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An Investigation of the Performance of Five Different Estimators in the Measurement Error Regression Model

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In a comprehensive paper by Riggs et al.(1978) the authors analyse the performances of numerous estimators for the regression slope in the measurement error model with positive measurement error variances $\sigma_\delta^2 > 0$ 0 for X and $\sigma_\tau^2 > 0$ for Y . In particular, using a Monte Carlo simulation, the authors demonstrate that the adjusted geometric mean estimator of Madansky (1959, Equation 4, p. 179), which requires knowledge of both σ_δ^2 and σ_τ^2, performs "much worse than" the maximum likelihood estimator in the normal structural measurement error model which requires only knowledge of the ratio $\kappa = \sigma_\tau^2 / \sigma_\delta^2$. The second moment estimator, β_1^{kap} coincides with the maximum likelihood estimator in the normal structural measurement error model (Madansky (1959)). In practice κ has to be estimated by $\tilde{\kappa}$. In this paper, we show that the bias of β_1^{kap} is not only dependent on the magnitude of the difference between κ and $\tilde{\kappa}$ but also on the magnitude of $\tilde{\kappa}\sigma_\tau^2 - \sigma_\delta^2$. We use a fourth moment estimator to smooth the jump discontinuity in the estimator of Copas (1972) as described in ODriscoll and Ramirez (2011) and use this estimator to find estimates for each error variance σ_δ^2 and σ_τ^2. Our Monte Carlo simulations show that the adjusted geometric mean estimator of Madansky performs much better than the ordinary least squares estimators OLS(y|x) and OLS(x|y) when the error variances are strictly positive and performs equally as well as the geometric estimator, β_1^{gm}, and the perpendicular estimator of Adcock (1878), β_1^{per}, with $\kappa = 1$.

Keywords: *Moment estimation; measurement errors; errors in variables.*

Introduction

With ordinary least squares $OLS(y|x)$ regression we have data

$$\{(x_1, Y_1|X = x_1), \ldots, (x_n, Y_n|X_n = x_n)\},$$

and we minimize the sum of the squared vertical errors to find the best-fit line $y = h(x) = \beta_0 + \beta_1 x$, where it is assumed that the independent or causal variable X is measured without error. In this paper it is assumed that X and Y are random variables with respective finite variances σ_X^2 and σ_Y^2, finite fourth moments and have the linear relationship $Y = \beta_0 + \beta_1 X$ It is also assumed that the x_i are a random sample from the random variable X. In the chapter of *Measurement Error Model* we outline the assumptions of the measurement error model and give second and fourth order moment equations. In the next chapter we give an expression for the asymptotic bias of the second moment estimator. We show that this bias is dependent on the magnitude of the error variance σ_δ^2 associated with the X variable and illustrate this result in Table 1 using a Monte Carlo simulation. We introduce a fourth order slope estimator and suggest that this estimator be used in the second order moment equations to estimate σ_δ^2 and σ_τ^2. When $\kappa = \sigma_\tau^2/\sigma_\delta^2$, the maximum likelihood estimator $\beta_1^{kap} = \tilde{\beta}1(\kappa)$ (see for example, Madansky (1959, Equation 3)) has been dubbed by Riggs (1978, p. 1320) as the Properly Weighted Perpendicular Least Squares Estimator. The geometric mean of Madansky's estimators from his Equations 1 and 2 is given in Equation 4 which we have dubbed as Madansky's Adjusted Geometric Mean Estimator and is denoted by β_1^{agm}. Using a second Monte Carlo simulation, we investigate the performance of each of the five estimators estimators β_1^{ver}, β_1^{hor}, β_1^{per}, β_1^{gm} and β_1^{agm}. In the next chapter we present two applications of β_1^{agm} and compare its performance to the results in the literature. In the final chapter of this paper we conclude that β_1^{agm} performs equally as well as β_1^{per} and β_1^{gm} and much better than the ordinary least squares estimators β_1^{ver} and β_1^{hor}.

