Strategyproof Matching with Minimum Quotas

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We study matching markets in which institutions may have minimum and maximum quotas. Minimum quotas are important in many settings, such as hospital residency matching, military cadet matching, and school choice, but current mechanisms are unable to accommodate them, leading to the use of ad hoc solutions. We introduce two new classes of strategyproof mechanisms that allow for minimum quotas as an explicit input and show that our mechanisms improve welfare relative to existing approaches. Because minimum quotas cause a theoretical incompatibility between standard fairness and nonwastefulness properties, we introduce new second-best axioms and show that they are satisfied by our mechanisms. Last, we use simulations to quantify (1) the magnitude of the potential efficiency gains from our mechanisms and (2) how far the resulting assignments are from the first-best definitions of fairness and nonwastefulness. Combining both the theoretical and simulation results, we argue that our mechanisms will improve the performance of matching markets with minimum quota constraints in practice.

CCS Concepts: • **Theory of computation → Algorithmic mechanism design**; • **Applied computing → Economics**

Additional Key Words and Phrases: Minimum quotas, lower bounds, school choice, strategyproofness, envy, fairness, deferred acceptance

ACM Reference Format:
Daniel Fragiadakis, Atsushi Iwasaki, Peter Troyan, Suguru Ueda, and Makoto Yokoo. 2015. Strategyproof matching with minimum quotas. ACM Trans. Econ. Comput. 4, 1, Article 6 (December 2015), 40 pages.
DOI: http://dx.doi.org/10.1145/2841226

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1. INTRODUCTION

The theory of matching has been extensively developed for markets in which the agents (students/schools, hospitals/residents, workers/firms) have maximum quotas that

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Fragiadakis and Troyan gratefully acknowledge support from the Stanford Institute for Economic Policy Research through the B.F. Haley and E.S. Shaw Fellowship (Troyan) and the Leonard W. Ely and Shirley R. Ely Fellowship (Fragiadakis). Iwasaki, Ueda, and Yokoo acknowledge financial support from JSPS Kakenhi Grant Number 24220003. Iwasaki acknowledges financial support from JSPS Kakenhi Grant Number 26280081.
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DOI: http://dx.doi.org/10.1145/2841226

cannot be exceeded.[1] However, in many real-world markets, *minimum* quotas are also present, and there is a lack of mechanisms that take minimum quotas into account. The main contribution of this article is to provide new strategyproof mechanisms that fill all minimum quotas while at the same time satisfying other important desiderata (fairness and efficiency) as much as possible.

There are many examples of matching problems with minimum quotas. School districts may need at least a certain number of students in each school for the school to operate, as in college admissions in Hungary [Biró et al. 2010]. In medical residency matching markets in many countries, rural hospitals suffer from doctor shortages, and authorities want to ensure that a minimum number of residents are assigned to each region. The United States Military Academy (USMA) solicits cadet preferences over assignments to various army branches, and each branch has minimum manning requirements. In the context of schools, minimum quotas are important not only in assigning students across schools but also in assigning students to classes *within* schools. For example, computer science students at Kyushu University must all complete a laboratory requirement. Students submit preferences over the labs, but each lab has a minimum and maximum quota. Matching with minimum quotas is also important to firms that want to assign employees to specific projects: for example, newly graduated doctors often must complete an intern year in which they rotate through various departments in a hospital. The hospitals consider doctor preferences when making assignments, but each department has a minimum staffing requirement that must be satisfied above all else.

Because standard matching mechanisms cannot accommodate minimum quotas explicitly, many markets will impose artificially lower maximum quotas. For example, in Japan, the Japan Residency Matching Program (JRMP) lowers the capacities of hospitals in urban areas such as Tokyo so that more doctors will be forced to apply to hospitals in rural areas.[2] Similarly, an October 1, 2007, memorandum from the army to USMA describes an assignment algorithm in which the quotas of popular branches are reduced to ensure that all branches will meet their manning requirements.[3]

We call the approach described earlier imposing *artificial caps*. Notice that imposing artificial caps will also obscure the presence of minimum quotas in markets in which they are truly present, but in which designers simply lack good tools to handle them. Though the ultimate goal is to satisfy the minimum quotas, artificial caps only do so implicitly, by eliminating positions ex ante, without regard to agent preferences. This leads to efficiency losses, since after the preferences are submitted and the algorithm is run, some institutions in high demand will end up below their true capacities, making it possible to reassign the agents and make everyone better off. Our mechanisms recover these efficiency losses by lowering capacities at popular institutions *only when it is actually necessary*, which depends on the submitted preferences. Importantly, we show that these efficiency gains can be realized without compromising any incentive or fairness properties.

More specifically, we start by introducing standard axioms that have been identified as key in the literature without minimum quotas: fairness, nonwastefulness, and strategyproofness.[4] Because of the importance of strategyproofness to many policymakers,

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[1]See Roth and Sotomayor [1990] for a comprehensive survey of many results in this literature.

[2]See Kamada and Kojima [2015].

[3]See Sönmez and Switzer [2013] and Sönmez [2013], which study other aspects of this market unrelated to minimum quotas.

[4]Fairness means that if one student envies the assignment of another, then the second student must have a higher priority at his or her assigned school than the first. It is also called "no justified envy." Nonwastefulness is an efficiency requirement that says that if a student prefers some school to his or her current assignment,

we take it as a key design requirement. With minimum quotas, matchings that are simultaneously fair and nonwasteful may not even exist, and so at least one must be weakened. Because different institutions may weigh the relative importance of fairness and nonwastefulness differently, we provide two mechanisms.

Our first new mechanism, extended-seat deferred acceptance (ESDA), works by dividing the seats into two classes. Each school is given a number of regular seats equal to its minimum quota, and a number of "extended" seats equal to the difference between its minimum and maximum quota. Students apply first to the regular seats and then to the extended seats according to their preferences. By restricting the number of extended seats that can be assigned in an appropriate manner, we ensure that all of the regular seats will be filled and all minimum quotas will be satisfied. The second mechanism, multistage deferred acceptance (MSDA), runs by first reserving a number of students such that, no matter how the remaining students are assigned, we will have enough students reserved to fill any remaining minimum quotas. The students not reserved are then assigned according to the standard deferred acceptance (DA) algorithm, and we calculate how many minimum quota seats remain. This process is then repeated until all students are assigned. Note that both of these mechanisms take the minimum quotas as an explicit input and use this information together with the student preferences to allocate the flexible seats (those above the minimum quota), in stark contrast to artificial caps, which bluntly eliminate a large block of seats without considering the actual preferences of the students.

We show that both ESDA and MSDA are strategyproof, which ensures that our predictions of agent behavior are very robust, and the predicted welfare gains from our mechanisms will actually be realized in equilibrium. With regard to the tradeoff between fairness and nonwastefulness, we show that ESDA is fair, while MSDA is nonwasteful. While it is impossible to satisfy both fairness and nonwastefulness simultaneously, they are still appealing normative properties to policymakers, and hence it is desirable to weaken them as little as possible. We analyze this in two ways. First, theoretically, we introduce new second-best definitions of fairness and nonwastefulness and show they are satisfied by our mechanisms. Second, we use simulations to quantify how far each mechanism is from the first-best concepts of fairness and nonwastefulness.

Finally, we compare our new mechanisms to artificial caps DA (ACDA), which has been used in several markets, such as the Japan medical residency and military cadet markets mentioned earlier. ACDA is strategyproof and fair (as is ESDA), but ACDA is strongly wasteful, while ESDA is weakly nonwasteful. Simulations show that both ESDA and MSDA waste significantly fewer seats than ACDA and are overwhelmingly preferred by the students, in the sense that the rank distributions of ESDA and MSDA first-order stochastically dominate that of ACDA. Thus, from a policy perspective, there seems to be little theoretical or empirical justification for using ACDA. If fairness is a larger concern, then ESDA should be used, while if nonwastefulness is paramount, then MSDA is the better choice.

We close by emphasizing that the main goal of this article is to provide practical mechanisms that can easily be implemented in markets with minimum quota constraints. Any such mechanism must inevitably give up either fairness *or* nonwastefulness, at least in its strongest form. This necessitates nonstandard ways of thinking about such properties, which is an additional contribution of this article. We follow the school choice literature in taking fairness and nonwastefulness as normative axioms, and so while one of them must be weakened, we hope to satisfy the weakened axiom as much

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then that school must be filled to capacity. Strategyproofness is an incentive constraint that ensures that truth telling is a dominant strategy; that is, it is not possible for students or parents to "game the system." See Section 2 for formal definitions.

as possible. By showing that our mechanisms satisfy second-best theoretical properties and using simulations to measure the losses in fairness and nonwastefulness quantitatively, we argue that while minimum quotas do lead to some impossibility results, the mechanisms we introduce still satisfy many desirable properties and should be useful in practical applications where minimum quotas are an important concern.

Related Literature

While minimum quotas seem like a natural extension to the standard matching models, there is little work on the topic, most likely because the problem becomes difficult and many of the results from the previous literature have been negative, starting with the rural hospital theorem first introduced in Roth [1986]. In the context of school choice, Kojima [2012] shows that some types of affirmative action quotas may actually hurt the very minorities they are supposed to help. Hafalir et al. [2013] use minority reserves (as opposed to majority quotas) to alleviate the problem identified by Kojima [2012], while Kominers and Sönmez [2012] generalize the mechanism of Hafalir et al. [2013] to allow for slot-specific priorities. Westkamp [2013] analyzes complex (maximum) quota constraints in the German university admissions system, while Braun et al. [2014] conduct an experimental analysis of the same system. The important difference between our setting and these previous works is that the minimum quotas are hard constraints that must be satisfied by any matching, while the papers listed previously either do not model minimum quotas explicitly (instead using a maximum quota for majority students as a way to implicitly reserve seats for minority students) or treat the minimum quotas as "soft" constraints that may or may not actually be satisfied at the final matching.

The paper most related to this one is Ehlers et al. [2014], who study diversity constraints in school choice. They show that if the constraints are interpreted as hard constraints, any mechanism that is fair and satisfies a definition they call constrained nonwastefulness cannot simultaneously be strategyproof. [5] They provide a trading-cycles-style mechanism similar in spirit to Erdil and Ergin [2008], but they also show that this mechanism is manipulable. Because of the difficulties introduced by hard constraints, they then reinterpret the diversity constraints in their model as soft constraints and are able to obtain more positive results. The model of soft constraints found in Ehlers et al. [2014] can be thought of as a generalization of Hafalir et al. [2013] to the case where there are more than two types of students; both are formally distinct from the model in this article, because they allow the minimum quotas to be violated at the final matching. We consider the case of hard constraints that must be satisfied for a matching to be feasible and are the first to provide strategyproof mechanisms in such a setting. We view this as an important contribution, as the matching literature has found strategyproofness to be a key property in a wide variety of settings.[6] Due to

[5] Alcalde and Romero-Medina [2014] study a similar weakening to the constrained nonwastefulness notion of Ehlers et al. [2014] in a model without floor constraints. They say a matching is λ-*equitable* if, whenever a student objects to it in favor of some other matching based on his or her priority being violated, some other student will object to this new matching. They then call a matching τ-*fair* if it is efficient and λ-equitable. Just as in Ehlers et al. [2014], τ-fairness is incompatible with strategyproofness [Kesten 2010]. We should also note that the terminology used by Alcalde and Romero-Medina [2014] is different from that used here: what they call *equity* is essentially what we call *fairness*, while they use the word "fair" to mean "equitable and efficient."

