FILLING LINKS AND SPINES IN 3-MANIFOLDS

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ABSTRACT. We introduce and study the notion of filling links in 3-manifolds: a link L is filling in M if for any 1-spine G of M which is disjoint from L, $\pi_1(G)$ injects into $\pi_1(M \smallsetminus L)$. A weaker "$k$-filling" version concerns injectivity modulo k-th term of the lower central series. For each $k \geq 2$ we construct a k-filling link in the 3-torus. The proof relies on an extension of the Stallings theorem which may be of independent interest. We discuss notions related to "filling" links in 3-manifolds, and formulate several open problems. The appendix by C. Leininger and A. Reid establishes the existence of a filling hyperbolic link in any closed orientable 3-manifold with $\pi_1(M)$ of rank 2.

1. INTRODUCTION

There is an old theme in 3-manifold topology that links or even knots can be built inside a general 3-manifold which in one sense or another are as robust as an embedded 1-complex can be. Here are several examples (in historical order):

1. Bing's theorem [Bi58] that a closed 3-manifold M is diffeomorphic to S^3 if and only if every knot K in M is contained ("engulfed") in a 3-ball.

2. "Disk busting curves": Theorem [My82]: for any compact 3-manifold M there is a knot K in M so that every essential sphere or disk meets K.

3. Suppose $X \longrightarrow B \longrightarrow V$ is a manifold (X)-bundle over a 3-manifold V. Then there is a link L in V so that B restricted to $V \smallsetminus L$ admits a flat topological connection, i.e. a foliation of class C^1 transverse to the fibers [Me18, Fr20].

In all three examples the theorem becomes trivial if instead of considering a knot or link, we replace those words in the statement with an embedded 1-complex. So the pattern seems to be that by knotting and linking one can make 1-submanifolds nearly as "filling" as a 1-complex: hard to engulf, hard to avoid, and hard to flatten over. Our first intent was to extend this theme to higher dimensions, but the natural questions have proved difficult and we just state a few of them for further thought:

Q1. If M is a smooth homotopy 4-sphere with the property that every smoothly embedded surface (or perhaps just every 2-sphere?) lies in a ball, is M invertible?

Q2. Is there an analog of (2) in high dimensions (being careful with respect to homotopy spheres)?

Q3. In higher dimensions can such bundles be flattened over the complement of a codimension 2 link, or even of a Cantor set?

In this paper we consider only 3-manifolds and attempt to add a 4th example of "filling" by a link or knot. To formulate the question, we start by discussing the relevant versions of the notion of a *spine* of a 3-manifold M:

(a) (The "rank" definition) A spine is a 1-complex in M of least first Betti number, surjecting onto $\pi_1(M)$. We always consider 1-complexes up to $I - H$ moves (also known as Whitehead moves); for this reason we also refer to them as handlebodies.

(b) ("local isotopy of handlebody") A spine is a handlebody in M up to isotopy with the property that it is onto on π_1 but no smaller handlebody obtained from compression of a non-separating disk is onto.

(c) ("local homotopy") The same definition as (ii) but replacing isotopy with homotopy.

The following question may be formulated for each of the definitions above.

Q4. Given a compact 3-manifold M, is there a link L in M so that whenever G is a spine in M and G is disjoint from L, then $\pi_1(G) \longrightarrow \pi_1(M \smallsetminus L)$ is injective?

If there is such a link L, we call it *filling* in M. As we prove below any 1-spine for M, if allowed to play the role of L would have this injectivity property. So as in the first three examples the question is whether a link can "do the work of", or "be as filling as" a 1-complex. This question and its relatives actually seem difficult to get a hold of and we are only able establish some partial results. We present them in the hope that others will find these "filling" questions of interest.

The definitions (b), (c) of a spine give the same answer as to whether a link (or more generally a 1-complex) is or is not filling. The reason is that the difference between homotopy and isotopy is finger moves. And these finger moves can miss the link or 1-complex L (that we are trying to decide if it is filling.) These finger moves in the complement of L do not change maps on π_1 so we get the same answers for both definitions (b) and (c) of spine, when it comes to deciding if L is filling.

We start with π_1-injectivity in the complement of a 1-*complex*, the case that is much easier than the complement of a *link*, similarly to examples 1 – 3 in the beginning of the introduction. In the following lemma, the 1-complex that is shown to be filling is a spine of a Heegaard handlebody:

Lemma 1. *Let $M = H \cup H^*$ be a Heegaard decomposition. Then for each definition (a, b, c) of a spine $G \subset M$, any embedding $i\colon G \longrightarrow M \smallsetminus H^*$ induces an injection $\pi_1 G \rightarrowtail \pi_1(M \smallsetminus H^*)$.*

Proof. Let π be the image of $\pi_1(G)$ in $\pi_1(M \smallsetminus H^*) \cong \pi_1(H)$. Being a subgroup of a free group, π is free. Also rank(π) is less than or equal rank$(\pi_1(G))$ since the map $\pi_1(G) \longrightarrow \pi$ is onto. If that map has a kernel, by the Hopfian property of free groups rank$(\pi) <$ rank$(\pi_1(G))$, contradicting minimal rank in the context of definition (a).

Now consider a spine satisfying definition (b) or (c). As in the previous paragraph, $\pi_1(G) \longrightarrow \pi$ is onto, implying rank$(\pi) <$ rank$(\pi_1(G))$. By [MKS66, Theorem 3.3], there is a basis B for $\pi_1(G)$ so that a subset of B generates ker$[\pi_1(G) \longrightarrow \pi]$.

Proposition 2. *Let G be a handlebody of genus $g > 0$ and assume that $b \in \pi_1(G)$ is a free basis element. Then G contains a properly embedded disk D so that*

$$\pi_1(G) = \langle b \rangle * \pi_1(G \smallsetminus D).$$

Proof of Proposition 2: Stated geometrically, the hypothesis says that there is a map $f \colon G \longrightarrow W$ to a wedge of g circles inducing an isomorphism on π_1 and taking b to the generator of the first S^1 factor S of W. Let $p \in S$ be a point on S different form the base-point, and $P \subset G$ be the transverse preimage, $f^{-1}(p)$, of p in G. By standard 3-manifold techniques, we may assume that P is an incompressible surface in G. (If there is an essential loop γ on P which is nullhomotopic in G, consider its null homotopy J. Choose an innermost essential circle α in J; α bounds a singular disk in $G \smallsetminus P$, which by Loop theorem/Dehn's lemma may be replaced with an embedded disk $E \subset G \smallsetminus P$ with essential boundary in P. Compressing along E reduces the complexity of P. This process must eventually make P incompressible.) Since f is an isomorphism on π_1 and P is the preimage of a point it follows that the map induced by inclusion: $\pi_1(P) \longrightarrow \pi_1(G)$ is the trivial map. So incompressibility implies that P is a disjoint union of disks. P is Poincaré dual to the homology class $[b]$ so it follows that at least one of these disks D is homologically essential, i.e. non-separating. This is the D claimed in the proposition. $\square$

By Proposition 2, a free generator in the kernel corresponds to a non separating reducing disk in G. Cutting along that disk gives a sub-handlebody still generating $\pi_1(M)$, contradicting minimality. This concludes the proof of Lemma 1. $\square$

Now we turn to the main problem, the existence of filling links. While the notion of "filling" is interesting for both definitions (a) and (b,c), in this paper we will focus on the slightly simpler rank definition (a). Logically it is harder to show that a link is filling with respect to (b, c), since this definition allows more "spines" which need to all inject. As we see below, the analysis in the easier case (a) is already quite subtle.