The Measurement Error Model

The observed data $\{(x_i, y_i), 1 \leq i \leq n\}$ are subject to error by $x_i = X_i + \delta_i$ and $y_i = Y_i + \tau_i$, where it is also assumed that δ is $N(0, \sigma_\delta^2)$, τ is $N(0, \sigma_\tau^2)$, $Cov(\delta_i, \delta_j) = 0$, $i \neq j$, $Cov(\tau_i, \tau_j) = 0$, $i \neq j$ and $Cov(\delta_i, \tau_j) = 0$, for all i and j. Let

$$s_{xx} = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n}, \quad s_{yy} = \sum_{i=1}^{n} \frac{(y_i - \bar{y})^2}{n} \text{ and } s_{xy} = \sum_{i=1}^{n} \frac{(x_i - \bar{x})(y_i - \bar{y})}{n}.$$

From Gillard and Iles (2009) it follows that second moment equations are

$$s_{xx} = \sigma_X^2 + \sigma_\delta^2, \quad s_{yy} = \beta_1^2 \sigma_X^2 + \sigma_\tau^2, \quad s_{xy} = \beta_1 \sigma_X^2 \qquad (1)$$

which yield the second moment estimators

$$\tilde{\sigma}_\delta^2 = s_{xx} - \frac{s_{xy}}{\beta_1} \text{ and } \tilde{\sigma}_\tau^2 = s_{yy} - \beta_1 s_{xy}. \tag{2}$$

Fourth moment equations are

$$s_{xxxy} = \beta_1 \mu_{X_4} + 3\beta_1 \sigma_X^2 \sigma_\delta^2; \; s_{xyyy} = \beta_1^3 \mu_{X_4} + 3\beta_1 \sigma_X^2 \sigma_\tau^2. \tag{3}$$

If the ratio of the error variances $\kappa = \sigma_\tau^2 / \sigma_\delta^2$ is assumed finite, then the second moment estimator for the regression slope is

$$\beta_1^{kap} = \tilde{\beta}_1(\kappa) = \frac{(s_{yy} - \kappa s_{xx}) + \sqrt{(s_{yy} - \kappa s_{xx})^2 + 4\kappa s_{xy}^2}}{2s_{xy}} \tag{4}$$

which coincides with the maximum likelihood estimator of the regression slope in the normal structural measurement error model (see Madansky (1959) and Davidov (2005)). If $\kappa = 1$ in Eqn. (4), then the moment estimator (often called the Deming (1943) Regression Estimator) is equivalent to the perpendicular estimator, β_1^{per}, first introduced by Adcock (1878). Similarly, if $\kappa = 0$ in Eqn. (4), then the moment estimator reduces to *OLS(X|Y)*, β_1^{hor}, and if $\kappa = \infty$ then the moment estimator reduces to *OLS(Y|X)*, β_1^{ver}.

Since the second sample moments converge in probability to their expectations (see Davidov (2005) and Mamun et al. (2013)), it follows from (1) that

$$\frac{s_{yy} - \kappa s_{xx}}{s_{xy}} \xrightarrow{p} \beta_1 - \kappa/\beta_1 \text{ and } \frac{s_{xx}}{s_{xy}} \xrightarrow{p} (1 + \sigma_\delta^2/\sigma_X^2)/\beta_1,$$

where $\sigma_\delta^2/\sigma_X^2$ is the noise to signal ratio of the model.

If the researcher knows the true error $\kappa = \sigma_\tau^2/\sigma_\delta^2$, then

$$\tilde{\beta}_1(\kappa) \xrightarrow{p} 0.5 \left((\beta_1 - \kappa/\beta_1) + \sqrt{(\beta_1 - \kappa/\beta_1)^2 + 4\kappa} \right) = \beta_1 \tag{5}$$

and there are no asymptotic bias problems. We will discuss the more realistic situation when κ is an unknown parameter and must be estimated by $\tilde{\kappa}$.