[6] See, for example, Roth [1991], Abdulkadiroğlu et al. [2005], Ergin and Sönmez [2006], Chen and Sönmez [2006], Pathak and Sönmez [2008], and Pathak and Sönmez [2013], who discuss the negative consequences of manipulable mechanisms. The use of strategyproof mechanisms also advances the so-called Wilson Doctrine [Wilson 1987], which argues that in order to analyze practical problems, economic models should reduce their reliance on common knowledge assumptions among the players. In particular, strategyproof mechanisms require no knowledge or beliefs about the preferences of others in order for students to formulate a

the impossibility result of Ehlers et al. [2014], to achieve strategyproofness, we must weaken one of nonwastefulness or fairness, and we study this tradeoff extensively in this article, both theoretically and using simulations. Fragiadakis and Troyan [2014] consider a model similar to the diversity constraints model of Ehlers et al. [2014] but provide a different class of mechanisms.

The problem of matching with minimum quotas has also been addressed in the computer science community. Biró et al. [2010] analyze college admissions in Hungary, in which colleges may declare minimum quotas for their programs, and study the difficulty of finding stable matchings when minimum quotas are introduced, but they do not provide explicit mechanisms or consider incentive or efficiency issues, as we do here. Hamada et al. [2014] also study matching with minimum quotas in the hospitals-residents problem, showing that minimizing the number of blocking pairs is an NP-hard problem when minimum quotas are imposed.

As a final possible application, consider the medical residency market studied first by Roth [1984]. In these markets, the shortage of doctors in rural areas is a well-known problem, and the so-called rural hospitals theorem suggests it is difficult to solve [Roth 1986; Martinez et al. 2000; Hatfield and Milgrom 2005]. Kamada and Kojima [2015] discuss one possible solution used in Japan: capping the number of residents who can be assigned to a given region. To the extent that these caps are simply an ad hoc way to ensure some true minimum quotas are satisfied, imposing these quotas directly and using one of our mechanisms is another possible approach.

The remainder of this articles is organized as follows. In Section 2, we present a basic model with minimum and maximum quotas and introduce some standard desiderata. We show how these desiderata become incompatible in the presence of minimum quotas, and so must be weakened in some way. Section 3 introduces the ESDA algorithm and discusses its properties, while Section 4 does the same for the MSDA algorithm. Section 5 uses computer simulations to quantitatively study ESDA, MSDA, and ACDA with respect to fairness, nonwastefulness, and student welfare. Section 6 concludes. All proofs are in the appendix, unless otherwise stated.

2. MATCHING WITH MINIMUM QUOTAS

2.1. Model

For convenience, we use the language of matching students and schools, but our model can be applied to many other types of markets, including those mentioned in the introduction.

A market consists of $(S, C, p, q, \succ_S, \succ_C)$. $S = \{s_1, s_2, \ldots, s_n\}$ is a set of n students, $C = \{c_1, c_2, \ldots, c_m\}$ a set of m schools ("colleges"). We use $p = (p_{c_1}, \ldots, p_{c_m})$ and $q = (q_{c_1}, \ldots, q_{c_m})$ to denote lists of minimum and maximum quotas, respectively, for each school, where $p_c \geq 0$, $q_c > 0$, $p_c \leq q_c$, and $n \geq q_c + \sum_{c' \neq c} p_{c'}$ for all $c \in C$ and $\sum_{c \in C} p_c < n < \sum_{c \in C} q_c$. The last two conditions are consistency conditions that relate the total number of students to the sizes of the minimum and maximum quotas.[7] Define $e = n - \sum_{c \in C} p_c$ to be the number of *excess students* above the sum of the minimum quotas.

best response (see also Bergemann and Morris [2005] for a discussion of robustness to beliefs in general mechanism design settings). It should be noted, however, that while economists have generally advocated for the replacement of manipulable mechanisms with strategyproof ones, the benefits are not without a cost: Miralles [2009], Abdulkadiroğlu et al. [2011], Featherstone and Niederle [2011], and Troyan [2012] have shown that nonstrategyproof mechanisms (the Boston mechanism in particular) may sometimes outperform strategyproof ones, at least in equilibrium.

[7]If $n > \sum_{c \in C} q_c$ or $n < \sum_{c \in C} p_c$, then there would be no way to assign all of the students without violating some quota. If $n = \sum_{c \in C} p_c$ or $\sum_{c \in C} q_c$, there is no flexibility in the seats to be assigned and the standard DA algorithm can be used. Our model only becomes interesting when choices must be made about where to

To make the problem interesting, we additionally assume that $m > 2$ and $p_c < q_c$ for at least two schools.[8]

Each student s has a strict preference relation $\succ_s$ over C, while each school c has a strict priority relation $\succ_c$ over S. Profiles of relations, one for each agent, are denoted $\succ_S = (\succ_s)_{s \in S}$ for the students and $\succ_C = (\succ_c)_{c \in C}$ for the schools. Let $\mathcal{P}$ denote the set of possible preference relations over C, and $\mathcal{P}^{|S|}$ denote the set of all preference profiles for all students. As is standard in the school choice literature, the school priorities are fixed and known to all students (in applications, priorities are often related to such things as the distance a student lives from a school or whether or not a student has a sibling attending the school). For now, we assume all students are acceptable to all schools and vice versa. This is a reasonable assumption in many contexts: in public school choice, school districts are legally required to assign every student a seat at some school, and school districts often assign students to schools they did not express any preference for (though they may still of course take their outside options); in military cadet matching, cadets are obligated to serve in the army, and so must express preferences over every possible branch.[9] This assumption is formally required to guarantee the existence of a feasible and individually rational matching, though in practice, our mechanisms can be run without it (see Section 5).

A **matching** is a mapping $\mu : S \cup C \to 2^S \cup C$ that satisfies (1) $\mu(s) \in C$ for all $s \in S$, (2) $\mu(c) \subseteq S$ for all $c \in C$, and (3) for any $s \in S$ and $c \in C$, we have $\mu(s) = c$ if and only if $s \in \mu(c)$. A matching is **feasible** if $p_c \leq |\mu(c)| \leq q_c$ for all $c \in C$. Let $\mathcal{M}$ denote the set of feasible matchings. A mechanism $\chi : \mathcal{P}^{|S|} \to \mathcal{M}$ is a function that takes as an input any possible preference profile of the students and gives as an output a feasible matching of students to schools. We write $\chi_i(\succ_S)$ for the assignment of agent $i \in S \cup C$.[10]

2.2. Properties

In this section, we discuss several properties that are important both in theory and for practical market design. All have been key considerations in the redesign of school choice mechanisms in many cities.

In order to define the first property (fairness), we must introduce the notion of a blocking pair.

Definition 2.1. Given a matching μ, student–school pair (s, c) is a **blocking pair** if $c \succ_s \mu(s)$ and $s \succ_c s'$ for some $s' \in \mu(c)$.

In words, student s would rather be matched to school c than his or her current match $\mu(s)$, and he or she has higher priority at c than some student s' who is currently assigned there; thus, s has a claim on a seat at c over student s'. In some papers, it is said that s has **justified envy** toward s', and we will sometimes use these terms interchangeably. Priorities are often based on criteria such as distance between a student and the school or test scores, and if one student justifiably envies another, he or she may be able to take legal action against the school district [Abdulkadiroğlu and

assign the flexible seats. If $n < q_c + \sum_{c' \neq c} p_{c'}$ for some c, then whenever school c was assigned q_c students, there would not be enough students left to fill the minimum quota seats at the other schools.

[8]These assumptions are likely to be satisfied in any real-world market of reasonable size. The special cases where these assumptions do not hold are dealt with in Appendix E.

[9]See the appendix for a more detailed discussion of markets where the assumption of complete preference listings is satisfied.

[10]Since student preferences are the only private information, we only explicitly write this as a function of $\succ_S$; however, this function of course implicitly depends on C, p, q, and $\succ_C$ as well.

Sönmez 2003]. Thus, an important goal of many school districts is for a matching to contain no such blocking pairs. When this is true, we say that the matching is fair.[11]

Definition 2.2. A matching μ is **fair** (or **eliminates all justified envy**) if no student–school pair (s, c) can form a blocking pair.

The next important property is nonwastefulness. Say that student s **claims an empty seat at school** c if $c \succ_s \mu(s)$ and $|\mu(c)| < q_c$.

Definition 2.3. A matching μ is **nonwasteful** if, whenever a student s claims an empty seat at school c, we have $|\mu(\mu(s))| = p_{\mu(s)}$.

Put another way, this definition says that if student s prefers school c to his or her assignment $\mu(s)$, school c has an empty seat, **and** school $\mu(s)$ has strictly more students than its minimum quota, then s should be moved to c.[12]

These properties also have counterparts for mechanisms: we say χ is fair (nonwasteful) if for every preference profile it produces a fair (nonwasteful) matching. The last important properties concern incentives for the students to report truthfully.

Definition 2.4. A mechanism χ is **strategyproof** if $\chi_s(\succ_S) \succeq_s \chi_s(\succ'_s, \succ_{S\setminus\{s\}})$ for all $\succ_S \in \mathcal{P}^{|S|}$, $s \in S$, and $\succ'_s \in \mathcal{P}$.

In words, a mechanism is strategyproof if no student ever has any incentive to misreport his or her preferences, no matter what the other students report. As discussed in the introduction, strategyproofness has been found to be a very important property in the success of matching mechanisms, for both positive and normative reasons. All of the mechanisms we provide will be strategyproof. In addition, our mechanisms will be immune to certain types of group manipulations.

Definition 2.5. A mechanism χ is **group strategyproof** if there does not exist a preference profile $\succ_S \in \mathcal{P}^{|S|}$, a group of students $S' \subseteq S$, and a preference profile $(\succ'_s)_{s \in S'} \equiv \succ'_{S'}$ such that $\chi_s(\succ'_{S'}, \succ_{S\setminus S'}) \succ_s \chi_s(\succ_S)$ for all $s \in S'$.

That is, there is no subset of students who can jointly misreport their preferences and make every member of the set strictly better off. Clearly, group strategyproofness implies strategyproofness. It is well known that without minimum quotas, DA is group strategyproof [Hatfield and Kojima 2009]. Both of our new mechanisms will also be group strategyproof.

2.3. Impossibility of a Simultaneously Fair and Nonwasteful Matching

In the standard school choice model with only maximum quotas, it is always possible to find matchings that are simultaneously fair and nonwasteful. However, in the presence

[11] For example, fairness was an important criterion to administrators of the Boston school district when they were redesigning their school assignment mechanism. See Abdulkadiroğlu et al. [2005].

