)^{2-\delta_v} \left(\sum_{x \in \mathbb{Z}^s} q^{2\chi_k(x) + \langle x,u\rangle} \prod_v z_v^{(2Mx+k-Mu)_v}\right).$$

From the above equation and the discussion preceding Section 7.1, we see that for every $j \in \mathbb{Z}$, the coefficient of $q^{\Delta_k + j}$ in $\widehat{Z}_k^{\#}$ is equal to

$$\sum_{\substack{x \in \mathbb{Z}^s \\ 2\chi_k(x) + \langle x,u\rangle = j}} \widehat{F}_{\Gamma,k}(x),$$

which verifies the desired equality. □

7.3. **Recovering the q-series.** In this section we show that, when the admissible family is $\widehat{F}$, the two-variable series specializes to $\widehat{Z}_k^{\#}(q)$ by setting $t = 1$. Calculations for S^3 and $\Sigma(2, 7, 15)$ are presented further below.

Definition 7.5. Fix a negative definite plumbing Γ and a spin^c structure $[k]$. Define

$$\widehat{\widehat{Z}}_{[k]}(q,t) := P^{\infty}_{\widehat{F},[k]} \tag{37}$$

which, as we recall from Definition 6.4, is an invariant of $(Y(\Gamma), [k])$ up to multiplication by a power of t.

Theorem 7.6. *With the above notation,*

$$\widehat{\widehat{Z}}_{[k]}(q,1) = \widehat{Z}_k^{\#}(q).$$

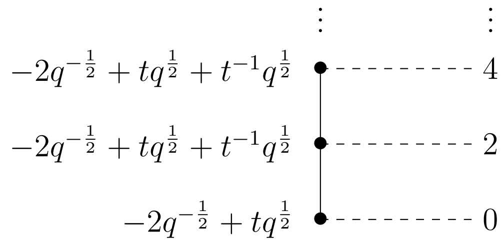
Proof. Fix $j \in \mathbb{Z}$. Using the notation in the proof of Theorem 6.3, the coefficient of $q^{\Delta_k + j}$ in $\widehat{\widehat{Z}}_{[k]}(q,1)$ is equal to

$$\sum_{x \in \partial \tilde{S}_j} \widehat{F}_{\Gamma,k}(x),$$

which by Proposition 7.4 equals the coefficient of $q^{\Delta_k + j}$ in $\widehat{Z}_k^{\#}(q)$. □

Example 7.7. Consider $Y = S^3$ represented as a plumbing Γ consisting of a single vertex v with weight $m_v = -1$. Let $k = -1 \in m + 2M\,\mathbb{Z}$ be a representative of the unique spin^c structure. We have

$$\chi_k(x) = \frac{1}{2}(x^2 + x), \quad \varepsilon_k(x) = -\frac{1}{2} + x^2, \quad \widehat{F}_{\Gamma,k}(x) = \widehat{F}_0(-2x).$$



$$\widehat{\widehat{Z}}_{[k]}(q,t) = -2q^{-\frac{1}{2}} + (t+t^{-1})q^{\frac{1}{2}}$$

FIGURE 6. The weighted graded root and two-variable series $\widehat{\widehat{Z}}_{[k]}(q,t)$ for S^3 corresponding to the admissible family $\widehat{F}$. Recall from grading conventions 3.6 that the numbers $0, 2, 4$ to the right of the graded root denote the Heegaard Floer gradings.

The formula in (14) implies $\widehat{F}_{\Gamma,k}(\pm 1) = 1$, $\widehat{F}_{\Gamma,k}(0) = -2$, and $\widehat{F}_{\Gamma,k}(x) = 0$ for all other values of x. Since $\chi_k^{-1}(1)$ contains the lattice points $-1, 0, 1$, we see that the weights stabilize at the sublevel set $\chi_k^{-1}((-\infty, 1])$. The weighted graded root and $\widehat{\widehat{Z}}_{[k]}$ are given in Figure 6.

In particular, specializing $\widehat{\widehat{Z}}_{[k]}(q,t)$ at $t = 1$ recovers the $\widehat{Z}$ invariant for S^3. The calculation above shows that the 2-variable series $\widehat{\widehat{Z}}$ introduced in this paper is different from the conjectured Poincaré series of the BPS homology [GPV17, Equation (6.80)], [GM21, Equation (18)].

Example 7.8. Consider the Brieskorn sphere $\Sigma(2,7,15)$, which can be represented as a negative definite plumbing, as shown in Figure 2. Since $\Sigma(2,7,15)$ is an integer homology sphere, we denote its unique spin^c structure by $\mathfrak{s}_0$.