The Empirical Bias of the Second Moment Estimator for an Incorrect Choice of κ

Empirical Bias

In practice, the researcher estimates κ by $\tilde{\kappa}$ with error $\epsilon = \kappa - \tilde{\kappa} \neq 0$. To develop an expression for the asymptotic bias $E(\tilde{\beta}_1(\tilde{\kappa}) - \beta_1)$, we recall Eqn. (4) and write

$$E(\tilde{\beta}_1(\tilde{\kappa}) - \beta_1) = E(\tilde{\beta}_1(\tilde{\kappa}) - \tilde{\beta}_1(\kappa)) + E(\tilde{\beta}_1(\kappa) - \beta_1) = E(\tilde{\beta}_1(\tilde{\kappa}) - \tilde{\beta}_1(\kappa)).$$

We define the *empirical bias* in using $\tilde{\kappa}$ to estimate κ as *empbias($\tilde{\kappa}$:κ)* = $\tilde{\beta}_1(\tilde{\kappa})$-$\tilde{\beta}_1(\kappa)$, which in terms of $\{s_{xx}, s_{yy}, s_{xy}\}$ is

$$\frac{1}{2s_{xy}} \left(-s_{xx}(\tilde{\kappa} - \kappa) + \sqrt{(s_{yy} - \tilde{\kappa} s_{xx})^2 + 4\tilde{\kappa} s_{xy}^2} - \sqrt{(s_{yy} - \kappa s_{xx})^2 + 4\kappa s_{xy}^2} \right). \tag{6}$$

The empirical bias is an estimate of the error that occurs in $\widetilde{\beta}_1(\kappa)$ as a result of using $\widetilde{\kappa}$ for the unknown error ratio κ. In our simulation study we record estimates for both $E_{bias} = E(\widetilde{\beta}_1(\widetilde{\kappa}) - \beta_1)$ and $E_{emp} = E(\widetilde{\beta}_1(\widetilde{\kappa}) - \widetilde{\beta}_1(\kappa))$ in Table 1. They are, as expected, nearly equal values, and are both close to the theoretical values for the bias, $bias(\widetilde{\kappa} : \kappa)$.

With $\epsilon = \kappa - \widetilde{\kappa} \neq 0$ and $\theta = 1 + \sigma_\delta^2/\sigma_X^2$, the bias, $bias(\widetilde{\kappa} : \kappa)$, in terms of $\{\beta_1, \epsilon, \kappa, \theta\}$ is then

$$-0.5\left(\beta_1 + \frac{\kappa}{\beta_1} + \frac{\theta\epsilon}{\beta_1}\right) + 0.5\left(\beta_1 + \frac{\kappa}{\beta_1} - \frac{\theta\epsilon}{\beta_1}\right)\sqrt{1 + \frac{4\epsilon\left(1 + \frac{\kappa\theta}{\beta_1^2}\right)}{\left(\beta_1 + \frac{\kappa}{\beta_1} - \frac{\theta\epsilon}{\beta_1}\right)^2}}. \qquad (7)$$

We note that $bias(\widetilde{\kappa} : \kappa) = 0$ only when $\epsilon = \kappa - \widetilde{\kappa} = 0$ as proven by Lindley, D., El-Sayyad, M. (1968).

Series Expansion for the Bias

The series expansion, $serbias(\widetilde{\kappa} : \kappa)$, of the bias may be written in terms of ϵ as

$$-\frac{\epsilon\beta_1(\theta - 1)}{\beta_1^2 + \kappa}\left(1 - \frac{\epsilon(\beta_1^2 + \kappa\theta)}{(\beta_1^2 + \kappa)^2} - \frac{\epsilon^2(-\kappa\theta + \theta\beta_1^2 - 2\beta_1^2)(\beta_1^2 + \kappa\theta)}{(\beta_1^2 + \kappa)^4}\right) + O(\epsilon^4). \qquad (8)$$

Since

$$\frac{\epsilon\beta_1(\theta - 1)}{\beta_1^2 + \kappa} = \frac{(\widetilde{\kappa} - \kappa)\sigma_\delta^2}{\left(\beta_1 + \frac{\kappa}{\beta_1}\right)\sigma_X^2},$$

Equation (8) shows that $bias(\widetilde{\kappa} : \kappa)$ is not alone dependent on the magnitude of $(\widetilde{\kappa} - \kappa)$, but is also dependent on the magnitude of σ_δ^2; that is, the magnitude of the bias is dependent on the magnitude of the difference $\widetilde{\kappa}\sigma_\delta^2 - \sigma_\tau^2$, and our claim in the Abstract is justified.