[12] Starting with Gale and Shapley [1962], most of the literature combines fairness and nonwastefulness into a single definition called stability, which is then given a *positive* interpretation (i.e., if a matching is unstable, agents can jointly deviate to circumvent the match and obtain a more preferred assignment). In many recent applications, organizers may actually have the power to prevent such blocking pairs from forming, such as cities that control all school seats in a district and can prevent a student from enrolling in a school other than his or her assigned one. While technically feasible, implementing an allocation that ignores preferences and priorities still seems very undesirable and justifies fairness and nonwastefulness as important *normative* axioms. See, for example, Balinski and Sönmez [1999], Abdulkadiroğlu and Sönmez [2003], and Abdulkadiroğlu et al. [2005]. This also applies to other markets such as military cadet matching, where the army has the power to assign cadets to any branch, but it is a rooted institutional feature that cadets who rank higher on the Order of Merit List deserve their more preferred assignments.

of minimum quotas, such matchings may not exist. This is shown in the following example, which is instructive in illustrating the main problem that arises.[13]

Let there be two students s_1 and s_2 and three schools c_1, c_2, and c_3. The student preferences and school priorities and quotas are given in the following table. Note that each school has one seat, and school c_1 has a minimum quota of 1. All other minimum quotas are 0.

	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
	s_2	s_2	s_1
	s_1	s_1	s_2
p_c	1	0	0
q_c	1	1	1

$\succ_{s_1}$	$\succ_{s_2}$
c_2	$\boxed{c_3}$
c_3	c_2
$\boxed{c_1}$	c_1

The minimum quota requirement at c_1 means that one of s_1 or s_2 must be assigned there; nonwastefulness then requires that the other student be assigned his or her most preferred school. However, the student assigned c_1 will then justifiably envy the other student. For example, in the allocation indicated by the boxes, s_1 prefers c_3 and has higher priority than s_2 there, and thus forms a blocking pair. The case where s_2 is assigned c_1 is similar. Because of this impossibility, one of these properties must be weakened if we are to satisfy the minimum quotas.

2.4. Artificial Caps Deferred Acceptance (ACDA)

Without minimum quotas, the well-known DA algorithm of Gale and Shapley [1962] is fair, nonwasteful, and strategyproof. However, DA may not satisfy the minimum quotas. Because of this, many markets opt to use the very simple solution of running deferred acceptance under artificially lower maximum quotas ("artificial caps"). By imposing sufficiently stringent artificial caps, it is possible to ensure that no matter how the students are allocated, the minimum quotas will be satisfied.[14]

As an example, consider a market of $n = 100$ students and $m = 10$ schools, each with $p_c = 5$ and $q_c = 20$. Now, imagine imposing artificial caps of $q^*_c = 10$ at each school and running the standard DA algorithm. By doing so, exactly 10 students will be assigned to each school, thereby satisfying all minimum and maximum quotas. However, this may be very wasteful if, for example, the first choice of every student is c_1, because many students could be moved to c_1 and made better off without violating any quotas. The problem is that the mechanism has no flexibility and eliminates seats without regard to student preferences. (Note also that to an outside observer it would appear that each school had only 10 seats with no minimum quotas. In reality, this is not the case, and only appears so because the school district has no good way of handling minimum quotas.)

Definition 2.6. Let $q^* = (q^*_{c_1}, \dots, q^*_{c_m})$ be a list of **artificial capacities** (that need not be equal to the true capacities q). Artificial capacities q^* **ensure a feasible matching** if $|\mu(c)| \leq q^*_c \quad \forall c \in C p_c \leq |\mu(c)| \leq q_c \quad \forall c \in C$.

In words, this definition says that whenever matching μ explicitly satisfies the artificial caps q^*, it also implicitly satisfies the true minimum and maximum quotas p and q. In general, there will be many choices q^* that ensure a feasible matching, and at least one such choice always exists: pick a feasible μ and set $q^*_c = |\mu(c)|$.

[13]This example was first noted in Ehlers et al. [2014].

[14]The deferred acceptance algorithm is well known in the literature, and it is also a special case of the new mechanisms we define in Section 3, and so we do not give its definition here. See Gale and Shapley [1962] for the original description, or Abdulkadiroğlu and Sönmez [2003] for a discussion of DA in the context of school choice.

The ACDA is defined as the standard DA algorithm under some artificial caps q^* that ensure a feasible matching. ACDA is a popular approach because it inherits two of the good properties of DA, strategyproofness and fairness, while also satisfying feasibility and being very simple to implement. The drawback of ACDA is that it will waste seats.

THEOREM 2.7. *Let q^* be a choice of artificial capacities that ensures a feasible matching. Artificial caps deferred acceptance under q^* is group strategyproof and fair and produces a feasible matching for any profile of submitted preferences; however, ACDA is wasteful.*

Feasibility follows by definition of q^*, while strategyproofness and fairness follow from the fact that the DA mechanism itself is strategyproof and fair. To see that ACDA may be wasteful, consider the example in Section 2.3, and impose an artificial cap of 0 at c_2 (the other capacities are unchanged). Then, consider the following preferences of the students:

$\succ_{s_1}$	$\succ_{s_2}$
c_2	$\boxed{c_1}$
$\boxed{c_3}$	c_2
c_1	c_3

ACDA gives the allocation shown in the boxes, which is wasteful because s_1 claims an empty seat at c_2. The problem is that ACDA eliminates the seat at c_2 ex ante (before preferences are submitted) to ensure that when running standard DA, at least one student will be assigned to c_1 for any possible preference profile. The new mechanisms that we introduce will correct this by allowing the extra seats to be assigned more flexibly, based on the submitted student preferences.

3. EXTENDED-SEAT DA (ESDA)

3.1. Definition of ESDA

To define our first new algorithm, we take the original market $(S, C, p, q, \succ_S, \succ_C)$ and define a corresponding "extended market": $(S, \tilde{C}, \tilde{q}, \tilde{\succ}_S, \tilde{\succ}_{\tilde{C}})$. When extending the market, the set of students is unchanged. For the schools, we divide each school c into two smaller schools: a *standard school*, which, with slight abuse of notation, we label c, and that has a maximum quota of $\tilde{q}_c = p_c$, and an *extended school* c^*, which has a maximum quota of $\tilde{q}_{c^*} = q_c - p_c$. Each school (standard and extended) uses the original priority relation of school c: $\tilde{\succ}_c = \tilde{\succ}_{c^*} = \succ_c$. Thus, the set of schools is now $\tilde{C} = C \cup C^* = \{c_1, \ldots, c_m, c_1^*, \ldots, c_m^*\}$ and the maximum quotas are $\tilde{q} = \{\tilde{q}_{c'}\}_{c' \in \tilde{C}}$. Note that the extended market has no minimum quotas. By assigning no more than $e = n - \sum_{c \in C} p_c$ students to extended schools, all standard schools will be filled to capacity, thereby satisfying all minimum quotas in the original market.[15]

For the students, preferences over $C \cup C^*$ are created by taking the original preference relation $\succ_s$ and inserting school c_j^* immediately after school c_j. That is,

$$\text{preference relation } \succ_s\text{: } c_j \succ_s c_k \cdots \text{ becomes } \tilde{\succ}_s : c_j \tilde{\succ}_s c_j^* \tilde{\succ}_s c_k \tilde{\succ}_s c_k^* \cdots$$

[15]In a problem with affirmative action for minority students, Hafalir et al. [2013] use a similar strategy of creating a new market by creating for each school c a "fictitious" school c' that favors minority students and analogously augmenting the student preferences over the original and fictitious schools. Their problem and resulting algorithm is formally distinct from ours, with the main difference being that the lower bounds in their model are only "soft" constraints, and so may or may not be satisfied at the final matching (see the introduction). This allows them to immediately apply the standard deferred acceptance algorithm directly to their augmented market. In our model, the minimum quotas are hard constraints, and so we must alter the deferred acceptance algorithm so that no more than e students are assigned to the extended schools, which ensures that at the end of our algorithm, the minimum quotas will always be satisfied.

The main issue that arises is how to assign the extended seats when more than e students have applied to them. To do so, we fix an ordering of the schools and let the schools accept students one by one in this order until e students have been accepted across all of the extended schools. The remaining students who have applied to the extended schools are rejected.

Extended-seat deferred acceptance
Fix$(\bar{q}_{c^*})_{c^* \in C^*}$ such that $\bar{q}_{c^*} \leq \tilde{q}_{c^*}$ and $\sum_{c^* \in C^*} \bar{q}_{c^*} \leq e$, and fix an ordering of the extended schools, which, for notational convenience, we denote $\{c_1^*, c_2^*, \ldots, c_m^*\}$. Let $\tilde{\mu}$ be the matching produced in the extended market.

(1) Begin with an empty matching such that $\tilde{\mu}(s) = \emptyset$ for all $s \in S$.
(2) Choose a student s who is not currently tentatively matched to any school. If no such student exists, end the algorithm.
(3) Let s apply to the most preferred school $\tilde{c} \in \tilde{C}$ according to $\tilde{\succ}_s$ that has not yet rejected him or her. If $\tilde{c}$ is a regular school, let school $\tilde{c}$ choose up to the $\tilde{q}_{\tilde{c}}$ highest-ranked students according to $\tilde{\succ}_{\tilde{c}}$ among those students who thus far have applied to $\tilde{c}$ but have not yet been rejected by $\tilde{c}$. School $\tilde{c}$ rejects any remaining students, and the algorithm returns to step 2. If $\tilde{c}$ is an extended school, proceed to step 4.
(4) In this step, we consider *all* extended schools. For each extended school $c^* \in C^*$, let S_{c^*} be the set of students tentatively held at c^* (i.e., those students who have applied to c^* but have not yet been rejected, including the student who applied in step 3). Let each school $c^* \in C^*$ choose up to the $\bar{q}_{c^*}$ highest-ranked students in S_{c^*} according to $\tilde{\succ}_{c^*}$; denote this set S'_{c^*}. Starting with a tentative match of S'_{c^*} for each school, let the schools choose, one by one, the best remaining student in its current applicant pool (i.e., those students in S_{c^*} that have not yet been chosen) unless either a school's true capacity $\tilde{q}_{c^*}$ is reached or the total number of students assigned to extended schools reaches the cap e. Formally, set $j = 1$ and:
   (a) If either (i) the number of students assigned to extended seats across all extended schools is equal to e, or (ii) for each extended school $c^* \in C^*$, the number of students chosen so far throughout step 4 is equal to $\min\{\tilde{q}_{c^*}, |S_{c^*}|\}$, then reject all remaining students not chosen by any extended school and return to step 2.
   (b) If not, let c_j^* choose its most preferred student in $S_{c_j^*}$ who has not yet been chosen, as long as the number of students chosen so far is strictly less than $\tilde{q}_{c^*}$. If $j < m$, increment j by 1. If $j = m$, set $j = 1$. Return to step 4(a).

The previous algorithm outputs a matching in the extended market $\tilde{\mu}$. We then take this outcome and map it to an outcome in the original market in the obvious way: if $\tilde{\mu}(s) = c$ or $\tilde{\mu}(s) = c^*$, then $\mu(s) = c$, and the final output of the ESDA algorithm is the matching μ.[16]

3.2. An Example of ESDA

We next present an example to show how the ESDA mechanism runs.

