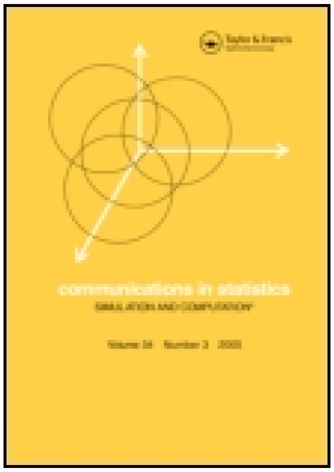

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Geometric View of Measurement Errors

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The slope of the best fit line from minimizing the sum of the squared oblique errors is the root of a polynomial of degree four. This geometric view of measurement errors is used to give insight into the performance of various slope estimators for the measurement error model including an adjusted fourth moment estimator introduced by Gillard and Iles (2005) to remove the jump discontinuity in the estimator of Copas (1972). The polynomial of degree four is associated with a minimun deviation estimator. A simulation study compares these estimators showing improvement in bias and mean squared error.

Keywords Maximum likelihood estimation; Measurement errors; Moment estimation; Oblique errors.

Mathematics Subject Classification 62J05; 62G05.

1. Introduction

With ordinary least squares OLS$(y\,|\,x)$ regression, we have data $\{(x_1, Y_1\,|\,X = x_1), \ldots, (x_n, Y_n\,|\,X = x_n)\}$ and we minimize the sum of the squared *vertical errors* to find the *best-fit* line $y = h(x) = \beta_0 + \beta_1 x$. With OLS$(y\,|\,x)$ it is assumed that the independent or causal variable is measured without error. The measurement error model has wide interest with many applications; for example, see Carroll et al. (2006) and Fuller (1987). The comparison of measurements by two analytical methods in clinical chemistry is often based on regression analysis. There is no causal or independent variable in this type of analysis. The most frequently used method to determine any systematic difference between two analytical methods is OLS$(y\,|\,x)$ which has several shortcomings when both measurement sets are subject to error. Linnet (1993) stated that "it is rare that one of the (measurement) methods is without error." Linnet (1999) further stated that "A systematic difference between two (measurement) methods is identified if the estimated intercept differs significantly from zero (constant difference) or if the slope deviates significantly

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from 1 (proportional difference)." Our article concentrates on how to determine whether or not there is a proportional difference between two measurement instruments using a Monte Carlo simulation. As in the regression procedure of Deming (1943), to account for both sets of errors, we determine a fit so that a function of both the squared vertical and the squared horizontal errors will be minimized. All of the estimated regression models we consider are contained in the parametrization (with $0 \leq \lambda \leq 1$) of the line from $(x, h(x))$ to $(h^{-1}(y), y)$.

We outline the Oblique Error Method in Sec. 2. In Sec. 3, we show how the geometric mean slope is a natural estimator for the slope in the measurement error (error-in-variables) model. Section 4 computes Madansky's moment estimators for varying slope estimators and shows a relationship between the maximum likelihood estimator in the measurement error model and our moment estimator. We give a case study to illustrate the effects that erroneous assumptions for the ratio of variance of errors can have on the maximum likelihood estimators. Section 5 discusses a fourth moment estimator and shows a circular relationship to the maximum likelihood estimator. Section 6 develops a minimum deviation estimator derived by minimizing Eq. (2) in Sec. 2 with respect to λ for fixed β_1. Section 7 contains our Monte Carlo simulations where we illustrate the effects that erroneous assumptions for the ratio of variance of errors can have on the maximum likelihood estimators and we compare the efficiencies of the above mentioned estimators. Supporting Maple worksheets are available from the link http://people.virginia.edu/~der/ODriscoll_Ramirez/.

2. Minimizing Squared Oblique Errors

From the data point (x_i, y_i) to the fitted line $y = h(x) = \beta_0 + \beta_1 x$, the vertical length is $a_i = |y_i - \beta_0 - \beta_1 x_i|$, the horizontal length is $b_i = |x_i - (y_i - \beta_0)/\beta_1| = |(\beta_1 x_i - y_i + \beta_0)/\beta_1| = |a_i/\beta_1|$, and the perpendicular length is $h_i = a_i/\sqrt{1+\beta_i^2}$. Using standard notation we set $S_{xx} = \sum_{i=1}^n (x_i - \overline{x})^2$, $S_{yy} = \sum_{i=1}^n (y_i - \overline{y})^2$, $S_{xy} = \sum_{i=1}^n (x_i - \overline{x})(y_i - \overline{y})$, correlation $\rho = S_{xy}/\sqrt{S_{xx}S_{yy}}$, $S_{xxxy} = \sum (x_i - \overline{x})^3 (y_i - \overline{y})$ and $S_{xyyy} = \sum (y_i - \overline{y})^3 (x_i - \overline{x})$.