One may find a *knot K* giving π_1-injectivity for a *fixed* embedding of a spine as follows. Consider a minimal genus Heegaard decomposition $M^3 = H \cup H^*$, and let K be a diskbusting curve in the handlebody H^*. If $\pi_1 H \longrightarrow \pi_1(M \smallsetminus K)$ had kernel, by the loop theorem there would be a compressing disk in H^* disjoint from K, a contradiction.

The problem of finding a filling link L (so an *arbitrary* embedding of a spine is π_1-injective in $M \smallsetminus L$) has a trivial solution for genus one 3-manifolds. In this case, $M^3 = H \cup H^*$ where H, H^* are solid tori, and a filling knot is given by the core circle of H^*.

The appendix by C. Leininger and A. Reid establishes the existence of a filling hyperbolic link in any closed orientable 3-manifold with $\pi_1(M)$ of rank 2. In particular, it follows that a closed orientable 3-manifold with finite fundamental group contains a filling link, see Corollary A.3.

In the main body of this paper we focus on the case of the 3-torus T^3, although it seems likely that the approach (which involves equivariant homological analysis in the universal cover of M^3) should work for hyperbolic 3-manifolds as well.

The problem of analyzing π_1-injectivity of *any* embedding of a spine G in the complement of a given link in the 3-torus T^3 turned out to be quite subtle. The kernel of $\pi_1 G \longrightarrow \pi_1 T^3$ is the commutator subgroup of the free group $\pi_1 G$ on 3 generators. A standard classical tool for showing injectivity of maps of the free group is the Stallings theorem, see Section 2.1. However the Stallings theorem does not directly apply in our context. Specifically, it does not apply to the map $\pi_1 G \longrightarrow \pi_1(T^3 \smallsetminus L)$ because it is not surjective on second homology, and it is injective, rather than an isomorphism, on H_1. The complexity of the problem reflects the fact that the image of the map on π_1 depends on the embedding of the handlebody G. One may attempt to apply the Stallings theorem to the map $[\pi_1 G, \pi_1 G] \longrightarrow K$, where K is the kernel $\pi_1(T^3 \smallsetminus L) \longrightarrow \pi_1(T^3)$ for a suitable choice of L. But injectivity of the infinitely generated first homology of the commutator subgroup is hard to establish when the embedding $G \longrightarrow T^3 \smallsetminus L$ changes by an arbitrary homotopy, as we discuss below. To our knowledge, the existence of a filling link in T^3 is an open problem, but we are able to find links with a weaker property. We call a link $L \subset M$ "k-filling" if the injectivity in Q4 above holds modulo the kth term of the lower central series, $\pi_1 G/(\pi_1 G)_k \rightarrowtail \pi_1(M \smallsetminus L)/\pi_1(M \smallsetminus L)_k$.

Theorem 3. *For any $k \geq 2$ there exists a k-filling link in the T^3.*

To prove this theorem we give an extension of the Stallings theorem using powers of an augmentation ideal, which applies uniformly to all embeddings $G \longrightarrow T^3 \smallsetminus L$, where the conclusion holds modulo a given term of the lower central series. Powers of the augmentation ideal of a group ring have been classically studied in low-dimensional topology, cf. [CG83, St75], and are related to the lower central series of the group. We analyze a different aspect of the theory, focusing on the connection between the lower central series of a group π and powers of the augmentation ideal of the group ring of $H_1(\pi)$.

It is interesting to note the similarity of Theorem 3 with the current state of knowledge of the topological 4-dimensional surgery theorem for free non-abelian groups (see [FQ90]). The underlying technical statement, the disk embedding conjecture, also has a variable homotopy, and the question is whether the map on π_1 can be made *trivial*. One can solve the problem modulo any term of the lower central series, but the question itself is open. (See [FK20] for recent developments.)

The proof of Theorem 3 follows from Lemmas 5, 7, 9 given in the following sections. Section 2 and Lemma 5 set up the equivariant homological framework for analyzing the

effect of homotopies of a spine in terms of powers of the augmentation ideal. Lemma 7 and its analogue for T^3 in section 3 construct links satisfying the conditions of Lemma 5. Lemma 9 in section 4 gives an extension of the Stallings theorem, relating powers of the augmentation ideal to the lower central series, needed to complete the proof of the theorem.

2. THE RELATIVE CASE: $T^2 \times I$

2.1. **Notation and background.** The coefficients of homology groups are set to be $\mathbb{Z}$ throughout the paper. We start by recalling the Stallings theorem. Given a group A, its lower central series is defined inductively by $A_1 = A$, $A_k = [A_{k-1}, A]$; $A_\omega = \cap_{k=1}^\infty A_k$.

Theorem 4 (Stallings' theorem [St65]). *Let $f \colon A \longrightarrow B$ be a group homomorphism inducing an isomorphism on H_1 and an epimorphism on H_2. Then f induces an isomorphism $A/A_k \longrightarrow B/B_k$ for all finite k, and an injective map $A/A_\omega \longrightarrow B/B_\omega$.*

Dwyer [Dw75] extended the theorem, relaxing the surjectivity to be onto H_2 modulo the k-th term of the *Dwyer filtration*:

$$\phi_n(A) \; = \; \ker[H_2(A) \longrightarrow H_2(A/A_n)]$$

Assuming that $f \colon A \longrightarrow B$ is an isomorphism on H_1, the result of [Dw75] is that f induces an isomorphism $A/A_{k+1} \longrightarrow B/B_{k+1}$ if and only if it induces an epimorphism $H_2(A)/\phi_k(A) \longrightarrow H_2(B)/\phi_k(B)$.

2.2. **Relative spine and equivariant homology in the universal cover.** In this section we consider the relative case where the construction of k-filling links is easier to describe, $M = T^2 \times I$. Fix the standard "relative spine" $G = (\{*\} \times I) \cup (T^2 \times \partial I)$, and the dual spine $G^* = S^1 \vee S^1 \subset T^2 \times \{1/2\}$. Their preimages in the universal cover are illustrated in figure 1.
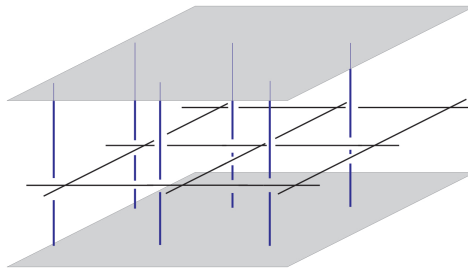

FIGURE 1. The preimage $\widetilde{G}$ in the universal cover $\mathbb{R}^2 \times I$ of the standard relative spine $G = (\{*\} \times I) \cup (T^2 \times \partial I)$ consists of the top and bottom shaded panels union the vertical line segments. The mid-level horizontal grid is the preimage of the dual spine $S^1 \vee S^1 \subset T^2 \times \{1/2\}$.