Example 1. There are five students $s_1, \ldots, s_5$ and three schools c_1, c_2, c_3. The preferences, priorities, and minimum and maximum quotas are shown in the following table:

[16]In the running of the mechanism, a choice must be made about how to set $\bar{q}_{c^*}$ for each c^*, which captures the number of students each c can accept before the round-robin picking procedure begins. This choice must be made so as to ensure that afterward, there are enough students remaining to satisfy the minimum quotas at all schools. One natural choice is $\bar{q}_{c^*} = 0$ for all $c^* \in C^*$, which corresponds to a situation in which each school is first allowed to accept a number of students equal to its minimum quota, and then schools continue to choose students one by one. However, nonsymmetric choices may also be used if we expect some schools to be more popular than others ex ante.

	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
	s_5	s_3	s_3
	s_3	s_4	s_4
	s_1	s_1	s_2
	s_2	s_2	s_5
	s_4	s_5	s_1
p	1	1	1
q	2	3	1

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$
c_2	c_2	c_1	c_2	c_1
c_1	c_1	c_2	c_3	c_2
c_3	c_3	c_3	c_1	c_3

To run ESDA, our extended market uses schools $C \cup C^* = \{c_1, c_2, c_3, c_1^*, c_2^*, c_3^*\}$. The maximum quotas are $\tilde{q}_{c_1} = \tilde{q}_{c_2} = \tilde{q}_{c_3} = 1$, and $\tilde{q}_{c_1^*} = 1, \tilde{q}_{c_2^*} = 2$ and $\tilde{q}_{c_3^*} = 0$. Note that there are no minimum quotas in the extended market. Recall that the ordering of the schools is $c_1 > c_2 > c_3$. The cap on the number of extended seats is $e = 2$. Set $\bar{q}_{c^*} = 0$ for all $c^* \in C^*$.

We additionally modify all students' preferences by inserting school c_j^* after school c_j. For example, the modified preferences of student s_1 are as follows:

$$\tilde{\succ}_{s_1} : c_2 \;\tilde{\succ}_{s_1} c_2^* \;\tilde{\succ}_{s_1} c_1 \;\tilde{\succ}_{s_1} c_1^* \;\tilde{\succ}_{s_1} c_3 \;\tilde{\succ}_{s_1} c_3^*.$$

This leads to the following extended market, where the changes are shown in red:

	$\tilde{\succ}_{c_1}$	$\tilde{\succ}_{c_1^*}$	$\tilde{\succ}_{c_2}$	$\tilde{\succ}_{c_2^*}$	$\tilde{\succ}_{c_3}$	$\tilde{\succ}_{c_3^*}$
	s_5	s_5	s_3	s_3	s_3	s_3
	s_3	s_3	s_4	s_4	s_4	s_4
	s_1	s_1	s_1	s_1	s_2	s_2
	s_2	s_2	s_2	s_2	s_5	s_5
	s_4	s_4	s_5	s_5	s_1	s_1
q	1	1	1	2	1	0

$\tilde{\succ}_{s_1}$	$\tilde{\succ}_{s_2}$	$\tilde{\succ}_{s_3}$	$\tilde{\succ}_{s_4}$	$\tilde{\succ}_{s_5}$
c_2	c_2	c_1	c_2	c_1
c_2^*	c_2^*	c_1^*	c_2^*	c_1^*
c_1	c_1	c_2	c_3	c_2
c_1^*	c_1^*	c_2^*	c_3^*	c_2^*
c_3	c_3	c_3	c_1	c_3
c_3^*	c_3^*	c_3^*	c_1^*	c_3^*

ESDA begins with s_1, s_2, and s_4 applying to school c_2 and students s_3 and s_5 applying to school c_1. Schools c_1 and c_2 tentatively accept s_5 and s_4, respectively. Everyone else is rejected.

Students s_1 and s_2 then apply to c_2^*, and s_3 applies to c_1^*. The extended schools then admit students from their applicant pools one by one. First, c_1^* admits its only applicant s_3, and then c_2^* admits student s_1 (since $s_1 \tilde{\succ}_{c_2^*} s_2$). At this point, two students have been admitted to extended schools, and so school c_2^* must reject the student it was tentatively holding, s_2.

Student s_2 continues by applying to c_1 but is rejected in favor of s_5, who is currently sitting at c_1. He or she then applies to c_1^*. We again allow the extended schools to admit students from their applicant pools one by one. This begins by c_1^* accepting s_3 (from its current applicant pool of $\{s_2, s_3\}$). We then move to school c_2^*, which again admits s_1. Once again, at this point two students have been admitted to extended schools, and so s_2, who is tentatively sitting at c_1^*, is rejected. He or she then proceeds to apply to c_3 and is admitted, and the algorithm ends. The output in the extended market is

$$\tilde{\mu} = \begin{pmatrix} c_1 & c_1^* & c_2 & c_2^* & c_3 & c_3^* \\ s_5 & s_3 & s_4 & s_1 & s_2 & \emptyset \end{pmatrix}.$$

Mapping this back to a matching in the original market, the final matching is

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_3, s_5\} & \{s_1, s_4\} & s_2 \end{pmatrix}.$$

3.3. Properties of ESDA

We now discuss the theoretical properties of the ESDA algorithm.[17]

THEOREM 3.1. *The ESDA mechanism is*
(*i*) *group strategyproof*
(*ii*) *fair.*

We show strategyproofness by first showing that no student can gain by misreporting his or her preferences in the extended market. Since this is a larger set of possible manipulations than in the original market, this means that no student can gain by misrepresenting his or her preferences in the original market either. At a formal level, the proof in the appendix relates our model to that of Kamada and Kojima [2015] and the matching with contracts model of Hatfield and Milgrom [2005]. Effectively, we can think of all of the extended seats as being controlled by one single "umbrella" school, which has a capacity of e. A contract in the Hatfield-Milgrom terminology then specifies a student and the specific extended school to which he or she is assigned, and these contracts are chosen according to the choice function of the umbrella school, which is determined by the choice of $\bar{q}_{c^*}$ and the ordering of the extended schools. Hatfield and Milgrom [2005] show that if the choice functions of all schools satisfy a key substitutes condition (as well as a condition they call the law of aggregate demand), then the DA algorithm in the matching with contracts model is strategyproof, while Hatfield and Kojima [2009] show that it is in addition group strategyproof (see also Hatfield and Kominers [2012], who obtain the same result in a more general model). The standard schools obviously have substitutable choice functions, and so the fact that the umbrella school controlling all of the extended seat contracts also has a substitutable choice function means that the mechanism is group strategyproof.[18]

For fairness, note that if a student s is rejected from a regular school, it must be because that school is filled to its capacity with higher-ranked students. Since the ranking of students assigned to a school only improves as the algorithm continues, at the end of the algorithm, s must be lower ranked than every student assigned to this school, and hence cannot form a blocking pair with any of these students. If, on the other hand, s is rejected from an extended school, it must be because either (1) that extended school is filled to capacity with higher-ranked students, in which case s cannot form a blocking pair for the same reason given previously, or (2) e other students are tentatively assigned to extended schools. For case (2), when the e^{th} student is accepted, the school from which student s is rejected is tentatively holding some (possibly empty) set of students who are higher ranked than s. Because of the fixed order in which extended schools are allowed to accept new students, no student ranked lower than any of the students currently held at this school will ever be admitted, and so s cannot form a blocking pair.

Because ESDA is fair, the impossibility result of Section 2.3 implies that it cannot be fully nonwasteful. However, there is still an intuitive sense in which ESDA is an improvement on ACDA, as it assigns the extended seats more flexibly, using information

[17]In a short extended abstract appearing in the Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems [Ueda et al. 2012], we discuss an alternative extended-seat DA algorithm. Though the terminology is the same, the two algorithms run entirely differently and satisfy different properties.

[18]Technically, a condition called *irrelevance of rejected contracts* [Aygün and Sönmez 2013] or *path independence* [Fleiner 2003] is necessary for this to hold. Aygün and Sönmez [2013] show that substitutability and the law of aggregate demand are sufficient this condition. The choice functions in our model satisfy both substitutability and the law of aggregate demand, and so strategyproofness follows. See the full proof in the appendix for further details.

contained in the submitted preferences. To see this, consider again the example from Section 2.4 where ACDA produced the outcome in the boxes:

$\succ_{s_1}$	$\succ_{s_2}$
c_2	$\boxed{c_1}$
$\boxed{c_3}$	c_2
c_1	c_3

Note that s_1 would prefer a seat at c_2. ESDA does indeed assign s_1 to c_2 and s_2 to c_1, thus making s_1 strictly better off without harming s_2.

In practice, if the school district had implemented the ACDA allocation, it may then get a request from student s_1 to be moved to c_2 and can grant this request without any objection from student s_2, who is already being assigned his or her favorite school. In a larger market, it is easy to construct matchings at which, if a school district grants a request for one student to move, a second student then requests the seat vacated by the first, a third student requests the seat vacated by the second, and so forth. Such a chain can end in one of two ways: (1) all such requests are granted without violating any minimum quotas or (2) eventually the school district receives a request that must be denied because otherwise the minimum quota would be violated. In case (2), since the school district cannot grant all requests while still ensuring feasibility, it may instead opt for a policy of not granting *any* student requests for an empty seat, as this will lead to an avalanche of other requests, not all of which can be granted.

The following definitions formalize these ideas. Say student s **weakly claims a seat at school** c if there exists a chain of students and schools $(c^0, s^1, c^1, s^2, \ldots, c^{J-1}, s^J)$, $J \geq 1$, such that (1) $|\mu(c^0)| < q_{c^0}$, (2) $c^{j-1} \succ_{s^j} c^j$, (3) $\mu(s^j) = c^j$, and (4) $c^{J-1} = c$ and $s^J = s$. When $J = 1$, weakly claiming a seat is equivalent to claiming an empty seat.

Definition 3.2. Given a matching μ, let $Z(\mu)$ be the set of students who weakly claim a seat at some school. If $|\mu(\mu(s))| > p_{\mu(s)}$ for all $s \in Z(\mu)$, then matching μ is **strongly wasteful**. If matching μ is not strongly wasteful, we say it is **weakly nonwasteful**.

Mechanism χ is weakly nonwasteful if it produces a weakly nonwasteful matching for every possible preference profile. Otherwise, χ is strongly wasteful.

THEOREM 3.3.
(*i*) *ACDA is strongly wasteful.*
(*ii*) *ESDA is weakly nonwasteful.*

Consider again the example in Section 2.4. The allocation produced by ACDA is strongly wasteful because it is possible to grant student s_1's request to move without denying the request of student s_2, who is already being assigned his or her favorite school (formally, $Z(\mu) = \{s_1\}$ and $|\mu(c_3)| > p_{c_3}$); this does not happen under ESDA, where student s_1 is assigned to c_2 (so $Z(\mu) = \emptyset$). Theorem 3.3 formalizes the intuition that ESDA is less wasteful than ACDA. In Section 5, we also study this question with simulations and find that quantitatively, ESDA wastes far fewer seats than ACDA and is preferred by the students. These results suggest that markets using ACDA would be better served switching to ESDA, since ESDA satisfies the same fairness and incentive properties as ACDA while wasting fewer seats and increasing student welfare.