Monte Carlo Simulation

We used Minitab for our simulation study setting the number of runs $N = 5000$, $\widetilde{\kappa} = 1$ and the sample size $n = 50$. The X data was generated from a normal $\beta_1 = 1$ and $\beta_0 = 0$. For the measurement error model, we used normal errors with mean equal to zero and variances $\{\sigma_\tau^2, \sigma_\delta^2\}$ varying over $\{1, 2, 3, 4, 9\}$. Typical values for the bias, $bias(\widetilde{\kappa} : \kappa)$ from Equation (7), and the Third Order series approximation, $serbias(\widetilde{\kappa} : \kappa)$ from Equation (8), are shown in Table 1. We note that the Third Order approximation yields, as expected, values close to the theoretical values for the bias, $bias(\widetilde{\kappa} : \kappa)$. We also record the results for the asymptotic bias $E(\widetilde{\beta}_1(\widetilde{\kappa}) - \beta_1)$, and the estimated empirical bias $E(\widetilde{\beta}_1(\widetilde{\kappa}) - \widetilde{\beta}_1(\kappa))$.

Table 1

$E_{bias} = E(\tilde{\beta}_1(\tilde{\kappa}) - \beta_1)$; $E_{emp} = E(\tilde{\beta}_1(\tilde{\kappa}) - \tilde{\beta}_1(\kappa))$; $bias(\tilde{\kappa} : \kappa)$; $serbias(\tilde{\kappa} : \kappa)$
with $\{\beta_1 = 1, \beta_0 = 0, \sigma_X^2 = 100, \kappa = \sigma_\tau^2/\sigma_\delta^2, \tilde{\kappa} = 1, n = 50, N = 5000\}$

σ_τ^2	σ_δ^2	κ	$\sigma_\tau^2 - \sigma_\delta^2$	E_{bias}	E_{emp}	$bias(\tilde{\kappa} : \kappa)$	$serbias(\tilde{\kappa} : \kappa)$
3	9	0.333	−6.0	-0.0276	-0.0303	-0.0296	-0.0330
2	4	0.500	−2.0	-0.0085	-0.0101	-0.0100	−0.0103
1	3	0.333	−2.0	-0.0093	-0.0102	-0.0100	−0.0112
1	2	0.500	−1.0	-0.0046	-0.0051	-0.0050	−0.0052
2	1	2.000	1.0	0.0048	0.0051	0.0050	0.0048
3	1	3.000	2.0	0.0109	0.0103	0.0100	0.0088
4	2	2.000	2.0	0.0100	0.0102	0.0100	0.0097
9	3	3.000	6.0	0.0321	0.0313	0.0304	0.0266

The rows of Table 1 are sorted in ascending order of the theoretical bias, $bias(\tilde{\kappa} : \kappa)$ displayed in Column 7. Column 8 shows that our approximation, $serbias(\tilde{\kappa} : \kappa)$, is a good estimate for $bias(\tilde{\kappa} : \kappa)$. Columns 5 and 6 also show that our simulation study produced very good results for $E(\widetilde{\beta}_1(\tilde{\kappa}) - \beta_1)$, and E $E(\widetilde{\beta}_1(\tilde{\kappa}) - \tilde{\beta}_1(\kappa))$.

We make the following observations. Firstly, with $\tilde{\kappa} = 1$, the ranking for the bias concurs with the ranking of the differences in the error variances $\sigma_\tau^2 - \sigma_\delta^2$ but does not concur with the ranking for $\kappa = \sigma_\tau^2/\sigma_\delta^2$ in terms of its closeness to $\tilde{\kappa}$; that is, the magnitude of the bias for the MLE estimator $\tilde{\beta}_1(\kappa)$ is not monotone in κ. Secondly, for equal $\kappa = 3/1$ in Row 6 and $\kappa = 9/3$ in Row 8, the respective biases 0.0101 and 0.0304 are approximately proportional to the respective differences of the error variances 2 and 6.

Since the asymptotic bias $E(\widetilde{\beta}_1(\tilde{\kappa}) - \beta_1)$ is dependent on the magnitude of σ_δ^2, we investigated the use of second and fourth moments to estimate each of the error variances.