For the oblique length from (x_i, y_i) to $(h^{-1}(y_i) + \lambda(x_i - h^{-1}(y_i)), y_i + \lambda(h(x_i) - y_i))$ (the horizontal error is $(1-\lambda)b_i = (1-\lambda)a_i/|\beta_1|$ and the vertical error is λa_i), the sum of squared horizontal, respectively, vertical, errors are given by $SSE_h(\beta_0, \beta_1, \lambda) = (1-\lambda)^2 (\sum_{i=1}^n a_i^2)/\beta_1^2$ and $SSE_v(\beta_0, \beta_1, \lambda) = \lambda^2 \sum_{i=1}^n a_i^2$. In a comprehensive article by Riggs et al. (1978), the authors placed great emphasis on the importance of equations being dimensionally correct and state that: "It is a poor method indeed whose results depend upon the particular units chosen for measuring the variables... and that invariance under linear transformations is equivalent to requiring the method to be dimensionally correct." So that our equation is dimensionally correct we consider

$$SSE_o(\beta_0, \beta_1, \lambda) = (1-\lambda)^2 \frac{SSE_h}{\tilde{\sigma}_\delta^2} + \lambda^2 \frac{SSE_v}{\tilde{\sigma}_\tau^2} \quad (1)$$

where $\{\tilde{\sigma}_\delta^2, \tilde{\sigma}_\tau^2\}$ are Madansky's moment estimators of the variance in the horizontal, respectively vertical, directions. In Sec. 3, we show that this is equivalent to using

$$SSE_o(\beta_0, \beta_1, \lambda) = (1-\lambda)^2 S_{yy} SSE_h + S_{xx} SSE_v. \quad (2)$$

Similar to that shown in O'Driscoll et al. (2008), the solution of $\partial SSE_o/\partial\beta_0 = 0$ is given by $\beta_0 = \bar{y} - \beta_1\bar{x}$ and the solutions of $\partial SSE_o/\partial\beta_1 = 0$ are the roots of the fourth degree polynomial equation in β_1, namely

$$P_4(\beta_1) = \lambda^2\sqrt{\frac{S_{xx}}{S_{yy}}}\frac{S_{xx}}{S_{yy}}\beta_1^4 - \lambda^2\frac{S_{xx}}{S_{yy}}\rho\beta_1^3 + (1-\lambda)^2\rho\beta_1 - (1-\lambda)^2\sqrt{\frac{S_{yy}}{S_{xx}}} = 0. \quad (3)$$

With $\lambda = 1$ we recover the minimum squared vertical errors with estimated slope β_1^{ver}, and with $\lambda = 0$ we recover the minimum squared horizontal errors with estimated slope β_1^{hor}.

For each fixed $\lambda \in [0, 1]$, there corresponds $\beta_1 \in [\beta_1^{ver}, \beta_1^{hor}]$ which satisfies Eq. (3), and conversely, for each fixed $\beta_1 \in [\beta_1^{ver}, \beta_1^{hor}]$, there corresponds $\lambda \in [0, 1]$ such that minimizing the sum of the squared oblique errors has estimated slope β_1. In particular, the geometric mean estimator $\beta_1^{gm} = \sqrt{S_{yy}/S_{xx}}$ has the oblique parameter $\lambda = 0.5$. We measure the angle θ_λ of the oblique projection associated with λ using the line segments (x, y) to $(x, h(x))$ and $(x, h(x))$ to $(h^{-1}(y), y)$. When the slope β_1 is close to one, for λ near one we anticipate θ_λ to be near 45° and for λ close to zero we anticipate θ_λ to be near 135°. The angles are computed from the Law of Cosines.

A similar argument to that of O'Driscoll et al. (2008) shows that $P_4(\beta_1)$ has exactly two real roots, one positive and one negative with the global minimum being the positive (respectively negative) root corresponding to the sign of S_{xy}. Riggs et al. (1978) in Eq. (119) also noted the role of the roots of a similar quartic equation in determining the slope estimators.