An analogue of lemma 1 shows that for any proper embedding of the line segment $\{*\} \times I$ into $((T^2 \times I) \smallsetminus G^*, T^2 \times \partial I)$ the fundamental group $\mathbb{Z}^2 * \mathbb{Z}^2$ of the resulting relative spine

injects into $\pi_1(T^2 \times I \smallsetminus G^*)$. This may be seen using the Stallings theorem. Note that $\pi_1 G \cong \mathbb{Z}^2 * \mathbb{Z}^2$ is a right-angled Artin group, and it is also a free product of surface groups; these classes of groups are residually nilpotent [DK92], [CO98].

The goal is to analyze the map on π_1 induced by inclusion when G^* is replaced by a link. This is illustrated in figure 2 where the link $L \subset T \times (0,1)$ is obtained by "resolving" the dual spine G^* into two disjoint essential circles. Different embeddings of the vertical interval are related by homotopies that may pass through link components and may be thought of as finger moves; the equivariant lift of an elementary finger move is illustrated on the right in figure 2.

We will continue denoting by G the standard embedding of the relative spine into $T^2 \times I$, and by $\widetilde{G}$ its preimage in the universal cover. The notation $G', \widetilde{G}'$ will be used when the embedding of the vertical arc is arbitrary, i.e. related to the standard embedding by finger moves. The fundamental group of $\widetilde{G}$, and also of $\widetilde{G}'$ is

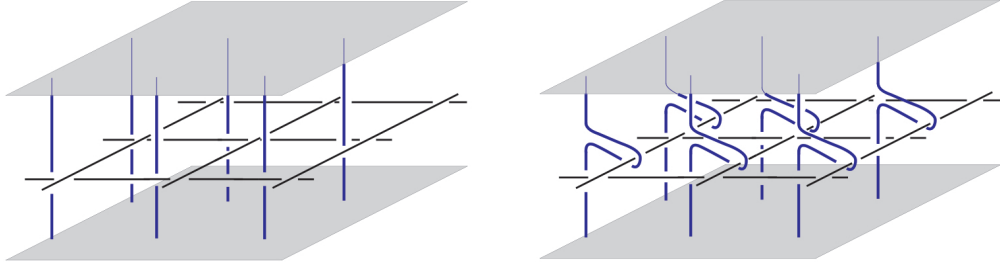$$K := \ker[\mathbb{Z}^2 * \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2].$$

FIGURE 2. An example of a link $L \subset T^2 \times (0,1)$, and a finger move.

Let L be any link in $T^2 \times (0,1)$ whose components are all essential in $\pi_1(T^2)$, and let $\widetilde{L}$ denote its preimage: a $\mathbb{Z}^2$-equivariant collection of lines in the universal cover. We restrict to the case where the components are all essential curves in $\pi_1(T^2)$ since in this case $\pi_1(\mathbb{R}^2 \times I \smallsetminus \widetilde{L})$ is free; links with nullhomotopic components would require an extension of the group theoretic analysis developed in this paper. The starting point is to analyze the injectivity of the map α in the commutative triangle

$$\pi_1(G) \xrightarrow{\ \alpha\ } \pi_1(T^2 \times I \smallsetminus L)$$
$$\beta \searrow \quad \Big\downarrow \gamma$$
$$\pi_1(T^2 \times I)$$

where all maps are induced by inclusions. Here α is induced by the *standard* embedding $G \subset T^2 \times I \smallsetminus L$; the general case of G' is discussed in section 2.3. It is clear that $\ker[\alpha]$

equals the kernel of the map $\ker[\beta] \longrightarrow \ker[\gamma]$. Here $\ker[\beta] = K$, and $\ker[\gamma] \cong \pi_1(\mathbb{R}^2 \times I \smallsetminus \widetilde{L})$. Therefore the focus is on the map

$$(1) \qquad\qquad K \longrightarrow \pi_1(\mathbb{R}^2 \times I \smallsetminus \widetilde{L}).$$

Denote by J the first homology of $\widetilde{G}$, $J = K/[K,K]$, and let H denote $H_1(\mathbb{R}^2 \times I \smallsetminus \widetilde{L})$. Since $\pi_1(\mathbb{R}^2 \times I \smallsetminus \widetilde{L})$ is a free group, if $J \longrightarrow H$ were injective, the Stallings theorem would imply that the map (1) is injective (for the standard spine G). We will give a construction of links L satisfying weaker injectivity, modulo powers of the augmentation ideal (see below), for any embedding $G' \subset T^2 \times I \smallsetminus L$.

Denote the map $J \longrightarrow H$ by Lk. The group H is generated by meridians $m(l)$, one for each line l in $\widetilde{L}$. The map Lk is given by $\mathbb{Z}^2$-equivariant linking, sending a 1-cycle c in $\widetilde{G}$ to a linear combination of meridians $\sum_i a_i m(l_i)$, where the coefficient $a_i \in \mathbb{Z}[\mathbb{Z}^2]$ is the linking "number" of c and l_i. Since there is a single generator $m(l)$ for each line l, when there is no risk of confusion we will write

$$Lk(c) = \sum_i a_i l_i.$$

An example calculating the linking map is given below.

As a module over $\mathbb{Z}[\mathbb{Z}^2]$, J is generated by the boundaries of two vertical "plaquettes", denoted P_x and P_y in figure 3a. We will think of elements of J as linear combinations of these plaquettes, with coefficients in $\mathbb{Z}[\mathbb{Z}^2]$. The translations in the directions perpendicular to P_x, P_y are denoted respectively by x, y. Note that the relation

$$(2) \qquad\qquad (1-x)P_x + (1-y)P_y = 0$$
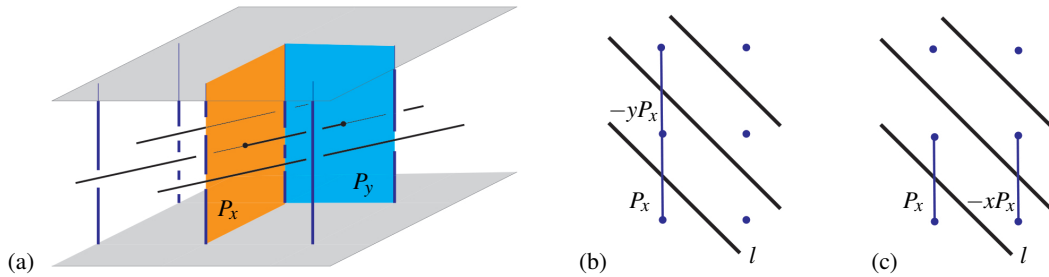
holds in J.

FIGURE 3.
(a): Plaquettes P_x, P_y generating J over $\mathbb{Z}[\mathbb{Z}^2]$.
(b), (c): Projection onto $\mathbb{R}^2$; dots represent the preimage of the edge $\{*\} \times I$ of the relative spine.

Figure 3 illustrates the case when the link L has a single component, the $(1,1)$-curve in the torus $T^2 \times \{1/2\}$. In this case the two translations act the same way on $\widetilde{L}$: for any line

l, $xl = yl$. Figures 3 (b, c) show the projection onto $\mathbb{R}^2$ of two elements of J: $(1-y)P_x$, $(1-x)P_x$. Denoting the line intersecting the plaquette P_x by l_0, we have

$$Lk((1-y)P_x) = (1-y)\,l_0, \quad Lk((1-x)P_x) = (1-x)\,l_0.$$

Since $(1-y)\,l_0 = (1-x)\,l_0$ in this example, the map $J \longrightarrow H$ is certainly not injective. Since the map Lk turned out to have kernel in this example, it is no surprise to observe that the boundary curve of $P_x \cup P_y$ in Figure 3 (a) is null homotopic in $\mathbb{R}^2 \times I \smallsetminus \widetilde{L}$.