Using Second and Fourth Moments to Estimate the Error Variances σ_τ^2 and σ_δ^2

O'Driscoll and Ramirez (2011) suggested using the fourth moment slope estimator

$$\beta_1^{mt} = \sqrt{(s_{xyyy} - 3s_{xy}s_{yy})/(s_{xxxy} - 3s_{xy}s_{xx})}$$

described by Gillard and Iles (2005, Eqn. 26) to smooth out the jump discontinuity between β_1^{ver} and β_1^{hor}. For $s_{xy} > 0$, the authors define β_1^{mt4} as:

$$\beta_1^{mt4} = \begin{cases} \beta_1^{gm} & \beta_1^{mt} \text{ undefined} \\ \beta_1^{ver} & \beta_1^{mt} < \beta_1^{ver} \\ \beta_1^{mt} & \beta_1^{ver} \leq \beta_1^{mt} \leq \beta_1^{hor} \\ \beta_1^{hor} & \beta_1^{mt} > \beta_1^{hor}. \end{cases}$$

The estimators $\tilde{\sigma}_\tau^2$ and $\tilde{\sigma}_\delta^2$ of Equation (2) are computed using $\widetilde{\beta}_1 = \beta_1^{mt4}$ and we constrain the moment estimators so that the error variances are non-negative. We then use these estimates in Madansky's adjusted geometric mean estimator

$$\beta_1^{agm} = sgn(s_{xy})\sqrt{\frac{s_{yy} - \tilde{\sigma}_\tau^2}{s_{xx} - \tilde{\sigma}_\delta^2}}. \tag{9}$$

Some authors such as Al-Nasser (2012) refer to Gillard and Iles (2009) who suggest using

$$\tilde{\kappa}(\beta_1^{mt4}) = \frac{(s_{yy} - \beta_1^{mt4} s_{xy})}{(s_{xx} - s_{xy}/\beta_1^{mt4})} \tag{10}$$

as an estimator for κ in Equation (4) to find β_1^{kap}. However, as noted in O'Driscoll and Ramirez (2011, Proposition 3), there is a circular relationship between $\tilde{\kappa}$ and $\beta_1^{kap} = \widetilde{\beta}_1(\tilde{\kappa})$ such that $\widetilde{\beta}_1(\tilde{\kappa}(\beta_1)) = \beta_1$; and in particular, $\widetilde{\beta}_1(\tilde{\kappa}(\beta_1^{mt4})) = \beta_1^{mt4}$.

In a second simulation study we again set $\beta_1 = 1$ and $\beta_0 = 0$. The X data was generated from a uniform distribution on $(0, 20)$ so that the error variances $\{\sigma_\tau^2, \sigma_\delta^2\}$ varying over $\{1, 4, 9\}$ would have a strong impact on the bias of the regression slope. Table 2 records the expected values and standard deviations of our estimators for σ_τ^2 and σ_δ^2 using β_1^{mt4} as an estimate for the slope β_1 in Equation (2).

Table 2
$E(\tilde{\sigma}_\delta^2), E(std(\tilde{\sigma}_\delta^2)), E(\tilde{\sigma}_\tau^2), E(std(\tilde{\sigma}_\tau^2))$
$\{\beta_1 = 1, \beta_0 = 0, \mathcal{L}(X) = U(0, 20), n = 50, N = 5000\}$

σ_δ^2	σ_τ^2	$E(\tilde{\sigma}_\delta^2)$	$E(std(\tilde{\sigma}_\delta^2))$	$E(\tilde{\sigma}_\tau^2)$	$E(std(\tilde{\sigma}_\tau^2))$
1	1	0.9542	0.6952	0.9485	0.6972
1	4	1.2637	1.2084	3.4798	1.4718
1	9	1.7060	1.7808	7.7635	2.5742
4	1	3.4872	1.4550	1.2518	1.2167
4	4	3.7429	2.1852	3.7925	2.2096
4	9	4.0009	2.9326	8.2035	3.4808
9	1	7.7670	2.6010	1.7161	1.8138
9	4	8.1291	3.5325	4.0944	2.9942
9	9	8.3387	4.5750	8.4323	4.5883

Tables 3A, 3B and 3C record the expected means, standard deviations and mean square errors for the five estimators $\beta_1^{ver}, \beta_1^{hor}, \beta_1^{per}, \beta_1^{gm}$ and β_1^{agm} where β_1^{mt4} is used in Equation (2) to estimate σ_τ^2 and σ_δ^2. Our values recorded in the set count are the number of times that β_1^{mt} was not defined, that σ_δ^2 had to be increased to zero and that σ_τ^2 had to be increased to zero respectively. Note

that β_1^{mt} requires the term under the radical to be non negative and hence for our simulation we chose the uniform distribution.