One can compute, cf. [GM21, Section 4.6]:

$$\begin{aligned}
\widehat{Z}_{\mathfrak{s}_0}(q) &= q^{1739/840} \sum_{n=0}^{\infty} \left[q^{(61+420n)^2/840} + q^{(149+420n)^2/840} + q^{(299+420n)^2/840} + q^{(331+420n)^2/840} \right] \\
&\quad - q^{1739/840} \sum_{n=0}^{\infty} \left[q^{(89+420n)^2/840} + q^{(121+420n)^2/840} + q^{(271+420n)^2/840} + q^{(359+420n)^2/840} \right] \\
&= q^{13/2} - q^{23/2} - q^{39/2} + q^{57/2} - q^{179/2} + q^{217/2} + q^{265/2} - q^{311/2} + \cdots
\end{aligned}$$

The beginning of the weighted graded root (a result of a computer calculation) for $\Sigma(2,7,15)$ is shown in Figure 1; additional weights are given in the table below.

Weight	Grading
$\frac{1}{2}(t^{-2}+1)q^{\frac{13}{2}} - t^{-1}q^{\frac{23}{2}} - t^{-1}q^{\frac{39}{2}}$	20
$\frac{1}{2}(t^{-2}+1)q^{\frac{13}{2}} - t^{-1}q^{\frac{23}{2}} - t^{-1}q^{\frac{39}{2}} + \frac{1}{2}q^{\frac{57}{2}}$	28

In particular, setting $t = 1$ in the weight at grading 28, one can see the first few terms of $\widehat{Z}_{\mathfrak{s}_0}(q)$ as result of stabilization, which is a consequence of Theorems 6.3, 7.6.

8. Spin^c conjugation

In this section we study the behavior of $\widehat{\widehat{Z}}$ and weighted graded roots under spin^c conjugation. Under the identification (2), conjugation is given by the map $[k] \to [-k]$. Both $\widehat{Z}$ and lattice cohomology, in particular graded roots, are invariant under conjugation. However, when considering our new theory of weighted graded roots, a different, more refined, story emerges which we now describe.

Let F be an $\mathcal{R}$-valued admissible family. Consider the following property.

$$F_n(-r) = (-1)^n F_n(r) \text{ for all } n \geq 0 \text{ and } r \in \mathbb{Z}\,. \tag{A3}$$

Proposition 8.1. *If F is an admissible family which satisfies property* (A3)*, then $P^{\infty}_{F,[k]}(q,t) \doteq P^{\infty}_{F,[-k]}(q,t^{-1})$ for all $k \in m + 2\,\mathbb{Z}^s$.*

Proof. Note that $k' := -k + 2Mu$ is a representative for $[-k]$. We will show that

$$F_{\Gamma,k}(x) q^{\varepsilon_k(x)} t^{\langle x,u\rangle} = F_{\Gamma,k'}(-x) q^{\varepsilon_{k'}(-x)} (t^{-1})^{\langle -x,u\rangle} \tag{38}$$

for all $x \in \mathbb{Z}^s$, and the claim follows. First,

$$\Delta_{k'} = \Delta_k, \qquad 2\chi_k(x) = 2\chi_{k'}(-x) - 2\langle x,u\rangle,$$

so $\varepsilon_k(x) = \varepsilon_{k'}(-x)$ for all $x \in \mathbb{Z}^s$. Next, $2Mx + k - Mu = -\left(2M(-x) + k' - Mu\right)$, so

$$F_{\Gamma,k}(x) = (-1)^{\sum_v \delta_v} F_{\Gamma,k'}(-x) = F_{\Gamma,k'}(-x),$$

where the first equality follows from property (A3) and the second is due to the sum of degrees in any graph being even. Lastly, $t^{\langle x,u\rangle} = (t^{-1})^{\langle -x,u\rangle}$ is automatic. □

Now note that $\widehat{F}$, introduced in Definition 7.1, satisfies property (A3).[1]

Corollary 8.2. $\widehat{\widehat{Z}}_{[k]}(q,t) \doteq \widehat{\widehat{Z}}_{[-k]}(q,t^{-1})$*. In particular, setting $t = 1$ recovers the conjugation invariance of $\widehat{Z}$.*

We now turn to weighted graded roots and illustrate, via two examples, some interesting behavior under spin^c conjugation. First, we briefly recall how graded roots transform under conjugation and describe the corresponding story in Heegaard Floer homology.