2.3. **Finger moves and the kernel of the linking map.** Next we extend this discussion to the case of an arbitrary embedding $G' \subset T^2 \times I \smallsetminus L$. Let j denote the inclusion of 1-cycles into cellular 1-chains of the preimage of the standard spine $\widetilde{G}$, $j\colon J \hookrightarrow C_1$. Here we consider $\widetilde{G}$ as a 1-complex where $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$ are contracted to points, and C_1 is generated by a single vertical line segment and as a module over $\mathbb{Z}[\mathbb{Z}^2]$. The linking map $Lk'\colon J' \longrightarrow H$ for any G' is given by $Lk + F \circ j$, where $F\colon C_1 \longrightarrow H$ is the "finger move" map which measures the difference in H between the standard embedding $\{*\} \times I$ and its homotopic image in G'. It follows that a 1-cycle c is in the kernel of Lk' if and only if

(3) $$Lk(c) = -F(j(c))$$

It is convenient to represent this using diagram (4) of $\mathbb{Z}^2$-equivariant maps. (Note that this diagram does *not* commute; the two maps are equal precisely on the kernel of Lk'.)

(4)

$$
\begin{array}{ccc}
c \in J & \xrightarrow{\;Lk\;} & H \\
{\scriptstyle j}\big\downarrow & \nearrow{\scriptstyle -F} & \\
C_1 & &
\end{array}
$$

2.4. **Powers of the augmentation ideal.** Consider elements of $\mathbb{Z}[\mathbb{Z}^2]$ as Laurent polynomials in two commuting variables x, y. Let I denote the augmentation ideal of $\mathbb{Z}[\mathbb{Z}^2]$. The following lemma provides a convenient tool for analyzing the injectivity of the linking map for an arbitrary spine, modulo powers of the augmentation ideal.

Lemma 5. *If*

$$i_k\colon I^k J/I^{k+1}J \longrightarrow I^k H/I^{k+1}H$$

is injective for some k, then for any relative spine $G' \subset T^3 \smallsetminus L$,

$$i'_k\colon I^k J'/I^{k+1}J' \longrightarrow I^k H/I^{k+1}H$$

is injective. Here i_k, i'_k are the map induced by the inclusions of G, G' into $T^3 \smallsetminus L$.

Proof. The maps in (4) are equivariant over $\mathbb{Z}[\mathbb{Z}^2]$; consider

(5)
$$
c \in I^k J / I^{k+1} J \xrightarrow{\;Lk_k\;} I^k H / I^{k+1} H
$$
$$
j_k \downarrow \qquad \nearrow {\scriptstyle -F_k}
$$
$$
I^k C_1 / I^{k+1} C_1
$$

The module J is generated by the plaquettes P_x, P_y, and their images under j in C_1 are
$$
j(P_x) = (1-y)Z, \; j(P_y) = (1-x)Z,
$$

where Z denotes a generator (vertical line segment) of the module C_1. Suppose $c \in I^k J$. Then $c = aP_x + bP_y$ for some $a, b \in I^k$, and
$$
j_k(c) = (a(1-y) + b(1-x))Z \in I^{k+1} C_1.
$$

It follows that $j_k(I^k J) \subset I^{k+1} C_1$, and so
$$
\ker[Lk_k + F_k \circ j_k] = \ker[Lk_k]
$$

$\square$

Corollary 6. *If*
$$
i_j \colon I^j J / I^{j+1} J \longrightarrow I^j H / I^{j+1} H
$$
is injective for all $1 \leq j \leq k$, then for any spine $G' \subset T^3 \smallsetminus L$,
$$
J' / I^{k+1} J' \longrightarrow H / I^{k+1} H
$$
is injective. So finger moves do nothing to this algebra; in other words they have no effect on injectivity modulo powers of the augmentation ideal.

2.5. Construction of the links.

In this section we construct links in $T^2 \times I$ satisfying the conditions in Corollary 6.

Lemma 7. *For any k there exists a link $L_k \subset T^2 \times I$ such that $i_j \colon I^j J / I^{j+1} J \longrightarrow I^j H / I^{j+1} H$ is injective for all $1 \leq j \leq k$.*

Proof. To illustrate the idea of the proof, consider small values of k. For $k = 0$, let L_0 be the 2-component link consisting of a symplectic basis $(1,0), (0,1)$ curves on the torus, shifted to disjoint levels in $T^2 \times I$. Two lines in their preimage in $\mathbb{R}^2 \times I$ are denoted l_x, l_y, figure 4. In this case J/IJ is 2-dimensional, generated by the plaquettes P_x, P_y, figure 3. The linking map $i_0 \colon J/IJ \longrightarrow H/IH$ is represented by the identity 2×2 matrix over the integers. (Here the linking numbers are elements of $\mathbb{Z}[\mathbb{Z}^2]/I \cong \mathbb{Z}$.)

For $k = 1$, L_1 is defined to be L_0 union the $(1, -1)$ curve on the torus, shifted to a different level in $T^2 \times I$; a line in its preimage is denoted l_{xy}. (In our convention the slopes are negative, as shown in Figures 3, 4.) In this case $IJ/I^2 J$ is 3-dimensional, spanned by $(1-x)P_x, (1-y)P_x, (1-x)P_y$ (recall the relation (2) $(1-x)P_x + (1-y)P_y = 0$).
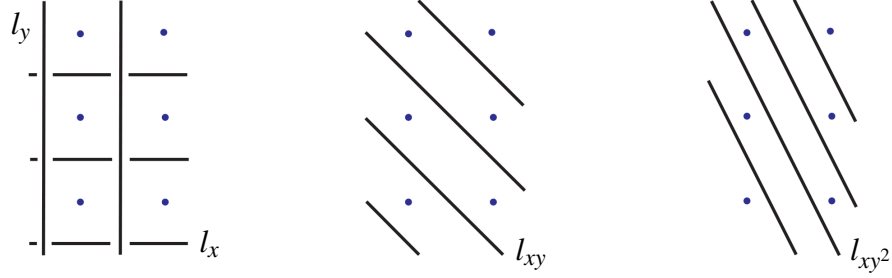
FIGURE 4. The preimage of the curves in $T^2 \times I$, used in the proof of lemma 7, in the universal cover: projection of $\mathbb{R}^2 \times I$ onto $\mathbb{R}^2$ is shown; as in figure 3 dots represent the preimage of the edge $\{*\} \times I$ of the relative spine.

IH/I^2H is generated by $(1-x), (1-y)$ times the lines l_x, l_y, l_{xy}. Note however that

$$(1-x)l_x = 0, \ (1-y)l_y = 0, \ (1-x)l_{xy} = (1-y)l_{xy}.$$

Therefore IH/I^2H is 3-dimensional. The 3×3 matrix over I/I^2 representing the linking map $i_1 \colon IJ/I^2J \longrightarrow H/I^2H$ is

	l_y	l_x	l_{xy}
$(1-x)P_y$	$1-x$	0	$1-x$
$(1-y)P_x$	0	$1-y$	$1-y$
$(1-x)P_x$	0	0	$1-x$

The map i_1 is seen to be injective. Note that even though the linking is measured over I/I^2, the entries in the first column are taken modulo the relation $1-x=0$, in the second column modulo $1-y=0$, and in the last column modulo $x=y$, because these relations hold for the lines defining the columns.