Table 3A
$\{\beta_1 = 1,\ \beta_0 = 0,\ \mathcal{L}(X) = U(0, 20),\ n = 50,\ N = 5000\}$

Estimator	$\{\sigma_\delta^2 = 1, \sigma_\tau^2 = 1\}$ $count = \{2, 736, 768\}$			$\{\sigma_\delta^2 = 4, \sigma_\tau^2 = 1\}$ $count = \{3, 82, 1371\}$			$\{\sigma_\delta^2 = 9, \sigma_\tau^2 = 1\}$ $count = \{20, 12, 1615\}$		
	Mean	St Dev	MSE	Mean	St Dev	MSE	Mean	St Dev	MSE
β_1^{ver}	0.9718	0.03422	0.00196	0.8957	0.04776	0.01315	0.7907	0.05699	0.04617
β_1^{hor}	1.0309	0.03664	0.00230	1.0329	0.05830	0.00448	1.0348	0.08248	0.00801
β_1^{per}	1.0009	0.04129	0.00171	0.9592	0.05351	0.00453	0.8916	0.06864	0.01646
β_1^{gm}	1.0009	0.03463	0.00112	0.9617	0.04995	0.00395	0.9040	0.06091	0.01293
β_1^{agm}	1.0013	0.04129	0.00171	0.9934	0.06637	0.00445	0.9797	0.09378	0.00921

Table 3B
$\{\beta_1 = 1,\ \beta_0 = 0,\ \mathcal{L}(X) = U(0, 20),\ n = 50,\ N = 5000\}$

Estimator	$\{\sigma_\delta^2 = 1, \sigma_\tau^2 = 4\}$ $count = \{7, 1328, 89\}$			$\{\sigma_\delta^2 = 4, \sigma_\tau^2 = 4\}$ $count = \{7, 356, 341\}$			$\{\sigma_\delta^2 = 9, \sigma_\tau^2 = 4\}$ $count = \{27, 105, 744\}$		
	Mean	St Dev	MSE	Mean	St Dev	MSE	Mean	St Dev	MSE
β_1^{ver}	0.9724	0.05428	0.00371	0.8963	0.06238	0.01464	0.7912	0.06860	0.04831
β_1^{hor}	1.1216	0.06080	0.01847	1.1235	0.07968	0.02160	1.1260	0.10460	0.02681
β_1^{per}	1.0476	0.05854	0.00569	1.0038	0.07244	0.00526	0.9331	0.08596	0.01119
β_1^{gm}	1.0442	0.05432	0.00490	1.0031	0.06451	0.00417	0.9428	0.07253	0.00853
β_1^{agm}	1.0133	0.06869	0.00490	1.0049	0.09508	0.00906	0.9963	0.12581	0.01584

Table 3C
$\{\beta_1 = 1,\ \beta_0 = 0,\ \mathcal{L}(X) = U(0, 20),\ n = 50,\ N = 5000\}$

Estimator	$\{\sigma_\delta^2 = 1, \sigma_\tau^2 = 9\}$ $count = \{23, 1640, 17\}$			$\{\sigma_\delta^2 = 4, \sigma_\tau^2 = 9\}$ $count = \{30, 763, 101\}$			$\{\sigma_\delta^2 = 9, \sigma_\tau^2 = 9\}$ $count = \{50, 328, 310\}$		
	Mean	St Dev	MSE	Mean	St Dev	MSE	Mean	St Dev	MSE
β_1^{ver}	0.9730	0.07688	0.00664	0.8969	0.08138	0.01726	0.7917	0.08451	0.05053
β_1^{hor}	1.2730	0.09325	0.08324	1.2751	0.11275	0.08837	1.2786	0.14135	0.09760
β_1^{per}	1.1295	0.08763	0.02445	1.0825	0.10024	0.01685	1.0067	0.11329	0.01288
β_1^{gm}	1.1123	0.07571	0.01834	1.0683	0.08281	0.01152	1.0041	0.08809	0.00778
β_1^{agm}	1.0307	0.10038	0.01102	1.0185	0.13041	0.01735	1.0127	0.16925	0.02880