Given a negative definite plumbing Γ and a spin^c structure $[k]$, the map $\mathbb{Z}^s \to \mathbb{Z}^s$, sending x to $-x$ induces an isomorphism $(R_k, \chi_k) \cong (R_{-k}, \chi_{-k})$ since $\chi_k(x) = \chi_{-k}(-x)$. Similarly, in Heegaard Floer homology, for any closed oriented 3-manifold Y and spin^c structure $\mathfrak{s}$, there is an isomorphism $HF^+(Y,\mathfrak{s}) \cong HF^+(Y,\bar{\mathfrak{s}})$, where $\bar{\mathfrak{s}}$ is the conjugate of $\mathfrak{s}$; see [OS04a, Theorem 2.4].

Moreover, for a self-conjugate spin^c structure we get an involution on the graded root and on Heegaard Floer homology. The involution on the graded root is induced by the map

$$\mathbb{Z}^s \to \mathbb{Z}^s, \quad x \mapsto -x - M^{-1}k.$$

Note here $M^{-1}k \in \mathbb{Z}^s$ since $[k] = [-k]$. For Heegaard Floer homology, the involution

$$\iota : HF^+(Y,\mathfrak{s}) \to HF^+(Y,\mathfrak{s})$$

comes from a chain map obtained by considering what happens when a pointed Heegaard diagram $(\Sigma, \alpha, \beta, z)$ representing Y is replaced with $(-\Sigma, \beta, \alpha, z)$. The involution ι is at the

[1]Although it will not be used, we note that $\widehat{F}^{\pm}$ from Definition 7.2 do not satisfy (A3).

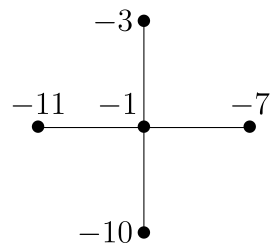
foundation of involutive Heegaard Floer homology, an extension of Heegaard Floer homology due to Hendricks-Manolescu [HM17].

For Γ an almost rational plumbing, Dai-Manolescu show that the two involutions described above are identified under the isomorphism given in Theorem 3.1 (see [DM19, Theorem 3.1]). Furthermore, they show that the graded root is symmetric about the infinite stem and the involution is the reflection about the infinite stem.

Example 8.3. Consider again the Brieskorn sphere $\Sigma(2,7,15)$. Note, the plumbing given in Figure 2 describing $\Sigma(2,7,15)$ is almost rational. Also, since $\Sigma(2,7,15)$ only has one spin^c structure, $\mathfrak{s}_0$, it is self-conjugate by default. Hence, the corresponding graded root is symmetric about the infinite stem and the involution is the reflection. However, as seen in Figure 1, the weighted graded root is no longer symmetric and the involution does not preserve all of the weights. There is a node at grading level 6 which has weight $\frac{1}{2}q^{13/2}$, whereas the node on the opposite side of the infinite stem has weight 0. The reason for this symmetry breaking is a result of the failure of $\widehat{F}_{\Gamma,k}(-x-M^{-1}k)q^{\varepsilon_k(-x-M^{-1}k)}t^{\langle -x-M^{-1}k,u\rangle}$ to equal $\widehat{F}_{\Gamma,k}(x)q^{\varepsilon_k(x)}t^{\langle x,u\rangle}$.

The following example shows that, unlike $\widehat{Z}$ and graded roots, the weighted graded root can distinguish conjugate spin^c structures. Moreover, it exhibits a new phenomenon different from that in Corollary 8.2.

Example 8.4. Let Γ be the plumbing pictured below:



Order the vertices so that v_1, v_2, v_3, v_4, v_5 correspond to the vertices with weights -1, -7, -10, -11, -3. Let $k = (-5, 5, 8, 9, 1)$. Consider the spin^c structure $[k]$ and its conjugate $[-k]$.

Initial segments of the weighted graded roots (a result of a computer calculation) corresponding to $[k]$ and $[-k]$ are pictured in Figure 7. As discussed in the beginning of this section, the $\widehat{Z}$ invariant and graded roots are invariant under spin^c conjugation. In this example the weighted graded roots not only distinguish $[k]$ and $[-k]$, they do this by more than just inversion of t in all the weights, compare with Corollary 8.2. (Note that the weights of graded roots are well defined up to an overall normalization by a power of t.) For example, the node at grading level $\frac{6722}{769}$ for $(R_{[k]}, \chi_{[k]}, P_{\widehat{F},[k]})$ is 0, while the corresponding node for $(R_{[-k]}, \chi_{[-k]}, P_{\widehat{F},[-k]})$ is $\frac{1}{2}q^{\frac{15009}{1538}}$.