The final example we consider is $k=2$. Using relation (2), consider the generators

$$(1-x)^2 P_x, (1-x)(1-y)P_x, (1-y)^2 P_x, (1-x)^2 P_y$$

of I^2J/I^3J. Let L_2 denote the 4-component link, obtained from L_1 by adding the $(1,-2)$ curve on the torus; a line in its preimage is denoted l_{xy^2} in figure 4. Like in the previous case, linking with l_{xy} is defined modulo $x=y$; it is convenient to represent the entries in the third column of the linking matrix below in terms of y.

The actions of x, y on the new line are related by $xl_{xy^2} = y^2 l_{xy^2}$, so linking with this line is defined modulo $x = y^2$. Note the equality

(6) $$(1-y^2) = (1-y)(1+y) = 2(1-y) \text{ in } I/I^2$$

Using (6), for example the linking number of $(1-x)^2 P_x$ and l_{xy^2} in I^2/I^3 equals $(1-x)^2 = (1-y^2)^2 = 4(1-y)^2$.

	l_y	l_x	l_{xy}	l_{xy^2}
$(1-x)^2 P_y$	$(1-x)^2$	0	$(1-y)^2$	$4(1-y)^2$
$(1-y)^2 P_x$	0	$(1-y)^2$	$(1-y)^2$	$(1-y)^2$
$(1-x)(1-y)P_x$	0	0	$(1-y)^2$	$2(1-y)^2$
$(1-x)^2 P_x$	0	0	$(1-y)^2$	$4(1-y)^2$

The lower right 2×2 block of the matrix is $\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$ times $(1-y)^2$, a non-singular matrix. Thus i_2 is injective.

Continuing the inductive construction, the link $L_k \subset T^2 \times I$ is defined to be L_{k-1} union the $(1,-k)$ curve on the torus, shifted in the I coordinate to be disjoint from L_{k-1}. The group $I^k J / I^{k+1} J$ is generated by $k+2$ elements:

$$\{ (1-x)^a (1-y)^b P_x, \ a+b = k, \ a,b \geq 0 \} \ \text{and} \ (1-x)^k P_y.$$

Using the equality $1 - y^j = j(1-y)$ in I/I^2, the linking number in I^k/I^{k+1} of $(1-x)^a(1-y)^{k-a} P_x$, and l_{xy^j} equals

$$(7) \quad (1-x)^a(1-y)^{k-a} = (1-y^j)^a(1-y)^{k-a} = j^a(1-y)^a(1-y)^{k-a} = j^a(1-y)^k.$$

Therefore the linking matrix has an upper triangular block decomposition with the diagonal 2×2 block $\begin{pmatrix} (1-x)^k & 0 \\ 0 & (1-y)^k \end{pmatrix}$ and $(1-y)^k$ times the $k \times k$ Vandermonde matrix V whose (m,n)-th entry is $V_{m,n} = m^{n-1}$. Its determinant is non-zero, showing that $i_k \colon I^k J / I^{k+1} J \longrightarrow I^k H / I^{k+1} H$ is injective. This concludes the proof of lemma 7. $\qquad \square$

3. THE 3-TORUS

This section extends the construction from the relative case $T^2 \times I$ above to T^3. After setting up the notation and giving details of the construction, we will outline the analogues of lemmas 5, 7.

Let $T^3 = H \cup H^*$ be the genus 3 Heegaard decomposition, and let G, G^* denote the spines of the two handlebodies. Denote $F := \pi_1 G$, the free group on three generators. The fundamental group of the preimage $\widetilde{G}$ in $\mathbb{R}^3$ is the commutator subgroup F_2. Denote by J the first homology of the cubical lattice $\widetilde{G}$ (the "jungle gym"), $J = F_2/[F_2, F_2]$. As in the previous section, we think of generators of J as plaquettes. As a module over $\mathbb{Z}[\mathbb{Z}^3]$, J is generated by three plaquettes P_x, P_y, P_z, figure 5. Denoting the three covering translations by x, y, z, observe the relation

$$(8) \qquad\qquad (1-x)P_x + (1-y)P_y + (1-z)P_z = 0,$$
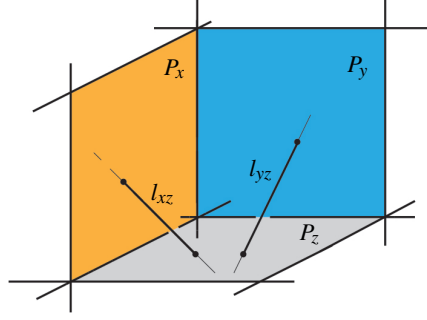
in J, analogous to (2).

FIGURE 5. Lines and plaquettes.

Consider a link $L \subset T^3 \smallsetminus G$ whose components are homologically essential in T^3. Its preimage, a $\mathbb{Z}^3$-equivariant collection of lines in $\mathbb{R}^3$, will be denoted $\widetilde{L} \subset \mathbb{R}^3$. Consider

$$K := \ker[\pi_1(T^3 \smallsetminus L) \longrightarrow \pi_1 T^3].$$

$K \cong \pi_1(\mathbb{R}^3 \smallsetminus \widetilde{L})$ is a free group. Denote by H the first homology of $\mathbb{R}^3 \smallsetminus \widetilde{L}$, $H = K/[K,K]$. Consider the map $Lk\colon J \longrightarrow H$ induced by the inclusion $G \hookrightarrow T^3 \smallsetminus L$. It is the $\mathbb{Z}[\mathbb{Z}^3]$-equivariant linking map, or in other words it is given by equivariant intersection between the plaquettes and the lines $\widetilde{L}$. Elements of $R := \mathbb{Z}[Z^3]$ are represented as Laurent polynomials in commuting variables x, y, z. Denote the augmentation ideal of R by I.

Let $i'\colon G' \longrightarrow T^3 \smallsetminus L$ be a spine homotopic to G. Such a homotopy may be thought of as finger moves of the three edges of G, intersecting the components of L during the homotopy. Lifting this to the universal cover, we have a $\mathbb{Z}[\mathbb{Z}^3]$-equivariant map $F\colon C_1(\widetilde{G}) \longrightarrow H$. Here the chain group C_1 of $\widetilde{G}$ is a free module over R of rank three. With this notation in place, the statement and the proof of lemma 5 and corollary 6 hold without any changes.

Next we adapt the construction of links L_k in lemma 7 to the setting of T^3. For $k = 0$ it suffices to consider a 3-component link L_0 obtained as a "resolution" of the 1-spine G^* of T^3. Its preimage in $\mathbb{R}^3$, the lines l_x, l_y, l_z, pair $\delta_{i,j}$ with the plaquette generators of J, and the map $J/IJ \longrightarrow H/IH$ is an isomorphism for L_0.