3. Measurement Error Model and Second Moment Estimation

We now consider the measurement error (errors-in-variables) model as follows. In this article, it is assumed that X and Y are random variables with respective finite variances σ_X^2 and σ_Y^2, finite fourth moments and have the linear functional relationship $Y = \beta_0 + \beta_1 X$. The observed data $\{(x_i, y_i), 1 \le i \le n\}$ are subject to error by $x_i = X_i + \delta_i$ and $y_i = Y_i + \tau_i$ where it is also assumed that δ is $N(0, \sigma_\delta^2)$ and τ is $N(0, \sigma_\tau^2)$. In our simulation studies we will use an exponential distribution for X.

It is well known, in a measurement error model, that the expected value for β_1^{ver} is attenuated to zero by the attenuating factor $\sigma_X^2/(\sigma_\delta^2 + \sigma_X^2)$, called the reliability ratio by Fuller (1987). Similarly, the expected value for β_1^{hor} is amplified to infinity by the amplifying factor $(\sigma_Y^2 + \sigma_\tau^2)/\sigma_Y^2$. Thus, for the measurement error model, when both the vertical and horizontal models are reasonable, a compromise estimator such as the geometric mean estimator β_1^{gm} is hoped to have improved efficiency.

Madansky's moment estimators for $\{\sigma_\delta^2, \sigma_\tau^2\}$ are

$$\begin{aligned} \tilde{\sigma}_\delta^2 &= \frac{S_{xx}}{n} - \frac{S_{xy}}{n\beta_1}, \\ \tilde{\sigma}_\tau^2 &= \frac{S_{yy}}{n} - \frac{\beta_1 S_{xy}}{n} \end{aligned} \quad (4)$$

from which it directly follows that β_1^{gm} is a fixed point of the ratio function $\beta_1 = \tilde{\sigma}_\tau(\beta_1)/\tilde{\sigma}_\delta(\beta_1)$.

However, Lindley and El-Sayyad (1968) proved that the expected value of β_1^{gm} is biased unless $\sigma_\tau^2/\sigma_Y^2 = \sigma_\delta^2/\sigma_X^2$.

We now return to the assertion made in Sec. 1. A natural standardized weighed average for the oblique model is shown in Eq. (1) and using the fixed point solution of the ratio function in this equation yields the equivalent model given in Eq. (2).

4. The Maximum Likelihood Estimator

If the ratio of the error variances $\kappa = \sigma_\tau^2/\sigma_\delta^2$ is assumed finite, then Madansky (1959), among others, showed that the maximum likelihood estimator for the slope is

$$\beta_1^{mle} = \frac{(S_{yy} - \kappa S_{xx}) + \sqrt{(S_{yy} - \kappa S_{xx})^2 + 4\kappa\rho^2 S_{xx}S_{yy}}}{2\rho\sqrt{S_{xx}S_{yy}}}. \tag{5}$$

It also follows that if $\kappa = 1$ in Eq. (5) then the MLE (often called the Deming Regression estimator) is equivalent to the perpendicular estimator, β_1^{per}, first introduced by Adcock (1878). In the particular case where $\kappa = S_{yy}/S_{xx}$, then β_1^{mle} has a λ value of 0.5. We note that S_{yy}/S_{xx} is a good estimator of $\sigma_y^2/\ \sigma_x^2$, but in general, it is not a good estimator of the error ratio $\kappa = \sigma_\tau^2/\ \sigma_\delta^2$. In Sec. 5, we discuss a moment estimator $\tilde{\kappa}$ for κ.

In Table 1, we record the corresponding obliqueness parameter λ for the maximum likelihood model for given typical values. Small values near 0 support OLS($x \mid y$), denoted by β_1^{hor}, and large values near 1 support OLS($y \mid x$), denoted by

Table 1
Values for λ for typical $\{\rho, \kappa, S_{xx}/S_{yy}\}$

$\kappa =$	0.500	0.500	0.500	0.500	1.000	1.000	1.000	1.000	2.000	2.000	2.000	2.000
$\rho =$	0.200	0.400	0.600	0.800	0.200	0.400	0.600	0.800	0.200	0.400	0.600	0.800
$S_{xx}/S_{yy} = 1/2$	0.033	0.111	0.197	0.273	0.089	0.223	0.316	0.375	0.500	0.500	0.500	0.500
$S_{xx}/S_{yy} = 1$	0.089	0.223	0.316	0.375	0.500	0.500	0.500	0.500	0.911	0.777	0.684	0.625
$S_{xx}/S_{yy} = 2$	0.500	0.500	0.500	0.500	0.911	0.776	0.684	0.625	0.967	0.889	0.803	0.727

