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A characterization of quotient algebras of $L^1(G)$

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Let G be a locally compact Abelian group; Γ the dual group of G; $C_0(\Gamma)$ the algebra of continuous functions on Γ which vanish at infinity; $C^B(\Gamma)$ the continuous, bounded functions on Γ; $M(G)$ the algebra of bounded Borel measures on G; $L^1(G)$ the algebra of absolutely continuous measures; and $M(G)^\wedge$ the algebra of Fourier–Stieltjes transforms.

Let E denote a compact subset of Γ, and let $A(E)$ be the set of all functions on E which are restrictions to E of functions belonging to $L^1(G)^\wedge$. The norm in $A(E)$ is the quotient norm; i.e.

$$\|f\|_{A(E)} = \inf\{\|g\|_{L^1(G)}: g \in L^1(G) \quad \text{and} \quad \hat{g}|E = f\}.$$

For $f \in C(E)$, let

$$\|f\|_{B(E)} = \sup\left\{\left|\int_E f\,d\lambda\right| : \lambda \in M(E) \quad \text{and} \quad \|\lambda^\wedge\|_\infty \leqslant 1\right\}.$$

Let $B(E)$ consist of those functions $f \in C(E)$ for which $\|f\|_{B(E)}$ is finite. Now

$$A(E) \subset B(E) \subset C(E) \quad \text{and} \quad \|f\|_{C(E)} \leqslant \|f\|_{B(E)} \leqslant \|f\|_{A(E)}, \quad \text{for} \quad f \in A(E).$$

THEOREM 1. *$B(E)$ is a commutative, semi-simple, self-adjoint Banach algebra with unit under point-wise operations.*

Proof. Katznelson and McGehee have noted that $B(E)$ is a Banach space (3). Let $\{f_n\}$ be a Cauchy sequence in $B(E)$ and f its limit point in $C(E)$. Let $0 \neq \lambda \in M(E)$, then

$$\left|\int_E (f-f_n)\,d\lambda\right| \leqslant \left|\int_E (f-f_m)\,d\lambda\right| + \|f_n - f_m\|_{B(E)}\|\lambda^\wedge\|_\infty.$$

Let $\epsilon > 0$, and N be such that if $n, m \geqslant N$, then $\|f_n - f_m\|_{B(E)} < \frac{1}{2}\epsilon$. Pick $m \geqslant N$ such that $\|f - f_m\|_{C(E)} < \epsilon/(2\|\lambda\|)$. Thus $\|f - f_n\|_{B(E)} < \epsilon$ for all $n \geqslant N$. Hence f is the limit point of $\{f_n\}$ in $B(E)$.

$B(E)$ is an algebra since for $\lambda \neq 0$,

$$\begin{aligned}\frac{\left|\int_E fg\,d\lambda\right|}{\|\lambda^\wedge\|_\infty} &= \frac{\left|\int_E fg\,d\lambda\right|}{\|(g\,d\lambda)^\wedge\|_\infty}\,\frac{\|(g\,d\lambda)^\wedge\|_\infty}{\|\lambda^\wedge\|_\infty} \leqslant \|f\|_{B(E)}\frac{\|(g\,d\lambda)^\wedge\|_\infty}{\|\lambda^\wedge\|_\infty}\\ &\leqslant \|f\|_{B(E)}\frac{\sup_{\gamma\in G}\left|\int \gamma g\,d\lambda\right|}{\|\lambda^\wedge\|_\infty}\\ &\leqslant \|f\|_{B(E)}\|g\|_{B(E)}\frac{\sup_{\gamma\in G}\|(\gamma\,d\lambda)^\wedge\|_\infty}{\|\lambda^\wedge\|_\infty}\\ &= \|f\|_{B(E)}\|g\|_{B(E)}.\end{aligned}$$

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Thus for $f, g \in B(E)$, $\|fg\|_{B(E)} \leqslant \|f\|_{B(E)}\|g\|_{B(E)}$. Similarly, $B(E)$ is self-adjoint since for $\lambda \neq 0$,

$$\frac{\left|\int \bar{f} d\lambda\right|}{\|\lambda^\wedge\|_\infty} = \frac{\overline{\left|\int f d\bar{\lambda}\right|}}{\|\lambda^\wedge\|_\infty} = \frac{\left|\int f d\bar{\lambda}\right|}{\|(\bar{\lambda})^\wedge\|_\infty}. \quad \blacksquare$$

It is natural to ask when is $A(E) = B(E)$ and when is $B(E) = C(E)$. If $B(E) = C(E)$, then the closed graph theorem implies that E is a Helson set; i.e. $A(E) = C(E)$. Thus $B(E) = C(E)$ if and only if E is a Helson set. (This was pointed out to the author by C. Dunkl.) If E is a compact subset of Γ which is of uniform positive measure, then $A(E) = B(E)$ (see (7) for a stronger result). Katznelson and McGehee(3) have given examples of compact sets such that $A(E) \neq B(E)$ (see also (8)). In all examples known to the author, $A(E)$ is, however, a *closed* subspace of $B(E)$ from which it follows that the $A(E)$ and $B(E)$ norms are equivalent on $A(E)$. For such E we characterize $A(E)$ in $C(E)$. Our proof yields a characterization of the $B(E)$-closure of $A(E)$ in $C(E)$.