Consider generators of I^k/I^{k+1} of the form

(9) $$C_{a,b,c}^k := (1-x)^a(1-y)^b(1-z)^c, \ a+b+c = k, \ a,b,c \geq 0.$$

It follows from equation (8) that

(10) $$\{C_{a,b,c}^k P_x, \ C_{a,b,c}^k P_y, \ C_{a,b,0}^k P_z\}$$

is a basis for $I^k J/I^{k+1}J$. Consider the $(2k+3)$-component link L_k given by L_0 union $(1,0,-j)$ and $(0,1,-j)$ curves in T^3, $j = 1,\ldots,k\}$. Their preimages are denoted l_{xz^j}, l_{yz^j}. The claim is that as in lemma 7, $i_j\colon I^j J/I^{j+1}J \longrightarrow I^j H/I^{j+1}H$ is injective for all $1 \leq j \leq k$. The proof is an extension of the analysis in section 2.5, which we outline next.

The linking pairing over I^k/I^{k+1} between the basis elements B_0 of $I^k J/I^{k+1}J$ of the form $\{C^k_{0,b,c}P_x,\ C^k_{a,0,c}P_y,\ C^k_{a,b,0}P_z\}$ and the lines l_x, l_y, l_z is non-degenerate. (For example, the linking number of $C^k_{0,b,c}P_x$ with l_x equals $(1-y)^b(1-z)^c$ and with l_y, l_z it is zero.) Since the linking pairing of l_x, l_y, l_z is trivial with all other basis elements (denote them $B_{\neq 0}$) of $I^k J/I^{k+1}J$, it suffices to analyze the linking pairing between $B_{\neq 0}$ and the lines $\{l_{xz^j},\ l_{yz^j}\}$. It decomposes as the direct sum of the linking matrix for

(11) $$\{C^k_{a\neq 0,b,c}P_x,\ a+b+c=k,\ a,b,c\geq 0\}\ \text{ and }\ \{l_{xz^j}, j=1,\ldots,k\}$$

and the analogous pairing for $\{C^k_{a,b\neq 0,c}P_y\}$ and $\{l_{yz^j}\}$. Consider (11); the analysis for y in place of x is directly analogous.

The linking number (element of I^k/I^{k+1}) of $C^k_{a,b,c}P_x$ with l_{xz^j} equals $(1-x)^a(1-y)^b(1-z)^c = j^a(1-y)^b(1-z)^{a+c}$, see (7). For different values of b, the linking numbers correspond to different basis elements of I^k/I^{k+1}. For a given value of b, the linking matrix of $\{C^k_{a,b,c}P_x, a=1,\ldots,k-c-b\}$ and $\{l_{xz^j}, j=1,\ldots,k-c-b\}$ is the Vandermond matrix times $(1-y)^b(1-z)^{a+c}$, as in the proof of lemma 7, showing that i_k is injective as claimed.

Remark. If a link L is replaced with the dual spine G^* (as in the setting of lemma 1), the analysis above substantially simplifies. Recall the following result, an instance of a more general statement in terms of the torsion-free derived series from [CH08]:

Theorem 8. [CH08, Corollary 2.2] *Suppose F is a free group, B is a finitely-related group, $\phi\colon F \longrightarrow B$ induces a monomorphism on $H_1(-;\mathbb{Q})$, and $H_2(B;\mathbb{Q})$ is spanned by $B^{(n)}$-surfaces. Then ϕ induces a monomorphism $F/F^{(n+1)} \subset B/B^{(n+1)}$.*

Here for a group B, its derived series is defined by $B^{(0)} = B$, $B^{(n+1)} = [B^{(n)}, B^{(n)}]$. The notion of $B^{(n)}$-surfaces (maps of surfaces into $K(B,1)$ where the image on π_1 is in $B^{(n)}$ [CH08, Definition 1.5]) gives an analogue of the Dwyer filtration in the derived setting.

Any embedding $i\colon G \longrightarrow T^3 \smallsetminus G^*$, induces an isomorphism on H_1, because both ϕ, ψ in the diagram below are isomorphisms:

(12)
$$
\begin{array}{ccc}
H_1(G) & \xrightarrow{\ \phi\ } & H(T^3) \\
{\scriptstyle i_*}\downarrow & \nearrow{\scriptstyle \psi} & \\
H_1(T^3 \smallsetminus G^*) & &
\end{array}
$$

Theorem 8 applies to $F = \pi_1(G), B = \pi_1(T^3 \smallsetminus G^*)$; the assumption on H_2 is satisfied trivially since $H_2(T^3 \smallsetminus G^*) = 0$. It follows that the map $J \longrightarrow H$ (where H now denotes the first homology of the preimage of G^* in $\mathbb{R}^3$) is injective, since $J = F^{(1)}/F^{(2)}$, $H = B^{(1)}/B^{(2)}$.

This approach does not work for links in place of the 1-complex G^* because $H_2(T^3 \smallsetminus L)$ is non-zero, and for a given link L it is a non-trivial problem to determine what the second homology is of the image on π_1 of an arbitrary map $G \longrightarrow T^3 \smallsetminus L$.

4. Powers of the augmentation ideal and the lower central series

The result of this section, in conjunction with corollary 6 and lemma 7 and their analogues for T^3 discussed in section 3, completes the proof of theorem 3. Let $i' \colon G' \longrightarrow T^3 \smallsetminus L$ denote any spine homotopic to the standard spine G, where L is a link whose components are all essential in $\pi_1 T^3$, as in section 3. The analysis below also applies to the relative case $T^2 \times I$, considered in section 2.

We start by recalling the notation and summarizing basic consequences of the topological setup. F denotes $\pi_1 G'$, the free group on three generators, and

$$K := \ker\big[\pi_1(T^3 \smallsetminus L) \longrightarrow \pi_1 T^3\big]$$

is isomorphic to $\pi_1(\mathbb{R}^3 \smallsetminus \widetilde{L})$, a free group. J' denotes the first homology of the preimage $\widetilde{G}'$ of G' in $\mathbb{R}^3$ and H denotes $H_1(\mathbb{R}^3 \smallsetminus \widetilde{L})$,

$$J' \cong F_2/[F_2, F_2], \quad H \cong K/[K, K].$$

J' and H are considered as modules over $\mathbb{Z}[\mathbb{Z}^3]$, and I^k denotes the k-th power of the augmentation ideal I of $\mathbb{Z}[\mathbb{Z}^3]$. The following statement is the main result of this section, relating the filtrations of J', H in terms of powers of the augmentation ideal and the lower central series of F, $\pi_1(T^3 \smallsetminus L)$.

Lemma 9. *Suppose $J'/I^k J' \longrightarrow H/I^k H$ is injective for some k. Then the map*

$$F/F_{k+1} \longrightarrow \pi_1(M \smallsetminus L)/\pi_1(M \smallsetminus L)_{k+1}$$

is injective.

It is convenient to introduce the following notation, so that the proof involves isomorphisms rather than injections. Consider

$$(13) \qquad\qquad \pi := \text{image} \big[F \xrightarrow{\; i'_* \;} \pi_1(T^3 \smallsetminus L)\big].$$

Since the composition $F \longrightarrow \pi \longrightarrow \pi_1 T^3$ is an isomorphism on H_1, $H_1 F \longrightarrow H_1 \pi$ is an isomorphism. Note that for $k > 1$, π_k is contained in K, and so is a free group. Consider

$$\overline{H} := \text{image}[J' \xrightarrow{\; i'_* \;} H] \cong (\pi \cap K)/[\pi \cap K, \pi \cap K].$$

Observe that $\pi \cap K = \pi_2$. Indeed, as noted above $\pi_2 \subset K$. To prove the opposite inclusion, consider $g \in \pi \cap K$. Then $g = i'_*(f)$ for some $f \in F$, see diagram (14). Since $f \in \ker[F \longrightarrow \pi_1 T^3] = F_2$, it follows that $g \in \pi_2$.

$$(14) \qquad\qquad
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi \cap K & \longrightarrow & \pi & \longrightarrow & \pi_1 T^3 & \longrightarrow & 1 \\
& & & \nwarrow & & & & & \\
& & & & F & & & &
\end{array}
$$

with i'_* labeling the diagonal map $F \to \pi$.