4. MULTISTAGE DA (MSDA)

4.1. Precedence Lists

The ESDA algorithm introduced in the last section satisfies the desirable properties of strategyproofness and fairness, and while it satisfies a stronger nonwastefulness property than artificial caps DA, ESDA will still result in some wasted seats. If nonwastefulness is a bigger concern to policymakers than fairness, then ESDA may not be a desirable mechanism to use. In this section, we define an alternative mechanism, called MSDA, which will be strategyproof and nonwasteful, though it will satisfy only a weaker definition of fairness.

Since we do not want to give up fairness entirely, we must introduce a new way to think about the concept. The problem is that any mechanism that is strategyproof and nonwasteful tends to produce too many blocking pairs according to the standard definition. Therefore, our new notion of fairness must declare some of these potential blocking pairs as invalid.

To see the crux of the problem, consider again the example from the impossibility result:

	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
	s_2	s_2	s_1
	s_1	s_1	s_2
p_c	1	0	0
q_c	1	1	1

$\succ_{s_1}$	$\succ_{s_2}$
$\textcircled{c_2}$	$\boxed{c_3}$
c_3	c_2
$\boxed{c_1}$	$\textcircled{c_1}$

The boxed and circled matchings are the only two that are both feasible and nonwasteful. In the boxed matching, s_1 forms a blocking pair with school c_3. In the circled matching, on the other hand, s_2 forms a blocking pair with c_1. However, the school district must somehow choose which student will be assigned to the undesirable school c_1 (i.e., must choose between the boxed and circled matchings) and then not allow this student to form a blocking pair. If the school district chooses the boxed matching, it is effectively giving precedence to student s_2 *at a district-wide level*; that is, the district is prioritizing $s_2 \succ s_1$, while if it chooses the circled matching, it is giving precedence to s_1, that is, prioritizing $s_1 \succ s_2$.

To formalize this idea, we introduce a district-wide **precedence list** $\succ_{PL}$ that ranks all students. Without loss of generality, we let $s_1 \succ_{PL} s_2 \succ_{PL} \cdots \succ_{PL} s_n$. Note that the precedence list does not correspond to any individual school. We then say that (s, c) form a ***PL*-blocking pair** if the following conditions are met for some $s' \in \mu(c)$:

(i) $c \succ_s \mu(s)$
(ii) $s \succ_c s'$ and $s \succ_{PL} s'$.

Using this definition, we can then define an analogous notion of fairness that eliminates all *PL*-blocking pairs.

Definition 4.1. A matching μ is **PL-fair** if no student–school pair (s, c) can form a *PL*-blocking pair. A mechanism χ is **PL-fair** if $\chi(\succ_S)$ is a *PL*-fair matching for every preference profile $\succ_S$.

Note that this is very similar to the standard definitions of blocking pairs and fairness, with the additional requirement that s's envy toward s' is only justified if he or she is ranked higher than s' on both the school-specific $\succ_c$ and the district-wide $\succ_{PL}$. In the previous example, if $s_1 \succ_{PL} s_2$, then the circled matching is *PL*-fair, since s_2 will no longer have justified envy toward s_1.

How to choose the precedence list will depend on the details of the setting, but in many cases, there are natural candidates. For example, the military has an Order of Merit List that ranks all cadets based on academic performance, physical fitness, and military performance. In many countries with centralized college admissions, students take a common entrance exam, which is a natural candidate for $\succ_{PL}$ [Abizada and Chen

2014].[19] In the Kyushu University problem of matching students to laboratories mentioned in the introduction, a listing of students based on grade point average is used.

In some cases, random numbers may also be used to form the district-wide precedence list. For example, in Boston, all families are assigned a random number (which can be thought of as $\succ_{PL}$), and 50% of seats at each school are reserved for students with walk zone priority at the individual school, while the other 50% are assigned according to $\succ_{PL}$ [Dur et al. 2013]. Thus, a student can effectively only justifiably envy another at a school c if he or she has higher walk-zone priority (is higher on $\succ_c$) **and** a higher random number (is higher on $\succ_{PL}$).[20]

The definition here is consistent with the normative interpretation of school priorities discussed in Section 2: all else equal, the school district desires to rely on the school priority lists as much as possible (perhaps due to political constraints) and only use $\succ_{PL}$ when necessary.[21] The goal of the MSDA mechanism defined later is to strike a similar balance, by relying on the school priority lists as much as possible and only deviating when necessary to satisfy the quotas.

4.2. Definition of MSDA

With the precedence list in hand, we can now describe our second mechanism, the **multistage deferred acceptance algorithm (MSDA)**. As the name suggests, MSDA is run in several stages. At the beginning of any stage, we temporarily reserve a group of students from the market and run standard DA on the remaining submarket. The number of students participating in the submarket is never too many to jeopardize the feasibility of the overall match; no matter how they are allocated, the sum of the minimum quotas remaining after the given stage will never exceed the number of students that remain unmatched after that stage. The assignments from the given stage are made final, and we reduce the minimum and maximum quotas accordingly. Now, we are left with a subproblem of unmatched students and updated minimum and maximum quotas. We repeat the process until all students are assigned. To determine what students will be reserved, we must use $\succ_{PL}$.

To formally define MSDA, we return to the original market $(S, C, p, q, \succ_S, \succ_C)$. Recall that without loss of generality, the precedence list ranks students $s_1 \succ_{PL} s_2 \succ_{PL} \cdots \succ_{PL} s_n$. Start by setting $R^0 = S$, $p_c^1 = p_c$, and $q_c^1 = q_c$ for all $c \in C$. Let $r^1 = \sum_{c\in C} p_c^1$ be the number of students that will be reserved in R^1.

Multistage deferred acceptance

Stage. $k \geq 1$

(1) Set $R^k = \{s_{n-r^k+1}, s_{n-r^k+2}, \ldots, s_n\}$; that is, R^k is the set of r^k students with the lowest priority according to $\succ_{PL}$.
   (a) If $R^{k-1} \setminus R^k \neq \emptyset$, run the standard DA mechanism on the students in $R^{k-1} \setminus R^k$ with maximum quotas for the schools equal to $(q_c^k)_{c\in C}$.

[19] See also Perach et al. [2007] and Perach and Rothblum [2010] for the use of precedence lists in problems of allocating university housing.

[20] An alternative (and equivalent) way to say this is that a student s's envy of the assignment of another student s' can be denied on the basis of $s' \succ_c s$ (s' has higher walk-zone priority than s) **or** $s' \succ_{PL} s$ (s' has a higher random number than s).

[21] The description of the priority system in Boston in the previous paragraph is also consistent with a normative viewpoint, as the school district views its approach as a desirable middle ground between completely neighborhood-based school assignment and equal access to all seats for everyone.

   (b) If $R^{k-1} \setminus R^k = \emptyset$, run the standard DA mechanism on the students in R^k with maximum quotas for the schools equal to $(p_c^k)_{c\in C}$.[22]
(2) Let μ^k be the matching from step $k-1$, and remove all students assigned from the market. If all students have been tentatively assigned a school, end the algorithm. If not, proceed to step 3.
(3) Define new quotas for each school:
   (a) $q_c^{k+1} = q_c^k - |\mu^k(c)|$,
   (b) $p_c^{k+1} = \max\{0, p_c^k - |\mu^k(c)|\}$.
   (c) $r^{k+1} = \sum_{c\in C} p_c^{k+1}$.
(4) Move to stage $k+1$.

Once the algorithm is completed (say, after stage K), we are left with a set of stage-specific matchings $\mu^1, \ldots, \mu^K$. The final matching output by the algorithm is the matching μ defined by $\mu(c) = \cup_{k=1}^K \mu^k(c)$ for all $c \in C$ and $\mu(s) = \mu^{k(s)}(s)$ for all $s \in S$, where $k(s)$ is the stage at which student s participated.

4.3. An Example of MSDA

We now provide an example of how MSDA works.

Example 2. We use the same instance of Example 1 to illustrate how MSDA works. Since the sum of the minimum quotas is $\sum_{c\in C} p_c = 3$, we temporarily remove students s_3, s_4, and s_5 according to $\succ_{PL}$. We then run the standard DA mechanism with no minimum quotas on students s_1 and s_2. At the end of this first stage, the assignments are as follows:

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ \emptyset & \{s_1, s_2\} & \emptyset \end{pmatrix}.$$

Thus, there are three students remaining, and neither c_1 nor c_3 have reached his or her minimum quotas. We temporarily remove s_4 and s_5 and run DA on s_3 alone. At the end of this stage, the assignments are as follows:

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ s_3 & \{s_1, s_2\} & \emptyset \end{pmatrix}.$$

Now, only c_3 has not reached its minimum quota, so we reserve s_5 and run DA on s_4 alone, who chooses to attend c_2. Finally, the only student remaining is s_5, and the minimum quota of 1 at c_3 still needs to be filled, so s_5 is assigned c_3. The final outcome is then

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ s_3 & \{s_1, s_2, s_4\} & s_5 \end{pmatrix}.$$

In the previous, we determine the number of students who must be removed in each round as the sum of the minimum quotas (step 3(c) of the algorithm). We may, however, be able to remove even fewer students and still ensure feasibility. For example, assume there are 15 students and 10 schools, with all minimum quotas equal to 1

[22]Most rounds will use step 1(a), except for (possibly) the last round of the algorithm. Step 1(b) ensures that the algorithm finishes. In 1(a), if, once we remove enough students to satisfy the minimum quotas, there are still students left ($R^{k-1} \setminus R^k \neq \emptyset$), we run the standard DA mechanism on those students who were not removed. However, if it happens at any stage that the number of students remaining is *exactly* equal to the minimum quotas, then "removing" these students would leave us with an empty set and the algorithm would run indefinitely. This explains step 1(b). Also note that the first time step 1(b) is executed will be the last round of the algorithm.

and all maximum quotas equal to 2. Then, we actually need only to remove four students, not 10, because no matter how the first 11 students are allocated, the minimum quotas of at least six schools must be satisfied, and the remaining four students can fill the minimum quotas of the other four schools. We have thus developed a dynamic programming-based method to precisely calculate the smallest possible value for r^k. This allows as many students as possible to participate in any given stage k, which makes MSDA close to standard DA (in the extreme case in which all students are allowed to participate in stage 1, MSDA = DA). This procedure is described in Appendix C. The MSDA algorithm using this procedure is exactly the same as described earlier, except that step 3(c) is replaced with the optimized value of r^k. We use this method of calculating r^k when running the simulations in Section 5.