From Table 3A with $\sigma_\delta^2 = 9$ and $\sigma_\tau^2 = 1$ the MSEs for β_1^{ver} and β_1^{hor} are 0.04617 and 0.00801 respectively. The MSE for β_1^{agm} is comparable to β_1^{hor} with value 0.00921 and is smaller than the MSEs for the other three slope estimators. From Table 3C with $\sigma_\delta^2 = 1$ and $\sigma_\tau^2 = 9$ the MSEs for β_1^{ver} and β_1^{hor} are 0.00664 and 0.08324. The MSE for β_1^{agm} is comparable to β_1^{ver} with value 0.01102 and is again smaller then the the MSEs for the other three slope estimators. Thus β_1^{agm} is robust when compared to a random choice between β_1^{ver} and β_1^{hor}.

Applications

Example 1

Riggs (1978) presented a small example with $n = 6$, $X = [-4, -3, -2, 0, 3, 6]$ and $Y = [-5, -1, -1, -2, 3, 6]$ and assumes that $\sigma_\tau^2 = 4.933$ is known. In the case where σ_τ^2 is known Madansky proposed the estimator $\beta_1^{M1} = (s_{yy} - \sigma_\tau^2)/s_{xy}$ which for this data set gave an unsuitable line fit with negative slope. However, in practice suitable constraints have to be placed on the estimator equations such that $0 < \tilde{\sigma}_\delta^2 < s_{xx}$ and $0 < \tilde{\sigma}_\tau^2 < s_{yy}$. Our estimated value for σ_τ^2 is 1.55 which yields $\beta_1^{M1} = 0.946$. In the case where σ_δ^2 is known Madansky proposed the estimator $\beta_1^{M2} = s_{xy}/(s_{yy} - \sigma_\delta^2)$ which yields $\beta_1^{M2} = 1.063$ with an estimated value for $\sigma_\delta^2 = .082$. (see Riggs(1978, Eqns. 35 and 37, p. 1320)). With these estimates of σ_δ^2 and σ_τ^2 Madansky's adjusted geometric mean estimator yields $\beta_1^{agm} = 0.9522$. Thus the anomaly with the original Madansky estimate does not exist with Madansky's adjusted geometric mean estimator.

Example 2

Mamun et al. (2013) introduced a nonparametric estimator to the slope for measurement error models. They considered the serum kanamycin data set from Kelly (1984) with $n = 20$ for which they found the estimator of the slope $\beta_1 = 1.0$ with $\sigma_\delta^2 = 3.4$ and $\sigma_\tau^2 = 5.4$. Her assumption of $k = 1$ is equivalent to $\tilde{\beta}_1 = \beta_1^{per}$, the perpendicular estimator of Adcock (1878). We compute the Jackknife estimator for the slope β_1^{agm} and intercept β_0^{agm} and their respective standard deviations and use this estimate to find variance estimators for σ_δ^2 and σ_τ^2. These values are shown in Column 4 in Table 4. The values in Column 5 are our Jackknife estimators for the perpendicular estimator used by Kelly (1984). Our methodology has as its only restriction that the error variances must be positive. The results are summarised in Table 4.

Table 4
Serum Kanamycin Data

	Kelly	Mamun et al.	agm	$\kappa = 1$
$\tilde{\beta}_0$	-1.16 (4.97)	-0.1(0.7)	-4.69 (1.01)	-1.23(1.16)
$\tilde{\beta}_1$	1.07 (0.26)	1.0 (NA)	1.24 (0.05)	1.07(0.06)
$\tilde{\sigma}_\delta^2$	4.6 (4.97)	3.4(3.4)	7.00 (1.03)	NA
$\tilde{\sigma}_\tau^2$	4.6 (4.97)	5.4(3.4)	0.65 (0.468)	NA