Therefore $\overline{H}$ could also be defined as $\overline{H} = \pi_2/[\pi_2, \pi_2]$.

Proof of lemma 9. The proof is by induction on k; the inductive assumption is that $F/F_k \longrightarrow \pi/\pi_k$ is an isomorphism. The overall strategy is motivated by the proof of the Stallings theorem [St65]; in particular we use the isomorphism $H_2(F/F_k) \cong H_2(\pi/\pi_k)$ which is a consequence of the inductive assumption. It is worth noting again that both F_2 and π_2 are free groups but the map $F_2 \longrightarrow \pi_2$ is not an isomorphism on H_1. Being an isomorphism on H_1 is equivalent to $J'/I^k J' \cong \overline{H}/I^k \overline{H}$ for *all* k. Rather the lemma has a weaker assumption, $J'/I^k J' \cong \overline{H}/I^k \overline{H}$ for *some* fixed k.

Therefore the main technical ingredient in the proof, the diagram of exact sequences (19), is formulated with the goal of comparing the lower central series to powers of the augmentation ideal and it differs from the one used in the Stallings theorem in [St65].

We start by setting up the relevant short exact sequences. Let ϕ_k denote the inclusion $F_k \subset F_2$ composed with the quotient map $F_2 \longrightarrow F_2/[F_2, F_2]$, and consider its kernel:

$$(15) \qquad 1 \longrightarrow F_k \cap [F_2, F_2] \longrightarrow F_k \xrightarrow{\phi_k} F_2/[F_2, F_2]$$

Denote the generators of F by x, y, z; the same letters will denote the covering translations of $\mathbb{R}^3$. To relate the map ϕ_k to the geometric discussion of the "jungle gym" J in the previous section, consider a basic example, the triple commutator $[[x, y], z] \in F_3$. The map ϕ_k is implemented by first expanding $[[x, y], z] = [x, y] \cdot ([x, y]^{-1})^z$. The first factor is mapped to the boundary of the plaquette P_z, figure 5. The second factor is mapped to the boundary of this plaquette with the opposite orientation and shifted one unit up, $-zP_z$. So $\phi_3([[x, y], z]) = (1 - z)P_z$. An arbitrary element of F_3 expands as a product of conjugates of elements of the form $[g_1, g_2]^{g_3} \cdot ([g_1, g_2]^{-1})^{g_4}$, where each $g_i \in F$. The map ϕ_3 takes $[g_1, g_2]$ to a cycle c in $\widetilde{G}'$, and

$$(16) \qquad \phi_3([g_1, g_2]^{g_3} \cdot ([g_1, g_2]^{-1})^{g_4}) \ = \ (g_3 - g_4) \cdot c \in IJ'.$$

It follows that the map ϕ_3 surjects onto IJ'. The analogous statements hold for F_k: the map ϕ_k takes a basic commutator $[\ldots [[x_1, x_2], x_3], \ldots, x_k]$, where $x_1, \ldots, x_k \in \{x, y, z\}$, $x_1 \neq x_2$, to the plaquette determined by $[x_1, x_2]$, miltiplied by $(1 - x_3) \ldots (1 - x_k) \in I^{k-2}$. More generally, the image of ϕ_k is in $I^{k-2}J \subset J$, and moreover any element of $I^{k-2}J$ is in the image of ϕ_k. Consider the exact sequence (15) for $k, k+1$:

$$(17) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & F_k \cap [F_2, F_2] & \longrightarrow & F_k & \xrightarrow{\phi_k} & I^{k-2}J & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & F_{k+1} \cap [F_2, F_2] & \longrightarrow & F_{k+1} & \xrightarrow{\phi_{k+1}} & I^{k-1}J & \longrightarrow & 1 \end{array}$$

The quotients of the respective groups form the short exact sequence

$$(18) \qquad 1 \longrightarrow \frac{F_k \cap [F_2, F_2]}{F_{k+1} \cap [F_2, F_2]} \longrightarrow \frac{F_k}{F_{k+1}} \longrightarrow \frac{I^{k-2}J}{I^{k-1}J} \longrightarrow 1$$

Recall that for $k > 1$, π_k is contained in K, and is a free group. We would like to compare the short exact sequence (18) for F to the corresponding one for π. The sequence for π is obtained by starting with the map $\pi_k \longrightarrow \pi_2/[\pi_2, \pi_2]$ as in (15). The same analysis as above, for example equation (16) with J' replaced with $\overline{H}$, shows that π_k surjects onto $I^{k-2}\overline{H}$. The vertical maps in the following diagram, relating the short exact sequences (18) for F, π, are induced by the inclusion $G' \subset T^3 \smallsetminus L$.

$$(19) \qquad \begin{array}{ccccccccc}
1 & \longrightarrow & \dfrac{F_k \cap [F_2, F_2]}{F_{k+1} \cap [F_2, F_2]} & \longrightarrow & \dfrac{F_k}{F_{k+1}} & \longrightarrow & \dfrac{I^{k-2}J}{I^{k-1}J} & \longrightarrow & 1 \\
& & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
1 & \longrightarrow & \dfrac{\pi_k \cap [\pi_2, \pi_2]}{\pi_{k+1} \cap [\pi_2, \pi_2]} & \longrightarrow & \dfrac{\pi_k}{\pi_{k+1}} & \longrightarrow & \dfrac{I^{k-2}\overline{H}}{I^{k-1}\overline{H}} & \longrightarrow & 1
\end{array}$$

The inductive assumption is that $F/F_k \longrightarrow \pi/\pi_k$ is an isomorphism, and the goal is to show that the middle vertical map β is an isomorphism, to propagate the inductive step. A separate assumption in the statement of the lemma, based on the link L, is that the map γ is an isomorphism. The goal is to show that α is an isomorphism; then it will follow that β is an isomorphism too, concluding the inductive step.