4.4. Properties of MSDA

We next turn to a discussion of the theoretical properties of MSDA. Theorem 4.2 shows that MSDA is group strategyproof and nonwasteful. Since MSDA is nonwasteful, the impossibility result of Section 2.3 immediately implies that it will not be fair. However, it will be *PL*-fair.[23]

THEOREM 4.2. *The MSDA mechanism is*
(*i*) *group strategyproof,*
(*ii*) *nonwasteful, and*
(*iii*) *PL-fair.*

Strategyproofness follows because within each stage the standard DA algorithm is strategyproof, and, fixing the preferences of the other students, no student s can affect the stage at which he or she participates in the algorithm. Nonwastefulness holds because the only time a student would be unable to get into a school with empty seats is the last round of the algorithm, in which case he or she is assigned to a school that will be filled exactly to its minimum quota and thus cannot be feasibly moved. For *PL*-fairness, note that if a student s is rejected from a school c, then school c was filled in the round s participates with students ranked higher than s according to $\succ_c$, and all students assigned to c in earlier rounds were ranked higher than s according to $\succ_{PL}$, and so s cannot form a *PL*-blocking pair with c.

While nonwasteful and fair matchings may not exist, nonwasteful and *PL*-fair matchings always exist (since MSDA finds them). *PL*-fairness is a weaker concept in that it limits the number of blocking pairs students are allowed to form, but it is the price of nonwastefulness. We attempt to quantify this price in two ways: theoretically and with computer simulations. First, in Section 5, we use simulations to count the number of standard blocking pairs (in the sense of Definition 2.1) the MSDA mechanism will produce. Second, we ask theoretically whether we can improve upon *PL*-fairness while still retaining strategyproofness and nonwastefulness. In Appendix B, we formalize a notion that allows us to rank mechanisms according to fairness by quantifying how many (standard) blocking pairs the school district must declare as illegitimate. The concept, which we call σ-fairness, is a generalization of the standard definition of fairness from Section 2 and the current definition of *PL*-fairness. Both are special cases, with *PL*-fairness being the least fair and the standard definition being the most fair. While we know from the impossibility that a nonwasteful mechanism will not be able to reach the highest level of σ-fairness, the question is whether it is possible for a nonwasteful mechanism to achieve some intermediate level of σ-fairness above

[23]This can be seen in the previous example, as s_5 has justified envy toward s_3, since he or she would prefer to attend c_1 and $s_5 \succ_{c_1} s_3$. Note, however, that $s_3 \succ_{PL} s_5$, and so this matching is *PL*-fair.

PL-fairness. We show in the appendix that this is unfortunately not the case: any nonwasteful mechanism is at most *PL*-fair (strategyproofness is not used in the proof). Thus, in this sense, MSDA is optimally fair in the class of nonwasteful mechanisms.

4.5. Relationship to the Serial Dictatorship

A popular mechanism in many assignment markets without school-specific priorities or minimum quotas is the **serial dictatorship**. Essentially, the serial dictatorship fixes an ordering of the students (e.g., $\succ_{PL}$) and lets the students choose, one by one in this order, their most preferred school that has seats remaining. In the standard model with no priorities or minimum quotas, the serial dictatorship is strategyproof and nonwasteful (fairness is not an issue when there are no school-specific priorities). When school priorities do exist, it is often argued that the serial dictatorship is a poor choice of mechanism because it completely ignores the school priorities, and hence is extremely unfair, as it will produce a large number of students with justified envy. However, as we have seen, when minimum quotas are present, to achieve nonwastefulness, we must give up on fairness, at least in its strongest form. The question arises, then, of how a serial-dictatorship-type mechanism will perform in a market with school priorities and minimum quotas.

First, we provide a formal definition of the serial dictatorship, modified to accommodate the minimum quotas.[24]

Serial dictatorship with minimum quotas (SD)
Fix an ordering of the students $\succ_{PL}$ (without loss of generality, we assume $s_1 \succ_{PL} s_2 \succ_{PL} \cdots \succ_{PL} s_n$). Let μ^0 be an empty matching, $p_c^1 = p_c$, $q_c^1 = q_c$, and $r^1 = \sum_{c \in C} p_c^1$. Proceed to **Stage** 1.

Stage k.

(1) If $n - k \geq r^k$, that is, the number of students still unassigned (excluding student s_k) is greater than or equal to r^k (i.e., the total number of minimum quotas that are not filled yet), then for student s^k, choose his or her most preferred school c where $q_c^k > 0$.
(2) If not, for student s^k, choose his or her most preferred school c where $p_c^k > 0$.
(3) Obtain μ^k by adding (s^k, c) to μ^{k-1}. If $n = k$, return μ^k as the final matching and end the mechanism. Otherwise, let $p_c^{k+1} = \max(0, p_c^k - 1)$, $q_c^{k+1} = q_c^k - 1$, and for $c' \neq c$, let $p_{c'}^{k+1} = p_{c'}^k$, $q_{c'}^{k+1} = q_{c'}^k$. Let $r^{k+1} = \sum_{c \in C} p_c^{k+1}$. Proceed to **Stage** $k + 1$.

Intuitively, the serial dictatorship with minimum quotas works exactly like the standard serial dictatorship, assigning students one at a time to their most preferred schools with seats remaining, up until the point at which there are exactly the same number of students remaining as there are minimum quota seats left to be filled. At this point, the mechanism departs from the standard serial dictatorship, by restricting the set of schools that the remaining students can choose from to be only those that have not yet reached their minimum quotas; otherwise, the feasibility constraints will not be satisfied.

[24]The serial dictatorship is a simple mechanism implemented in myriad settings, both formal and informal, in which a group of objects needs to be allocated to a group of agents without the use of money. For a formal analysis of the standard serial dictatorship (without minimum quotas), see Abdulkadiroğlu and Sönmez [1998]. To our knowledge, we are the first to formally analyze this extension of the serial dictatorship to the case of minimum quotas. For an analysis of a different type of minimum quota serial dictatorship that allows for school closures, see Monte and Tumennasan [2013].

THEOREM 4.3. *The serial dictatorship with minimum quotas is*
(*i*) *group strategyproof,*
(*ii*) *nonwasteful, and*
(*iii*) *PL-fair.*

Comparing this result with Theorem 4.2, we see that both MSDA and SD satisfy the same properties.[25] The next question is whether there is any reason to pick one mechanism over the other. Recall that in order to achieve nonwastefulness, it is necessary to weaken fairness. *PL*-fairness allows some justified envy (based only on the school priorities) to form. When it is not possible to eliminate all justified envy, a useful criterion to use to rank mechanisms is the number of students with justified envy. The fewer such students there are, the fewer complaints/attempts to circumvent the match there will be, leading to a more successful market over the course of many years. For this reason, several papers have used this as a useful metric to compare mechanisms. For example, given fixed preferences and priorities, Biró et al. [2010] and Hamada et al. [2014] search for an allocation that minimizes the number of blocking pairs, while Abdulkadiroğlu et al. [2009] use the number of blocking pairs produced as a way to rank mechanisms by their degree of stability in the context of the New York City High School Match. Motivated by this, we ask whether it is possible to rank SD and MSDA based on the amount of blocking pairs (equivalently, justified envy) produced.

Intuitively, one would expect MSDA to outperform SD, because SD ignores the school-specific priorities completely, while MSDA tries to use them as much as possible. The strongest possible result would be to say that MSDA produces fewer students with justified envy than SD for every possible instance of a matching market (i.e., for all possible preferences and priorities). However, the examples that follow show that this is not the case; for some instances, MSDA will produce fewer, while for others, SD will.[26]

Example 3. Consider a market with $S = \{s_1, s_2, s_3, s_4\}$,$C = \{c_1, c_2, c_3\}$, $p = (0, 0, 2)$, and $q = (1, 1, 3)$. Let $\succ_{PL}$ be such that $s_1 \succ_{PL} s_2 \succ_{PL} s_3 \succ_{PL} s_4$. First, consider the following preferences and priorities:

$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
s_2	s_2	s_3
s_1	s_3	s_4
s_3	s_4	s_2
s_4	s_1	s_1

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
c_1	c_1	c_2	c_2
c_2	c_2	c_3	c_3
c_3	c_3	c_1	c_1

The matching produced by SD is

$$\mu^{SD} = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & \{s_3, s_4\} \end{pmatrix},$$

while the matching produced by MSDA is

$$\mu^{MSDA} = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_1 & \{s_3, s_4\} \end{pmatrix}.$$

It is simple to check that under μ^{SD}, only one student s_2 has justified envy (toward s_1), while under μ^{MSDA}, there are two students s_3, s_4 with justified envy (both toward s_1).

[25]We thank an anonymous referee for suggesting this result.

[26]We thank an anonymous referee for asking this question. We focus only on the comparison between SD and MSDA because we have already shown that ESDA produces zero students with justified envy (at the price of nonwastefulness).

For other profiles, SD may produce more students with justified envy than MSDA. Consider the following profile:

$\tilde{\succ}_{c_1}$	$\tilde{\succ}_{c_2}$	$\tilde{\succ}_{c_3}$	$\tilde{\succ}_{s_1}$	$\tilde{\succ}_{s_2}$	$\tilde{\succ}_{s_3}$	$\tilde{\succ}_{s_4}$
s_2	s_2	s_3	c_1	c_1	c_1	c_1
s_3	s_3	s_4	c_2	c_2	c_3	c_3
s_4	s_4	s_2	c_3	c_3	c_2	c_2
s_1	s_1	s_1				

The matching produced by SD is

$$\tilde{\mu}^{SD} = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & \{s_3, s_4\} \end{pmatrix},$$

while the matching produced by MSDA is

$$\tilde{\mu}^{MSDA} = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_1 & \{s_3, s_4\} \end{pmatrix}.$$

It is simple to check that under $\tilde{\mu}^{SD}$, there are three students $s_2, s_3,$ and s_4 with justified envy (all toward s_1), while under $\tilde{\mu}^{MSDA}$, no students have justified envy.

The previous example shows that there is no way to uniformly rank MSDA and SD with respect to the number of students with justified envy for every profile of preferences and priorities. However, there are two other reasonable metrics that will yield positive results: the worst case and the average case. Under both metrics, we will find that MSDA outperforms SD.

We start here by analyzing the worst case; the average case will be studied using simulations in Section 5. Since it is impossible to eliminate all justified envy because MSDA and SD are nonwasteful, we ask what is the worst possible outcome that may arise from MSDA or SD, in the sense of producing the greatest number of students with justified envy.[27] For instance, the second part of Example 3 shows that the worst case for SD can be quite poor: three of the four students have justified envy.

Formally, let the function $JE^{SD}(\succ_S, \succ_C)$ denote the number of students with justified envy under SD at preference-priority profile $(\succ_S, \succ_C)$, and define

$$W^{SD} = \max_{(\succ_S, \succ_C)} JE^{SD}(\succ_S, \succ_C).$$

In words, W^{SD} is the worst-case number of students with justified envy under SD over all preference-priority profiles. Define $JE^{MSDA}(\succ_S, \succ_C)$ and W^{MSDA} similarly for the MSDA algorithm. Recall that r^1 is the number of students reserved in the first stage of the MSDA algorithm, where here, r^1 is the optimized value that is determined using the dynamic programming procedure described in Appendix C. Finally, let $\underline{q} = \min_{c \in C} q_c$.