Table 2
Error ratios for Madansky's moment estimators for varying β_1

	$\tilde{\sigma}_\delta^2$	$\tilde{\sigma}_\tau^2$	$\frac{\tilde{\sigma}_\tau^2}{\tilde{\sigma}_\delta^2}$
β_1^{ver}	0	$\frac{1-\rho^2}{n}S_{yy}$	∞
β_1^{hor}	$\frac{1-\rho^2}{n}S_{xx}$	0	0
β_1^{gm}	$\frac{1-\rho}{n}S_{xx}$	$\frac{1-\rho}{n}S_{yy}$	$\frac{S_{yy}}{S_{xx}}$
β_1^{per}	$\frac{1}{2}\frac{S_{xx}+S_{yy}-\sqrt{(S_{xx}-S_{yy})^2+4\rho^2 S_{xx}S_{yy}}}{n}$	$\frac{1}{2}\frac{S_{xx}+S_{yy}-\sqrt{(S_{xx}-S_{yy})^2+4\rho^2 S_{xx}S_{yy}}}{n}$	1
β_1^{mle}	$\frac{1}{2}\frac{S_{xx}+\frac{S_{yy}}{\kappa}-\sqrt{(S_{xx}-\frac{S_{yy}}{\kappa})^2+4\rho^2 S_{xx}\frac{S_{yy}}{\kappa}}}{n}$	$\frac{1}{2}\frac{\kappa S_{xx}+S_{yy}-\sqrt{(\kappa S_{xx}-S_{yy})^2+4\rho^2\kappa S_{xx}S_{yy}}}{n}$	κ

β_1^{ver}. For fixed $\{\kappa, \rho\}$, the values for the obliqueness parameter λ in each column of Table 1 increase indicating the model moves from β_1^{hor} towards β_1^{ver}. With $\kappa = S_{yy}/S_{xx}$, $\beta_1^{mle} = \beta_1^{gm}$ as shown by the cells of Table 1 with $\lambda = 0.500$.

The Madansky's moment estimators $\{\tilde{\sigma}_\delta^2, \tilde{\sigma}_\tau^2\}$ depend on the choice of β_1. In Table 2, we record the effect of varying slopes on the moments and their ratio when computable.

In the next section, we introduce a second moment estimator for κ and a fourth moment estimator for β_1.

5. Fourth Moment Estimation

When κ is unknown, Solari (1969) showed that the maximum likelihood estimator for the slope β_1 does not exist, as the maximum likelihood surface has a saddle point at the critical value. Earlier, Lindley and El-Sayyad (1968) suggested, in this case, that the maximum likelihood method fails as the estimator would be the geometric mean estimator which converges to the wrong value. Sprent (1970) pointed out the result of Solari does not imply that the maximum likelihood principle has failed, but rather that the likelihood surface has no maximum value at the critical value.

Copas (1972) offered some advice for using the maximum likelihood method. He assumed the data has rounding-off errors in the observations which allows for an approximated likelihood function to be used, and that this approximated likelihood function is bounded. His estimator for the slope has the rule

$$\beta_1^{cop} = \begin{cases} \beta_1^{ver} & \text{if } S_{yy} < S_{xx} \\ \beta_1^{hor} & \text{if } S_{yy} > S_{xx} \end{cases}, \tag{6}$$

so the ordinary least squares estimators are used depending on whether $|\beta_1^{gm}| < 1$ or $|\beta_1^{gm}| > 1$.

The Copas estimator is *not* continuous in the data as a small change in data can switch the direction of the inequality $S_{yy} < S_{xx}$ which will cause a jump discontinuity in the estimator β_1^{cop}. To achieve continuity in the data, we adjust the range of the fourth moment estimator β_1^{mom} described in Gillard and Iles (2005) to account for admissible values for $\{\sigma_\delta^2, \sigma_\tau^2\}$; see also Gillard and Iles (2009).