Recall that $F \longrightarrow \pi$ is surjective and $F/F_k \longrightarrow \pi/\pi_k$ is assumed to be an isomorphism. It follows that $F_2/F_k \longrightarrow \pi_2/\pi_k$ is an isomorphism, and so is $H_2(F_2/F_k) \longrightarrow H_2(\pi_2/\pi_k)$. Consider Hopf's characterization of H_2 of a group Free/R:

$$(20) \qquad H_2(F/R) \cong \frac{R \cap [F, F]}{[F, R]}$$

Apply this to $H_2(F_2/F_k) \cong H_2(\pi_2/\pi_k)$ to get

$$(21) \qquad \frac{F_k \cap [F_2, F_2]}{[F_k, F_2]} \cong \frac{\pi_k \cap [\pi_2, \pi_2]}{[\pi_k, \pi_2]}$$

In the Stallings' proof of his theorem, the second homology groups are terms in the 5-term exact sequence [St65, Theorem 2.1]; in our context the relation is not as immediate: the map α in (19) and the isomorphism (21) are related by the short exact sequences

$$(22) \qquad \begin{array}{ccccccccc}
1 & \longrightarrow & \dfrac{F_{k+1} \cap [F_2, F_2]}{[F_k, F_2]} & \longrightarrow & \dfrac{F_k \cap [F_2, F_2]}{[F_k, F_2]} & \longrightarrow & \dfrac{F_k \cap [F_2, F_2]}{F_{k+1} \cap [F_2, F_2]} & \longrightarrow & 1 \\
& & \downarrow & & \downarrow \cong & & \downarrow \alpha & & \\
1 & \longrightarrow & \dfrac{\pi_{k+1} \cap [\pi_2, \pi_2]}{[\pi_k, \pi_2]} & \longrightarrow & \dfrac{\pi_k \cap [\pi_2, \pi_2]}{[\pi_k, \pi_2]} & \longrightarrow & \dfrac{\pi_k \cap [\pi_2, \pi_2]}{\pi_{k+1} \cap [\pi_2, \pi_2]} & \longrightarrow & 1
\end{array}$$

The middle vertical map is an isomorphism by (21). It follows that the left vertical map is injective. Since $F \longrightarrow \pi$ is surjective, the left map is in fact an isomorphism. It follows that α is an isomorphism. $\qquad\square$

4.1. **Discussion and questions.** For each $k > 1$, the combination of corollary 6, lemma 7 and lemma 9 gives a k-filling link $L_k \subset T^3$: for any 1-spine G' of T^3 which is disjoint from L_k,

$$F/F_k \longrightarrow \pi_1(T^3 \smallsetminus L_k)/\pi_1(T^3 \smallsetminus L_k)_k$$

is injective. (As above, F denotes $\pi_1 G'$.) Denoting by π the image of F as in (13), this means $F/F_k \cong \pi/\pi_k$. It is interesting to note that by Dwyer's theorem (see section 2.1) this implies that $H_2(\pi)$ is contained in the $(k-1)$st term of the Dwyer filtration $\phi_{k-1}(\pi)$. This is true for *any* map $G' \subset T^3 \smallsetminus L_k$, a fact that seems quite non-trivial to prove directly since $H_2(T^3 \smallsetminus L_k) \nsubseteq \phi_3(T^3 \smallsetminus L_k)$.

The cardinality of the links L_k grows linearly with k, and the problem of finding a genuine filling link L in T^3 remains open. In this paper we considered only links L with essential components in T^3, ensuring that the complement in the universal cover has free fundamental group. To use this approach in the general case, where some of the components of the link are contractible, would require a non-trivial extension of the group-theoretic analysis developed above.

To illustrate the last point, we present a specific example, a "dense chain mail link", for which the property of being filling is not known to us. Consider the unit cubical T^3, and take an ε-net of points in $T^3 \times S^2$, the Grassmanian of all tangent 2-planes to T^3, for some small ε. Now define the link L to be the link of circles of radius $1/10$ in T^3 centered at and oriented by each point of the net. By perturbing the net we may assume L is an embedded link. Now suppose G is a spine of T^3 embedded disjointly from L. Each component of L bounds a disk in T^3; of course the disks for nearby components intersect. Thin finger move homotopies can be used to push G off these disks, ensuring trivial linking number with L. (Generically these finger moves will acquire intersections with other disks, but the new intersections come in ± 1 pairs and thus do not contribute to linking numbers.) The linking number gives information about the induced map on homology but the question of injectivity modulo the third and higher terms of the lower central series is open.

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APPENDIX A. FILLING LINKS IN 3-MANIFOLDS OF RANK 2

by Christopher J. Leininger[1] and Alan W. Reid[2]

In this appendix we prove the following result which provides the first examples of closed orientable 3-manifolds with Heegaard genus > 1 that contain a filling link.

Theorem A.1. *Let M be a closed orientable 3-manifold such that $\pi_1(M)$ has rank 2. Then M contains a filling hyperbolic link.*

Closed orientable 3-manifolds of Heegaard genus 2 are examples of manifolds covered by Theorem A.1, so an immediate corollary is:

Corollary A.2. *Let M be a closed orientable 3-manifold such that M has Heegaard genus 2. Then M contains a filling hyperbolic link.*

Note that there are closed orientable 3-manifolds M for which $\pi_1(M)$ has rank 2, but the Heegaard genus is 3 (see [BZ84]). However, an interesting case of Corollary A.2 is the following.

Using Perelman's resolution of the Geometrization Conjecture [KL08]) it is known that all closed orientable 3-manifolds with non-trivial finite fundamental group have Heegaard genus 1 or 2. To see this, the resolution of the Geometrization Conjecture proves that a closed 3-manifold M with finite fundamental group is covered by S^3, in which case M is a Seifert fibered space. Assuming that M is not S^3, or a Lens Space, then M is a Seifert fibered space over S^2 with three exceptional fibers (see [Ja80, VI.11]), from which a genus 2 Heegaard splitting may be constructed directly (see [BZ84, Proposition 1.3] for example). The existence of filling links in manifolds of genus 1 was noted after the proof of Proposition 2, so Theorem A.1 now shows.

Corollary A.3. *Let M be a closed orientable 3-manifold such that $\pi_1(M)$ is finite and non-trivial. Then M contains a filling link.*

Remark A.4. An alternative proof of Corollary A.3 bypassing the use of Heegaard genus is the following. Using [Mi57, Section 3], one obtains a classification of finite groups that act freely on S^3. Perelman's resolution of the Geometrization Conjecture eliminates the one class of finite groups from [Mi57] that are not subgroups of SO(4). The five families of non-cyclic subgroups of SO(4) can all be seen to have rank 2.

Proof of Theorem A.1. We begin with a preliminary remark. The hypothesis that M is closed, orientable and $\pi_1(M)$ has rank 2 implies that $\pi_1(M)$ is non-Abelian. The reason is this: From [He76, Theorem 9.13] the only abelian groups occurring as the fundamental group of a closed orientable 3-manifold are $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, and $\mathbb{Z}^3$, and these are all excluded by the rank hypothesis.

Let $L \subset M$ be a hyperbolic link with at least 3 components. To find such a link, one may start with any link L' in M with at least 2 components, then appealing to [My82, Corollary 6.3] we may choose a knot K in $M \setminus L'$ with hyperbolic complement, and set $L = L' \cup K$.

Since $M \setminus L$ is hyperbolic, it follows from [JS79, Theorem VI.4.1] that the possibilities for the image H of $\pi_1(G)$ in $\pi_1(M \setminus L)$ are:

(1) H is free of rank 2, or
(2) H is free abelian of rank ≤ 2, or
(3) H has finite index in $\pi_1(M \setminus L)$.

If H is free abelian of rank 1 or 2, then the image in $\pi_1(M)$ via the homomorphism induced by inclusion $M \setminus L \to M$ is the quotient of a free abelian group of rank at most 2, and is in particular abelian. Since G is a spine, H surjects $\pi_1(M)$, which contradicts the fact that $\pi_1(M)$ is non-abelian. Therefore, case (2) is impossible. Case (3) can also be eliminated as follows. Recall that the first Betti number of a 3-manifold is at least half the first Betti number of its boundary, and thus $\mathbb{H}^3/H$ has at most two cusps. This is a contradiction since $M \setminus L$ has at least 3 cusps. We therefore conclude that the only possibility for H is free of rank 2 and the proof is complete. □

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