THEOREM 4.4. *In the worst case, the number of blocking pairs produced by MSDA is weakly less than that produced by SD:* $W^{MSDA} \leq W^{SD}$. *If, in addition,* $r^1 < n - \underline{q}$, *then MSDA strictly outperforms SD in the worst case:* $W^{MSDA} < W^{SD}$.

Thus, this theorem says that in the worst case, MSDA always does at least as well as SD, and, under a small additional condition that will likely be satisfied in practice, MSDA will strictly outperform SD. Using the definition of r^1 and recalling that

[27] It is obvious that the best possible outcome is no students with justified envy. While this may occur under some profiles, we know from Section 2 that this will not hold under all profiles when we require nonwastefulness.

$e = n - \sum_{c \in C} p_c$ is the number of excess students above the sum of the minimum quotas, a simple sufficient condition for MSDA to strictly outperform SD is $\underline{q} < e$; that is, the smallest school is not large enough to accommodate all of the excess students. Intuitively, this will hold in markets where there are many more students than can be accommodated in any one school alone. In fact, in the proof of Theorem 4.4, we show that $W^{MSDA} \leq r^1 \leq n - \underline{q} \leq W^{SD}$. Therefore, the difference $(n - \underline{q}) - r^1$ is a lower bound on how much worse the worst case of SD will be. For example, consider the market described in Section 4.3 of 15 students and 10 schools, with $p_c = 1$ and $q_c = 2$ for all $c \in C$. Then, as argued there, the optimized value of r^1 is $r^1 = 4$, while $n - \underline{q} = 13$, and so the worst case under SD will have at least nine more students (out of 15 total) with justified envy than the worst case under MSDA. As another example, we will see in the next section a market of $n = 400$ students where $W^{MSDA} = 7$, while $W^{SD} = 385$, and so the worst case for SD can indeed be much worse than that for MSDA.

5. SIMULATIONS

To summarize our results thus far, we have provided two new strategyproof mechanisms that each satisfy one of our important normative axioms: institutions who place a higher weight on fairness should opt for ESDA, while those that find nonwastefulness paramount should opt for MSDA. Since both are important concerns, it is still desirable to satisfy the weakened property as much as possible. In this section, we attempt to quantify how far the mechanisms are from the first-best definitions of fairness and nonwastefulness using simulations. That is, even though a school district may opt for a nonwasteful mechanism, it still likely desires to eliminate as many blocking pairs as possible, and vice versa for a school district that opted for a fair mechanism. The simulations will also serve to compare our mechanisms with other popular approaches, namely, standard DA, ACDA, and SD.

We consider a market of $n = 400$ students and $m = 50$ schools. The maximum quotas are equal to 15 at each school, while the minimum quotas are the same across schools and will be varied from 1 to 7.[28] For the ACDA mechanism, we choose artificial caps of $q^*_c = 8$ for all c.[29] We choose these parameters to keep the simulations tractable while allowing us to explore the effects of changes in the size of the minimum quota over the full range of possibilities (from $p_c = 1$ to $p_c = 7$) while keeping the maximum quotas fixed.

In order to study how correlation in student preferences affects outcomes, the student preferences are constructed using a linear combination of a common value vector of cardinal utilities and a private value vector for each student. For each student, we randomly draw a private vector of cardinal utilities uniformly at random from the set $[1, 50]^m$. Label this vector u_s. We also draw a vector of common cardinal utilities $u_C = (u_{c_1}, \ldots, u_{c_m})$ (how this is done will be explained shortly) and then construct the cardinal preferences for student s as $\alpha u_C + (1 - \alpha)u_s$ for some $\alpha \in [0, 1]$. Higher values of α thus correspond to more correlation in student preferences. We then convert these cardinal preferences into ordinal preferences for use in the mechanism. School priorities $\succ_c$ are drawn uniformly at random, and the precedence list is without loss of generality set at $s_1 \succ_{PL} \cdots \succ_{PL} s_n$.[30]

We next discuss how we construct the common component of student preferences u_C. We do this in one of two ways. For the *uniform case,* we set $u_{c_j} = 50 - (j - 1)$. For the

[28]Minimum quotas of 7 at each school is the largest possible (symmetric) value that still allows for some flexibility. If $p_c = 8$ at all c, then $\sum_{c \in C} p_c = 400 = n$, and standard DA can be used.
[29]Similar to the previous footnote, this is the largest (symmetric) value for the artificial caps that will ensure a feasible matching.
[30]See also Hafalir et al. [2013], who run simulations in a similar manner.

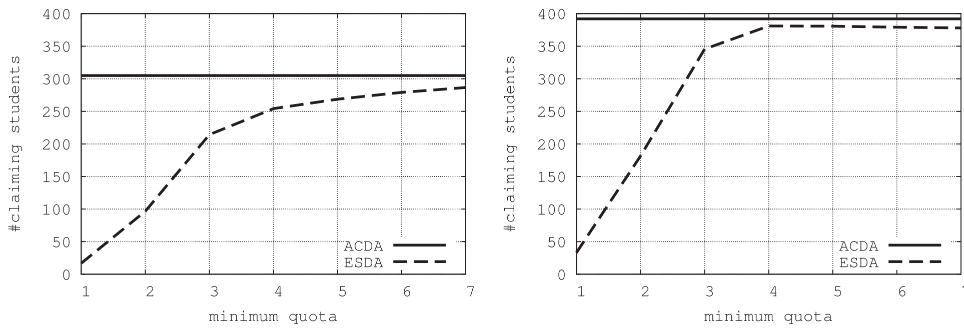

Fig. 1. Nonwastefulness as a function of minimum quota for $\alpha = 0.3$ (left) and $\alpha = 0.6$ (right). The results shown are for the exponential case.

exponential case, we set $u_{c_j} = 50e^{-(j-1)}$.[31] We consider the exponential case as a way to simulate situations in which there are some schools that are very popular, which is common in many matching settings. In particular, schools with lower indices j will tend to be much more popular than schools with higher indices.

Experiment 1: Nonwastefulness. We first study mechanisms by their level of wastefulness. As we have shown, MSDA is nonwasteful, so the major comparison here is between ESDA and ACDA. Recall that formally we were able to show that ESDA was weakly nonwasteful, while ACDA was not. Here, however, we compare the mechanisms by counting the number of students who claim an empty seat at some school. For all correlations and all values of the minimum quota, ESDA had fewer students claiming an empty seat than ACDA. For the uniform case, while ESDA does indeed unambiguously outperform ACDA, the difference is not as large and does not vary much with the quota (data not shown). The results for the exponential case are shown in Figure 1, where we can see that the size of the minimum quotas has an effect on how wasteful ESDA is: when the minimum quotas are large, ESDA is close to (but still outperforms) ACDA; as the minimum quotas decrease, ESDA begins to significantly outperform ACDA by wasting far fewer seats. This makes intuitive sense, as when the minimum quotas are lower, there is more flexibility, and so the rigid artificial caps begin to perform significantly worse.

Figure 2 analyzes how different correlations α affect the number of students claiming an empty seat (for these simulations, we fix $p_c = 3$ at all schools). As expected, the number of students claiming an empty seat increases as the correlation increases. Again, we can see that ESDA outperforms ACDA for all correlations, for both the uniform and exponential cases, with the difference between the two increasing as the correlation is decreased.

Experiment 2: Rank Distribution. Another metric that may be of interest is the distribution of the student rankings of the school they received. We study this by plotting cumulative distribution functions of the (average) number of students who received their k^{th} or higher-ranked school under each mechanism. For example, in Figure 3, under MSDA with $\alpha = 0.3$, about 40% of students get their first choice, about 65% of students get their first or second choice, 75% get their first or second or third choice, and so on. This is an important statistic because many school districts

[31]These common vectors could also be drawn randomly from the uniform/exponential distributions, but, since all students have the same common component, the differences in ordinal preferences are driven by the idiosyncratic component, and choosing the common component in this way does not have qualitative effects on the results (while at the same time simplifying the coding and debugging).

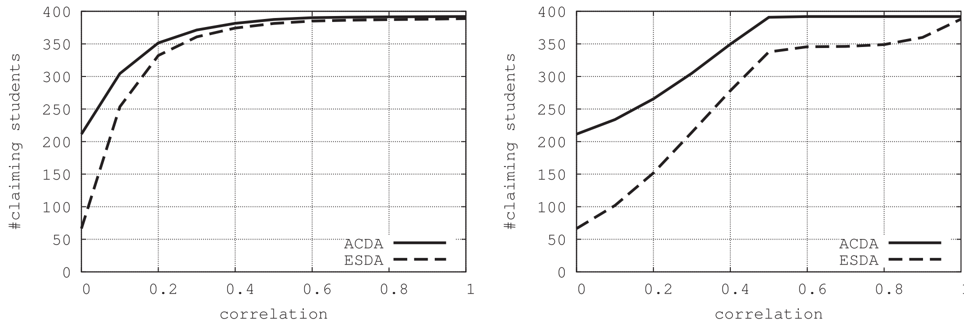

Fig. 2. Nonwastefulness as a function of correlation α for the uniform case (left) and exponential case (right). The results shown are for $p_c = 3$ at all $c \in C$.

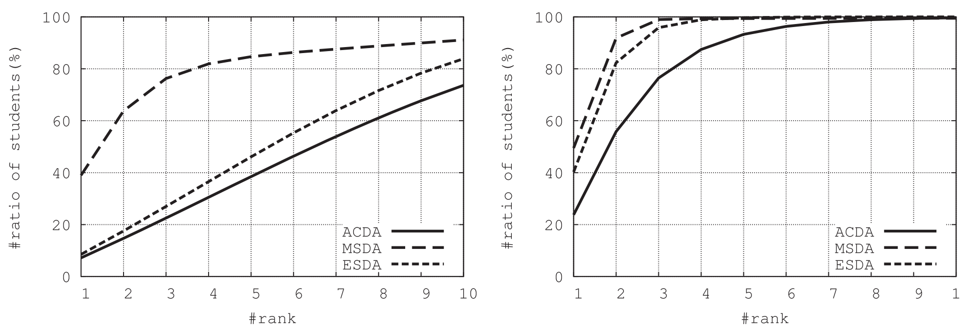

Fig. 3. CDFs of student welfare for the uniform case (left) and exponential case (right). The results shown are for $\alpha = 0.3$ and $p_c = 3$. For visualization purposes, we only plot the rank distributions up to $k = 10$, but the CDFs of ESDA and MSDA do in fact dominate ACDA for all $k = 1, \ldots, 50$.

and other organizations release data on rank distributions as a measure of mechanism performance. Thus, if the rank distribution of one mechanism first-order stochastically dominates another, then, all else equal, a school district would likely prefer to use the stochastically dominant mechanism. Additionally, because our mechanisms are strategyproof, we can be confident that under our mechanisms, the rank distributions will be an accurate reflection of the true student preferences.