The basic second moment estimators for $\tilde{\sigma}_\delta^2$ and $\tilde{\sigma}_\tau^2$ are shown in Eq. (4). Since variances must be positive, we have the admissible range for the moment estimator for $\tilde{\beta}_1$ as

$$\beta_1^{ver} = \frac{S_{xy}}{S_{xx}} < \tilde{\beta}_1 < \frac{S_{yy}}{S_{xy}} = \beta_1^{hor}. \tag{7}$$

Following Gillard and Iles (2005), a fourth moment estimator, $\tilde{\beta}_1$, is given by

$$\tilde{\beta}_1 = \sqrt{\frac{S_{xyyy} - 3S_{xy}S_{yy}}{S_{xxxy} - 3S_{xy}S_{xx}}}. \tag{8}$$

For example, consider the (x, y) data set $\{(1, 1), (2, 3), (3, 2), (4, 4)\}$ with $\beta_1^{gm} = 1$. The estimator β_1^{cop} has a jump discontinuity at (x_4, y_4) since for $(x_4, y_4) = (4, 3.99)$, $\beta_1^{cop} = 0.7970$, and for $(x_4, y_4) = (4, 4.01)$, $\beta_1^{cop} = 1.2528$. The corresponding values for $\tilde{\beta}_1$ are {0.9971, 1.0029}, respectively, demonstrating the smoothing achieved using the adjusted fourth moment estimator.

As pointed out by the referee, fourth moment estimators will require larger sample sizes in comparison with lower order moment estimators and for this estimator to be feasible both the numerator and denominator of Eq. (8) must be significantly different from zero. In keeping with this recommendation for the underlying distribution we used in our simulation studies the exponential distribution, whose kurtosis is significantly different from zero, sample sizes of 100 and found that $\tilde{\beta}_1$ was well defined around 99% of the time. If X and Y are highly correlated, then $S_{xyyy}/S_{xy}S_{yy} - 3$ and $S_{xxxy}/S_{xy}S_{xx} - 3$ are crude estimators of the kurtosis. When the kurtosis is near zero, these estimators can have different signs and the radicand in Eq. (8) will be negative, in which case we recommend using the geometric mean estimator.

To satisfy Eq. (7) we define β_1^{mom} as

$$\beta_1^{mom} = \begin{cases} \beta_1^{ver} & \text{if } \tilde{\beta}_1 \leq \beta_1^{ver} \\ \tilde{\beta}_1 & \text{if } \beta_1^{ver} \leq \tilde{\beta}_1 \leq \beta_1^{hor} \\ \beta_1^{hor} & \text{if } \tilde{\beta}_1 \geq \beta_1^{hor} \end{cases}. \tag{9}$$

This is a Copas-type estimator with the moment estimator $\tilde{\beta}_1$ used to "smooth out" the jump discontinuity inherent in the Copas estimator. We next study the circular relationship between this adjusted fourth moment estimator and the maximum likelihood estimator with fixed κ.

We will define the moment estimator $\kappa(\beta_1)$ as a function of β_1, then use this value to compute $\beta_1^{mle}(\kappa)$ as a function of κ. Finally, we note that $\beta_1^{mle}(\kappa(\tilde{\beta}_1)) = \tilde{\beta}_1$, showing the circular relationship between the estimators $\{\tilde{\beta}_1, \beta_1^{mle}\}$. Thus, our moment estimator also has the functional form of the maximum likelihood estimator with fixed κ.

Set $\tilde{\kappa}(\tilde{\beta}_1) = \tilde{\sigma}_\tau^2/\tilde{\sigma}_\delta^2$ so

$$\tilde{\kappa}(\tilde{\beta}_1) = \frac{S_{yy} - \tilde{\beta}_1 \rho \sqrt{S_{xx}S_{yy}}}{S_{xx} - \rho/\tilde{\beta}_1 \sqrt{S_{xx}S_{yy}}}. \tag{10}$$

We use $\tilde{\kappa}(\tilde{\beta}_1)$ in Eq. (5) to determine $\beta_1^{mle}(\tilde{\kappa}(\tilde{\beta}_1))$. As $\tilde{\beta}_1 \to \beta_1^{hor}$ the numerator in Eq. (10) tends to zero so $\tilde{\kappa}(\tilde{\beta}_1) \to 0$ and $\beta_1^{mle}(\tilde{\kappa}(\tilde{\beta}_1)) \to \beta_1^{hor}$; similarly, as $\tilde{\beta}_1 \to \beta_1^{ver}$ the denominator in Eq. (10) tends to zero so $\tilde{\kappa}(\tilde{\beta}_1) \to \infty$ and $\beta_1^{mle}(\tilde{\kappa}(\tilde{\beta}_1)) \to \beta_1^{ver}$. A stronger result is given in the following Proposition.