Note that $\widehat{\widehat{Z}}(q,t)$ is the limit of the Laurent polynomial weights, whose coefficients stabilize in every bidegree according to Theorem 6.3. The weighted graded roots carry the unstable information as well; this explains the discrepancy between this example and Corollary 8.2. On a more detailed level, the reason for the discrepancy by more than just inversion of t is due to the failure of $\widehat{F}_{\Gamma,-k}(-x)q^{\varepsilon_{-k}(-x)}(t^{-1})^{\langle -x,u\rangle}$ to equal $\widehat{F}_{\Gamma,k}(x)q^{\varepsilon_k(x)}t^{\langle x,u\rangle}$. Equation (38) in the proof of Proposition 8.1, where $k' = -k + 2Mu$, was sufficient for showing

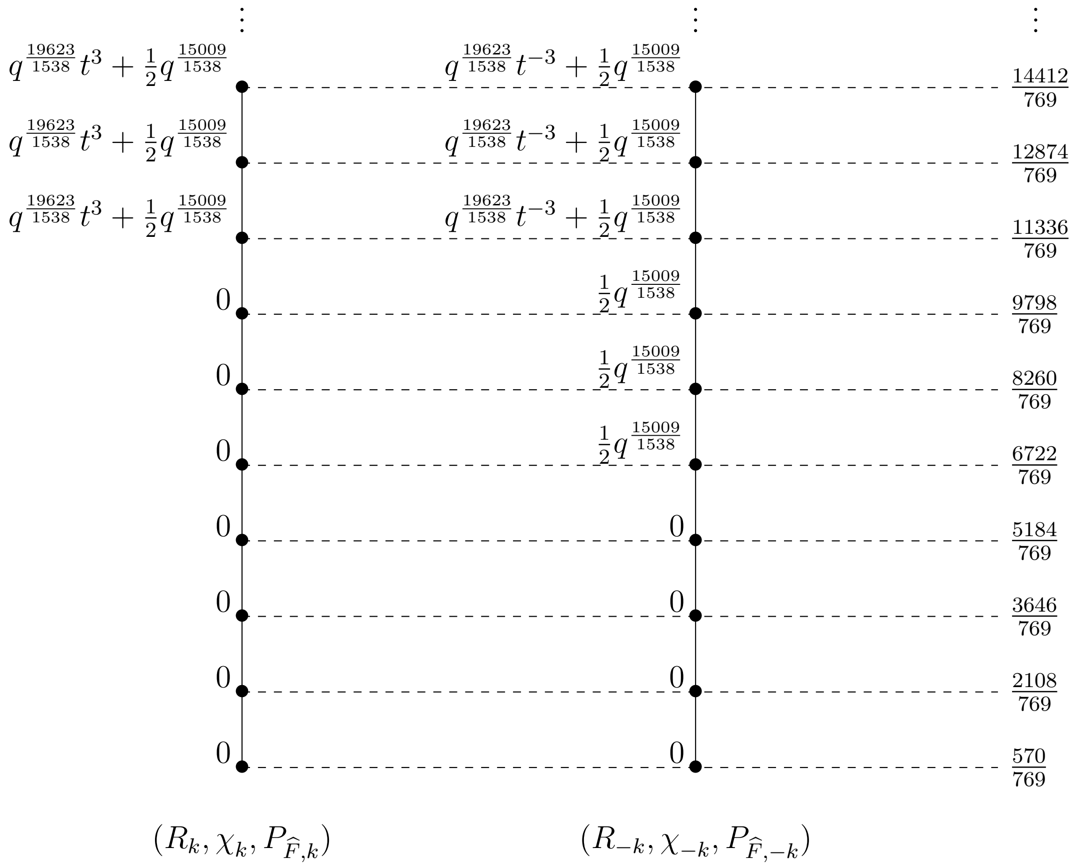


FIGURE 7. Note, $d(-Y(\Gamma), [(-5, 5, 8, 9, 1)]) = d(-Y(\Gamma), [(5, -5, -8, -9, -1)]) = \frac{570}{769}$. We use the normalization discussed in Remark 5.6.

$\hat{\hat{\hat{Z}}}_{[k]}(q,t) \doteq \hat{\hat{\hat{Z}}}_{[-k]}(q,t^{-1})$ because the sum is taken over all lattice points $x \in \mathbb{Z}^s$. However, the weights on the nodes of the graded root are sums over lattice points in some connected component of a sublevel set of χ_k for $(R_k, \chi_k, P_{\widehat{F},k})$, and of χ_{-k} for $(R_{-k}, \chi_{-k}, P_{\widehat{F},-k})$. But the map $x \mapsto -x$ takes the connected components of χ_k sublevel sets to connected components of χ_{-k} sublevel sets, not connected components of χ_{-k+2Mu} sublevel sets.

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Department of Mathematics, University of Virginia, Charlottesville VA 22904-4137
Email address: `ra5aq@virginia.edu, pkj4vj@virginia.edu, krushkal@virginia.edu`