Serum Kanamycin Data
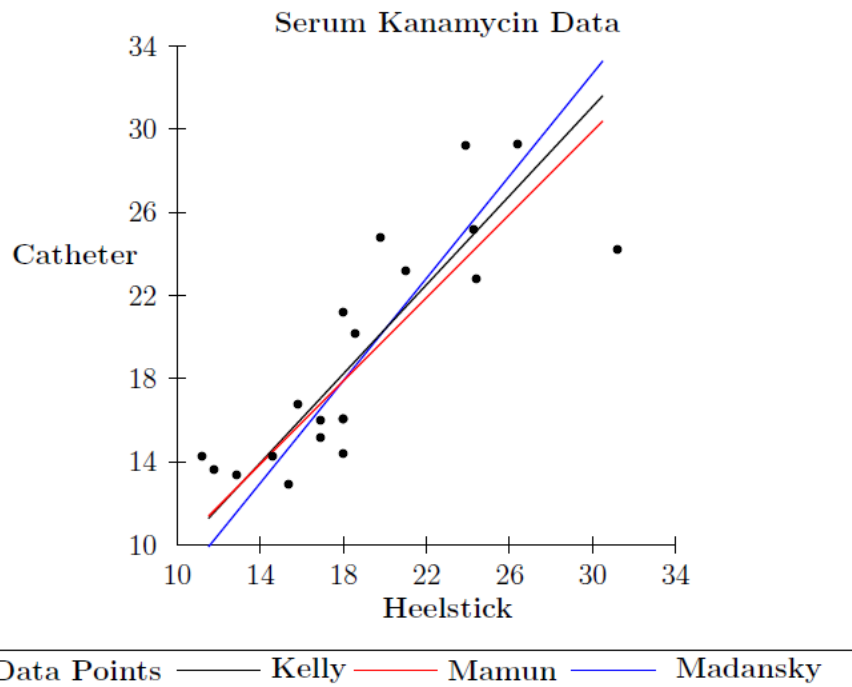
From the data of this particular sample, the adjusted geometric mean estimator for β_1 gives the linear calibration between the serum kanamycin levels in blood samples drawn simultaneously from an umbilical catheter and a heel venipuncture and is shown in the graph.

Conclusion

It has been demonstrated that the bias of the second moment estimator for the regression slope in the measurement error model is dependent on the magnitude of $\tilde{\kappa}\sigma_\tau^2 - \sigma_\delta^2$. Our simulation studies suggest that using second and fourth moments to estimate each of the error variances in Equation (9) yields improvement in bias and mean square error over the ordinary least squares estimators β_1^{ver} and β_1^{hor}, the geometric mean estimator, β_1^{gm}, and the perpendicular estimator β_1^{per}.

References

Adcock, R. J. (1878), A problem in least-squares, *The Analyst*, 5: 53–54.

Al-Nasser, A. (2012), On using the maximum entropy median for fitting the un-replicated functional model between the unemployment rate and the human development index in the Arab states, *J. Appl. Sci.*, 12: 326–335.

Davidov, O. (2005), Estimating the slope in measurement error models - a different perspective, *Statistics & Probability Letters*, 71: 215-223.

Deming, W. E. (1943), *Statistical Adjustment of Data*, New York: Wiley.

Gillard, J., Iles, T. (2005), Method of moments estimation in linear regression with errors in both variables, *Cardiff University School of Mathematics Technical Report*, Cardiff, Wales, UK.

Kelly, G. (1984), The influence function in the errors in variables problem, *The Annals of Statistics*, 12: 87-100.

Lindley, D., El-Sayyad, M. (1968), The Bayesian estimation of a linear functional relationship, J. Roy. Stat. Soc. B, Methodological, 30: 190–202.

Madansky, A. (1959). The fitting of straight Lines when both variables are subject to error. Journal of American Statistical Association 54:173-205.

Mamun, A., Hussin, G., Zubairi, Y., Imon, R. (2013), Maximum likelihood estimation of linear structural relationship model parameters assuming the slope is known, ScienceAsia 39 (2013): 561-565.

O'Driscoll, D., Ramirez, D. (2011), Geometric view of measurement errors, *Commun. Statistics–Simulation Comp.*, 40: 1373–1382.

Riggs, D., Guarnieri, J., Addelman, S. (1978). 'Fitting straight lines when both variables are subject to error', Life Sciences, 22: 1305-1360.