Proposition 5.1. *For each* β_1, $\beta_1^{mle}(\tilde{\kappa}(\beta_1)) = \beta_1$ *and in particular* $\beta_1^{mle}(\tilde{\kappa}(\tilde{\beta}_1)) = \tilde{\beta}_1$.

Proof. In Eq. (5), solve $\beta_1^{mle}(\kappa) = \beta_1$ for $\kappa = \kappa_0$, and then check that κ_0 is the same as in Eq. (10).

An example helps to demonstrate the smoothing achieved with the moment estimator β_1^{mom}. Assume $\{\rho = 0.5, S_{xx} = 1, S_{xxxy} = 10, S_{xyyy} = 5\}$. Equation (7) requires that $0.13029 \leq S_{yy} \leq 1.31862$. As S_{yy} varies over the admissible values for S_{yy}, $\tilde{\kappa}(\tilde{\beta}_1)$ varies over $[0, \infty]$ and $\tilde{\beta}_1$ varies over $[\beta_1^{ver}, \beta_1^{hor}]$ and $\beta_1^{mle}(\tilde{\kappa}(\tilde{\beta}_1)) = \tilde{\beta}_1$, a surprising result as shown in Table 3.

Table 3
Slope estimates with $\{\rho = 0.5, S_{xx} = 1, S_{xxxy} = 10, S_{xyyy} = 5\}$

S_{yy}	β_1^{ver}	$\tilde{\beta}_1$	β_1^{hor}	$\tilde{\kappa}(\tilde{\beta}_1)$	β_1^{mle}
0.1303	0.1805	0.7219	0.7219	0.0000	0.7219
0.2000	0.2236	0.7222	0.8944	0.0558	0.7222
0.4000	0.3164	0.7145	1.2649	0.3123	0.7145
0.6000	0.3873	0.6977	1.5492	0.7412	0.6977
0.8000	0.4472	0.6734	1.7889	1.4850	0.6734
1.0000	0.5000	0.6417	2.0000	3.0760	0.6417
1.2000	0.5477	0.6020	2.1909	9.6582	0.6020
1.3186	0.5742	0.5742	2.2966	∞	0.5742

6. Minimum Deviation Estimation

From Sec. 1, with fixed β_1 the solution of $\partial SSE_o/\partial\lambda = 0$ is given by $\lambda = S_{yy}/(S_{yy} + \beta_1^2 S_{xx})$. Substituting β_1^{mom} for β_1 in this result for λ produces a Minimum Deviation type estimator which we denote by β_1^{md}, with $\beta_1^{ver} \leq \beta_1^{md} \leq \beta_1^{hor}$.

7. Monte Carlo Simulation

Riggs et al. (1978) stated that "no one method of estimating β_1 is the best method under all circumstances." To determine the efficiency of the above estimators we conduct a Monte Carlo simulation which uses X with an exponential distribution with mean $\mu_X = 10$ (and $\sigma_X = 10$) and $Y = X$ so $\beta_1 = 1$ and $\beta_0 = 0$. Both X and Y are subject to errors σ_δ^2, respectively, σ_τ^2 where $(\sigma_\delta^2, \sigma_\tau^2) \in \{1, 4, 9\} \times \{1, 4, 9\}$. The sample size n is chosen as 100.

The first simulation, with the number of replications $R = 100$, summarized in Table 4, reports on the bias in the MLE estimator in using a misspecified value of κ. For $(\sigma_\delta^2, \sigma_\tau^2) \in \{1, 4, 9\} \times \{1, 4, 9\}$, κ ranges with ratios from 1:9 to 9:1 The true error

Table 4
Percentage Bias of MLE estimator for the assumed ratios $\kappa^\#$ for varying values of $\kappa = \sigma_\tau^2/\sigma_\delta^2$