We pair $\langle A(E), M(E)\rangle$ by $\langle f, \mu\rangle = \int_E f d\mu$. If $\langle f, \mu\rangle = 0$, for all $\mu \in M(E)$, in particular for point measures, then $f = 0$. If $\langle f, \mu\rangle = 0$ for all $f \in A(E)$, then $\mu = 0$ since $A(E)$ is sup-norm dense in $C(E)$. Let w denote the weak topology on $M(E)$ from this pairing; i.e. $\lambda_\alpha \xrightarrow{\alpha} 0$ in w if and only if $\int_E f d\lambda_\alpha \xrightarrow{\alpha} 0$ for all $f \in A(E)$. Finally, let $B_n^\wedge$ denote the set of $\mu \in M(E)$ such that $\|\mu^\wedge\|_\infty \leqslant n$.

THEOREM 2. *Let $A(E)$ be a closed subspace of $B(E)$. For $f \in C(E)$, the following are equivalent*:

(A) $f \in A(E)$.

(B) $\mu \to \int_E f d\mu$ *is w-continuous on $B_n^\wedge$.*

Proof. We use a technique introduced in (5).

Now $B_n^\wedge$ is a convex circled set in $M(E)$. It is w-bounded since

$$|\langle f, \mu\rangle| \leqslant \|\mu^\wedge\|_\infty \|f\|_{A(E)}.$$

It is w-closed since the restrictions of the continuous characters to E are in $A(E)$.

Let T denote the topology on $A(E)$ of uniform convergence on the sets $B_n^\wedge$. Our hypothesis asserts that T is the $B(E)$-norm topology.

The Grothendeick completeness theorem ((4), p. 149) yields the result since $A(E)$ is $B(E)$-norm closed. $\blacksquare$

If $A(E)$ is not a closed subspace of $B(E)$, then the above proof yields a characterization of the $B(E)$-closure of $A(E)$ in $C(E)$.

Let E be such that the set $M_0(E)$, consisting of $\mu \in M(E)$ such that $\mu \in C_0(G)$, has the property that for $f \in C(E)$ if $\int_E f d\mu = 0$ for all $\mu \in M_0(E)$, then $f = 0$. We call E a Tauberian set. For example, if E is of uniform positive measure, then E is Tauberian.

MAIN THEOREM 3. *Let E be a compact, Tauberian set. Then $A(E) = B(E)$, and for $f \in C(E)$, the following are equivalent:*

(A) $f \in A(E)$.

(C) *If $\mu_n \in M(E)$, $\|\mu^\wedge\|_\infty \leqslant 1$, and $\mu_n^\wedge \xrightarrow{n} 0$ pointwise on G, then $\int_E f\,d\mu_n \xrightarrow{n} 0$.*

Proof. Let $f \in B(E)$. Consider the linear functional on $M_0(E)^\wedge$ defined by $\mu^\wedge \to \int_E f\,d\mu$. Since $f \in B(E)$, the linear functional is sup-norm continuous and has an extension to $C_0(G)$. Thus there is $\lambda \in M(G)$ such that

$$\int_E f\,d\mu = \int_G \mu^\wedge\,d\lambda = \int_E \lambda^\wedge\,d\mu,$$

for all $\mu \in M_0(E)$. Since E is Tauberian, $f = \lambda^\wedge | E$ and hence $f \in A(E)$.

Assume (A). Thus there is $g \in L^1(G)$ such that $\hat{g}|E = f$. Thus for $\{\mu_n\}$ as in (C), we have that

$$\int_E f\,d\mu_n = \int_E \hat{g}\,d\mu_n = \int_\Gamma \hat{g}\,d\mu_n = \int_G \mu_n^\wedge\,dg \xrightarrow{n} 0$$

by the Lebesgue dominated convergence theorem.

Assume (C). If $A(E)$ is separable, then the w topology on $B_n^\wedge$ is metrizable and it follows easily that $\mu \to \int_E f\,d\mu$ is w-continuous on $B_n^\wedge$.

Since we are following our proof from ((5), p. 329), we just outline this proof.

Represent the continuous bounded functions on G, $C^B(G)$, as operators on $C_0(G)$ by $T_f(g) = fg$, $f \in C^B(G)$, $g \in C_0(G)$. Let WO denote the weak operator topology on $C^B(G)$ via this representation and SO the strong operator topology, (2). The SO topology is Buck's 'strict' topology, and it agrees with the compact-open topology on sup-norm bounded sets, ((1), p. 98). Viewing $M(E)$ as a subspace of $C^B(G)$ via the Fourier–Stieltjes transformation induces the WO and SO topologies on $M(E)$. It suffices by Theorem 2, to show that $\mu \to \int_E f\,d\mu$ is w-continuous on $B_n^\wedge$.