$\{\kappa^\#, \kappa\}$	1:9	1:4	4:9	1:1	4:4	9:9	9:4	4:1	9:1
				$\{\beta_1 = 1, \beta_0 = 0, n = 100, R = 100\}$					
1:9	0.166	0.502	2.164	0.870	3.663	7.995	8.723	3.592	9.282
1:4	−0.914	−0.012	0.811	0.666	2.807	6.087	7.351	3.067	8.265
4:9	−2.066	−0.564	−0.643	0.445	1.878	3.999	5.838	2.496	7.137
1:1	−4.067	−1.541	−3.184	0.051	0.218	0.266	3.083	1.467	5.058
4:4	−4.067	−1.541	−3.184	0.051	0.218	0.266	3.083	1.467	5.058
9:9	−4.067	−1.541	−3.184	0.051	0.218	0.266	3.083	1.467	5.058
9:4	−5.957	−2.495	−5.590	−0.342	−1.417	−3.330	0.338	0.437	2.936
4:1	−6.956	−3.016	−6.856	−0.561	−2.310	−5.230	−1.161	−0.136	1.748
9:1	−7.840	−3.489	−7.973	−0.763	−3.119	−6.899	−2.513	−0.663	0.657

ratios of κ are recorded in the first row and the assumed error ratios $\kappa^{\#}$, which are used to compute β_1^{mle}, are recorded in the first column, both in ascending order.

As expected, the values for $\kappa^{\#} = \kappa$ show the smallest bias, and in each column for a given κ the bias shows that the estimated slope moves from over estimating the true value to under estimating the true value of $\beta_1 = 1$. This was anticipated since for $\kappa^{\#}$ near zero the maximum likelihood estimator favors β_1^{hor} which over estimates $\beta_{1;}$ and correspondingly, for large $\kappa^{\#}$ the maximum likelihood estimator favors β_1^{ver} which under estimates β. If we assume that $\kappa^{\#} = 1$, we would expect that as the the true error ratio κ increases above 1 the bias would increase accordingly. However, this is not the case as can be seen from Row 4 of Table 4 indicating that the bias is not only dependent on the difference between the true error ratio κ and the assumed error ratio $\kappa^{\#}$ but also on the magnitude of each of the errors σ_δ^2 and σ_τ^2. In practice, the researcher may not have any knowledge of $\{\sigma_\delta^2, \sigma_\tau^2\}$ and may assign a value of 1 to $\kappa^{\#}$. If the true error ratio κ is in fact 1/16, then Table 7 (from our second simulation) shows that the (MSE, Bias) values for β_1^{per} are (7.406, −37.480) while those for β_1^{mom} are (5.717, −2.813) indicating a substantial improvement.

We conducted a second large scale Monte Carlo simulation study with $R = 1{,}000$ to demonstrate the improvement in the adjusted fourth moment estimator β_1^{mom} over the Copas estimator which has a jump discontinuity. Simulations for other slope estimators have been reported by Hussin (2004). We used an exponential distribution for X with $\mu_X = 10$, and set $\beta_1 = 1$ and $\beta_1 = 0$. The values for the error standard deviations were $(\sigma_\delta, \sigma_\tau) \in \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$, the sample size was $n = 100$ and the number of replications $R = 1{,}000$. We report in Tables 5–7 the MSE and the Bias for the estimators $\{\beta_1^{ver}, \beta_1^{hor}, \beta_1^{per}, \beta_1^{gm}, \beta_1^{mom}, \beta_1^{cop}, \beta_1^{md}\}$ for $(\sigma_\delta, \sigma_\tau) \in \{(1, 2), (1, 3), (1, 4)\}$. Similar results hold for $(\sigma_\delta, \sigma_\tau) \in \{(2, 1), (3, 1), (4, 1)\}$ Note that in each case the adjusted fourth moment estimator β_1^{mom} is more efficient than the Copas estimator. To see this we compare the pairs of values (MSE, Bias) in the three tables. For β_1^{mom} these are {(1.001, −0.830), (2.786, −1.807), (5.717, −2.813)} and for Copas these are {(2.378, −2.410), (8.769, −7.347), (23.018, −13.848)}. For the three reported simulations in Tables 5, 6, and 7, nearly all of the three replications with $R = 1{,}000$ had the adjusted fourth moment estimator β_1^{mom} well-defined with exceptions occurring with frequency {1.1%, 1.7%, 2.4%}, respectively. In these rare cases, we

Table 5
X is $Exp(10)$, $\beta_1 = 1$, $\beta_0 = 0$, $R = 1{,}000$, $n = 100$ $(\sigma_\tau = 1, \sigma_\delta = 2)$

	$MSE * 10^{-3}$	%Bias	λ	θ_λ
*OLS** reports average MSE and average absolute Bias for $\{\beta_1^{ver}, \beta_1^{hor}\}$				
β_1^{ver}	2.001	−3.843	1.000	46.12
*OLS**	1.336	2.518	NA	NA
β_1^{hor}	0.670	1.193	0.000	136.12
β_1^{per}	0.688	−1.396	0.507	89.99
β_1^{gm}	0.653	−1.360	0.500	90.78
β_1^{mom}	1.001	−0.830	0.339	108.27
β_1^{cop}	2.378	−2.410	0.651	74.47
β_1^{md}	0.646	−1.336	0.497	91.06

Table 6
X is $Exp(10)$, $\beta_1 = 1$, $\beta_0 = 0$, $R = 1{,}000$, $n = 100$ ($\sigma_\tau = 1, \sigma_\delta = 3$)

	$MSE * 10^{-3}$	$\%Bias$	λ	θ_λ
OLS^* reports average MSE and average absolute Bias for $\{\beta_1^{ver}, \beta_1^{hor}\}$				
β_1^{ver}	8.370	−8.459	1.000	47.53
OLS^*	4.847	4.831	NA	NA
β_1^{hor}	1.324	1.203	0.000	137.53
β_1^{per}	2.688	−3.954	0.520	89.60
β_1^{gm}	2.423	−3.760	0.500	92.19
β_1^{mom}	2.786	−1.807	0.318	110.94
β_1^{cop}	8.769	−7.347	0.848	58.14
β_1^{md}	2.309	−3.584	0.490	93.196

followed the rule of using β_1^{gm} for the slope estimator. Furthermore, in about half of the runs $\{49, 45, 48\}$, the fourth moment estimator $\tilde{\beta}_1$ satisfied the admissible conditions in Eq. (7) and we used the bounds for the adjusted fourth moment estimator β_1^{mom} to account for inadmissible values.

If the researcher does have information on the relative size of the errors, then he may choose either of $\{\beta_1^{ver}, \beta_1^{hor}\}$ with β_1^{hor} favored when σ_δ^2 is much bigger than σ_τ^2. Without prior knowlwdge of the errors ratio, a fairer comparison of each of the above estimators is to use OLS$(y\,|\,x)$ and OLS$(x\,|\,y)$ each 50% of the time. Thus, in the Tables we report the average for the MSE and the average of the absolute deviation of the biases for the two OLS estimators. These average (MSE, Bias) values from the tables are $\{(1.336, 2.518), (4.847, 4.831), (12.46, 7.858)\}$ showing the improved efficiency of β_1^{mom}. As anticipated, the minimum deviation estimator β_1^{ml} achieves further improvement in reduction of (MSE, Bias) with values $\{(0.646, -1.336), (2.309, -3.584), (5.578, -6.288)\}$.

Table 7
X is $Exp(10)$, $\beta_1 = 1$, $\beta_0 = 0$, $R = 1{,}000$, $n = 100$ ($\sigma_\tau = 1, \sigma_\delta = 4$)

	$MSE * 10^{-3}$	$\%Bias$	λ	θ_λ
OLS^* reports average MSE and average absolute Bias for $\{\beta_1^{ver}, \beta_1^{hor}\}$				
β_1^{ver}	22.791	−14.376	1.000	49.43
OLS^*	12.46	7.858	NA	NA
β_1^{hor}	2.134	1.339	0.000	139.43
β_1^{per}	7.406	−7.480	0.539	89.95
β_1^{gm}	6.242	−6.880	0.500	94.08
β_1^{mom}	5.717	−2.813	0.286	114.51
β_1^{cop}	23.018	−13.848	0.950	52.71
β_1^{md}	5.578	−6.288	0.480	96.04

8. Summary

We modified the fourth moment estimator of the slope from Gillard and Iles (2005) to show how to remove the jump discontinuity in the estimator given by Copas (1972). We show how the moment estimators $\{\beta_1^{mom}, \tilde{\sigma}_\delta^2, \tilde{\sigma}_\tau^2\}$ can be used to determine an MLE estimator which surprisingly is the original moment estimator of the slope. Our simulations support our claim that both $\{\beta_1^{mom}, \beta_1^{md}\}$ are more efficient than the average of the OLS estimators.

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