Now $\mu_\alpha \xrightarrow{\alpha} 0$ in w if and only if $\int_E f\,d\mu_\alpha \xrightarrow{\alpha} 0$ for all $f \in A(E)$ if and only if $\int_E \lambda^\wedge\,d\mu_\alpha \xrightarrow{\alpha} 0$ for all $\lambda \in M(G)$ since E is compact. If follows that the weak topology is equivalent to the WO-topology on $B_n^\wedge$. Thus we show that $\mu \to \int_E f\,d\mu$ is WO-continuous on $B_n^\wedge$. This is equivalent to showing that $\mu \to \int_E f\,d\lambda$ is SO-continuous on $B_n^\wedge$.

From (C) it follows that $\mu \to \int_E f\,d\mu$ is SO-continuous on the sets

$$C_n^\wedge = \{\lambda \in M_0(E) : \|\lambda^\wedge\|_\infty \leqslant n\}$$

and hence WO-continuous on $C_n^\wedge$ and hence w-continuous on $C_n^\wedge$. We extend $\mu \to \int_E f\,d\mu$ from $M_0(E)$ to $M(E)$, ((4), p. 49). The extended linear functional has the form $\mu \to \int_E g\,d\mu$, $g \in A(E)$, and is w-continuous on $B_n^\wedge$. Since E was chosen to be a Tauberian set, $f = g$. ∎

The technique used to characterize $A(E)$ in $C(E)$ can be used to characterize other linear spaces. Let $M_d(G)$ denote the discrete measures on G. We may pair $M_d(G)^\wedge$ and $M(\Gamma)$ by $\langle \mu^\wedge, \lambda\rangle = \int_\Gamma \mu^\wedge d\lambda$. Let τ denote the weak topology on $M(\Gamma)$ from this pairing. Let $B_n = \{\lambda \in M(\Gamma): \|\lambda\| \leqslant n\}$. Let $AP(\Gamma)$ denote the algebra of continuous almost periodic functions on Γ.

THEOREM 4. *Let G be a locally compact Abelian group. For $f \in C^B(\Gamma)$ the following are equivalent*:

(A) *f is almost periodic.*

(B) $\lambda \to \int_\Gamma f d\lambda$ *is τ-continuous on B_n.*

(C) *If $\{\lambda_\alpha\} \subset B_n$ and $\lambda_\alpha^\wedge \xrightarrow{\alpha} 0$ pointwise on G, then $\int_\Gamma f d\lambda_\alpha \xrightarrow{\alpha} 0$.*

Proof. Now B_n is a convex circled set in $M(\Gamma)$. It is τ-bounded since

$$|\langle \mu^\wedge, \lambda\rangle| \leqslant \|\mu^\wedge\|_\infty \|\lambda\|.$$

It is τ-closed since

$$\sup\left\{\left|\int_\Gamma \mu^\wedge d\lambda\right| : \mu \in M_d(G), \|\mu^\wedge\|_\infty \leqslant 1\right\} = \|\lambda\|.$$

Let T' denote the topology on $M_d(G)^\wedge$ of uniform convergence on the sets B_n. Since $\sup\left\{\left|\int_\Gamma \mu^\wedge d\lambda\right| : \lambda \in M(\Gamma), \|\lambda\| \leqslant 1\right\} = \|\mu^\wedge\|_\infty$, T' is the sup-norm topology.

The Grothendieck completeness theorem yields that (A) is equivalent to (B) since the sup-norm completion of $M_d(G)^\wedge$ in $C^B(\Gamma)$ is $AP(\Gamma)$.

That (B) is equivalent to (C) is immediate. ∎

The previous result is due to R. Edwards ((9), p. 254). His method is based on the bipolar theorem. Finally, we give a simple proof of a known result on almost periodic functions ((10), p. 449).

THEOREM 5. *Let f be a continuous almost periodic function on the locally compact, non-compact, Abelian group Γ. Let K be a compact subset of Γ. Then*

$$\sup\{|f(x)| : x \in K\} \leqslant \sup\{|f(x)| : x \notin K\}.$$

Proof. Since Γ is non-compact, we let Γ' denote the Bohr compactification of Γ ((6), p. 30). Let $\beta: \Gamma \to \Gamma'$ denote the natural continuous isomorphism of Γ into Γ'. Since K is compact and β is continuous, $\beta(K)$ is also compact in Γ'.

Suppose by way of contradiction that

$$\alpha = \sup\{|f(x)| : x \in K\} > \gamma = \sup\{|f(x)| : x \notin K\}.$$

Extend f to a continuous function, f', on Γ'. Since $\beta(K)$ is closed in Γ', $|f'(y)| \leqslant \gamma$ for $y \in \Gamma' \backslash \beta(K)$. Thus K contains an open subset U of Γ. It follows that β is a homeomorphism on U. Since β is a homomorphism, β is a homeomorphism on all of Γ. Thus Γ is compact. This contradiction completes the proof. ∎

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