Figure 3 shows these plots for both the uniform and exponential cases (once again, we set $p_c = 3$ at all schools). It can be seen that both ESDA and MSDA do in fact first-order stochastically dominate ACDA. This dominance holds when α and p_c are varied as well. The magnitude of this dominance becomes larger as the correlation in the student preferences (α) is decreased, and is also larger for the exponential case. Thus, according to this metric, students will on average unambiguously prefer our mechanisms to ACDA, in the first-order stochastic dominance sense described previously. Intuitively, this is happening because ESDA and MSDA are allocating the extra seats more flexibly, taking into account student demand, and so are able to provide students with higher choices, while ACDA simply eliminates these seats ex ante, forcing these students to be assigned to lower-ranked schools. Thus, if a school district is interested in producing a good rank distribution, it should choose one of our mechanisms over ACDA.

Experiment 3: Justified Envy. Since ACDA and ESDA are fair, they will always produce zero students with claims of justified envy. Since MSDA is nonwasteful, we know that it cannot eliminate all justified envy (in the sense of Definition 2.1), but

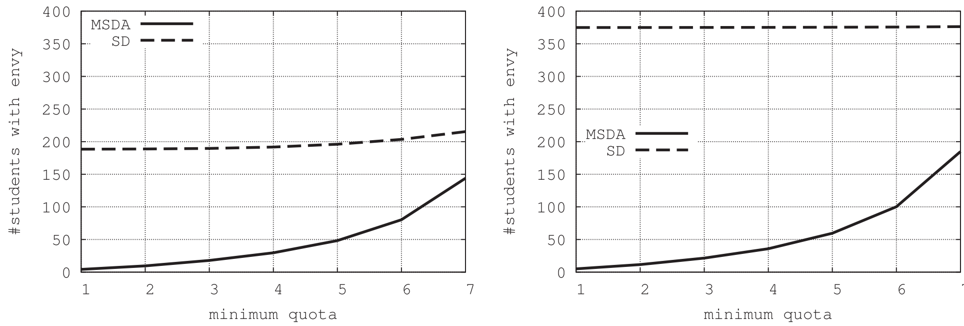

Fig. 4. The number of students who have a claim of justified envy as a function of the minimum quota for $\alpha = 0.3$ (left) and $\alpha = 0.6$ (right). The results shown are for the exponential case.

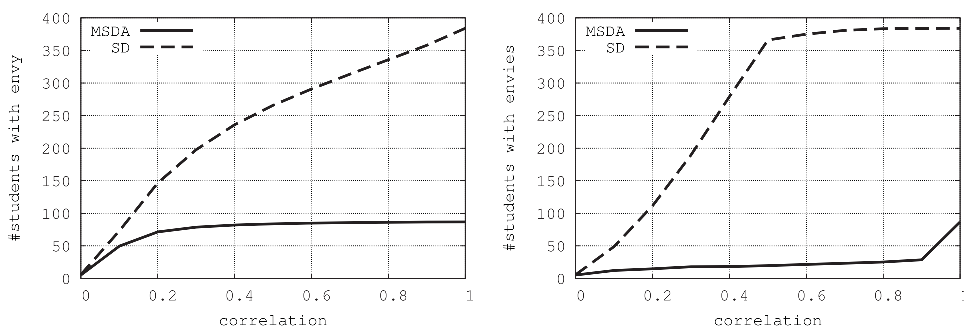

Fig. 5. The number of students who have a claim of justified envy as a function of correlation α for the uniform case (left) and exponential case (right). We set $p_c = 3$ at all c.

we can use simulations to try to understand how many students will have a claim of justified envy, and how this depends on parameters such as the preference correlation and the size of the minimum quotas. Figure 4 shows the results for the exponential case (the results for the uniform case are similar). From the figures, we see that MSDA performs well when the minimum quotas are low, with the number of claims increasing as the minimum quotas are increased. This is intuitive, as when the minimum quotas are relatively low, most of the students can participate in stage 1 of the algorithm, and no students who participate in stage 1 will have any justified envy.

In addition, we can use simulations to complement the worst-case analysis from Section 4.5 by comparing the average number of students with justified envy under MSDA to that under SD. Recall that in Section 4.5, we showed that MSDA always outperforms SD in the worst case. The simulations show that MSDA also performs significantly better than SD on average. The intuition for this is that the MSDA algorithm is very closely related to the DA algorithm, and so takes the school priorities $\succ_c$ into account, unlike SD, which completely ignores school priorities. It may also be of interest to note that for the markets studied in the simulations, the worst case for SD is $W^{SD} = 385$ (and this bound is actually being achieved in the right panel of Figure 4), while the worst case for MSDA is much smaller (e.g., when $p_c = 1$ for all c, $W^{MSDA} = 7$).

Figure 5 analyzes the number of students with justified envy as a function of the correlation α, fixing the minimum quotas at $p_c = 3$. We see that the level of justified envy under MSDA rises slightly as the correlation is increased but asymptotes fairly quickly, and MSDA performs reasonably well even at high correlations. This is in contrast to SD, where the level of justified envy rises much more sharply with the correlation. Thus, while both MSDA and SD will be nonwasteful, MSDA will produce far more students with justified envy.

Allowing for Unacceptable Schools. To formally ensure that a feasible matching exists, we must require all students to rank all schools, and all schools to find all students acceptable. While these assumptions are indeed satisfied in many markets (see Appendix D), they may not always hold. Without such assumptions, there may simply be no assignment that is formally feasible (consider the simple case where all students declare all schools as unacceptable). However, even in these cases, it is still possible to run our mechanisms. While they may end up leaving some minimum quotas unmet, depending on how many students have a viable outside option (e.g., a private school), our model may still be a good approximation and fill "most" of the minimum quotas. The choice of mechanism will obviously affect the number of minimum quota seats left empty, and so policymakers may still be interested in using a mechanism that does a better job at filling the minimum quotas; this is the idea behind imposing (artificial) regional caps in the Japanese medical resident matching problem [Kamada and Kojima 2015]. While artificial caps will indeed fill more minimum quotas, its price is that more students will be left unassigned. In this section, we allow students to declare schools as unacceptable in the simulations. The most important takeaway is that our ESDA mechanism fills roughly the same number of minimum quota seats as ACDA, but, because it is more flexible, it once again outperforms ACDA, in the sense of leaving fewer students unassigned.

In public school choice, each student is *entitled by law* to a seat at some school. If all schools listed on a student's rank order list are full, then the district assigns the student to some school for which he or she did not declare a preference (such assignments are called *administrative assignments* in New York City [Abdulkadiroğlu et al. 2005]). A common approach is to complete all students' submitted preferences by listing any schools not actively ranked in order of distance from the students' homes. With this in mind, we can expand the simulations by randomly drawing one additional cardinal utility value for each student, representing that student's outside option. Then, we complete every student's preference relation as described earlier and run each algorithm on the profile of completed preference relations. If a student is assigned a school that he or she disprefers to his or her outside option, he or she simply withdraws from the match, as in real-world school choice programs.

Allowing students to withdraw clearly means we cannot guarantee that all minimum quotas will be satisfied, but the choice of mechanism will have an important effect on both the number of schools that fall short of their minimum quotas and the number of students who are unassigned. Figure 6 shows the results for the uniform case with $\alpha = 0.3$ (the results for other cases are similar). As expected, DA with no caps does a very poor job at filling the minimum quota seats, with the number of unassigned seats rising sharply as a function of the minimum quota. This is exactly what leads policymakers (e.g., in the army or Japanese government) to impose artificial caps in the first place. ACDA does indeed perform better in terms of filling the minimum quotas, but this comes at the cost of leaving more students unassigned. What is important to note, though, is that ESDA performs roughly equivalently to ACDA in terms of filling the number of minimum quota seats but strictly outperforms ACDA with respect to the number of unassigned students. Because of the flexibility of ESDA, we are able to assign more students without hurting the minimum quotas. MSDA performs intermediately on both metrics: it leaves fewer students unassigned than ACDA or ESDA (but more than DA with no caps), but also leaves more minimum quota seats empty than does ACDA or ESDA (but not as many as DA with no caps).

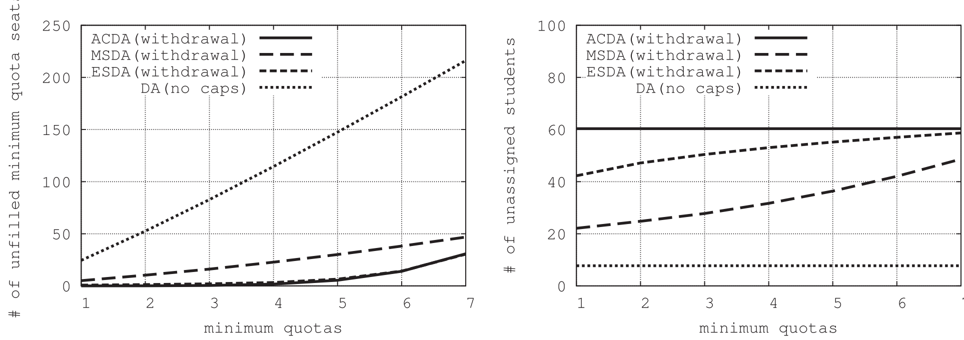

Fig. 6. The number of unfilled minimum quota seats (left) and number of unassigned students (right) as a function of the minimum quota. The results shown are for the uniform case with $\alpha = 0.3$.

There is of course a tradeoff between the number of minimum quotas filled and the number of students unassigned. As can be seen in the figures, though, the cost of DA, in terms of the number of unfilled minimum quotas, rises sharply, while the cost of our mechanisms, in terms of the number of students left unassigned, stays relatively low. Thus, even in situations where agents may leave the market, our mechanisms can still be used, and in fact ESDA outperforms ACDA on these metrics. By taking the minimum quotas explicitly into account, our mechanisms strike a balance between unassigned students and empty seats.

6. CONCLUSION

In this article, we construct two strategyproof mechanisms that provide solutions to minimum quota matching problems: the fair ESDA algorithm and the nonwasteful MSDA mechanism. Both adhere closely to the popular DA algorithm, making adjustments for the fact that standard DA may not satisfy the minimum quotas. In fact, they approach the standard DA algorithm as the minimum quotas approach extreme values. When $p_c = 0$ for all $c \in C$, MSDA reduces to standard DA, as no students are reserved and everyone just participates in round 1 of the algorithm. ESDA also reduces to DA when $p_c = 0$, because every seat is an extended seat, and $e = N$. Therefore, the extended seat cap is only reached when every student is assigned a seat somewhere, which means the algorithm has ended. So, a school never rejects a student unless it has reached its true maximum quota q_c, and ESDA is equivalent to DA. When the minimum quotas are high, ESDA also approaches standard DA, since there are more regular seats and fewer extended seats. When $p_c = q_c$ for all $c \in C$, ESDA is once again equivalent to DA.

We provide two mechanisms because of the impossibility of achieving fairness and nonwastefulness simultaneously in the presence of minimum quotas. One of these properties must be weakened, and we provide two classes of mechanisms, one for each side of the tradeoff: ESDA is fair, while MSDA is nonwasteful. We show that both of our mechanisms are strategyproof, which simplifies the strategic game the agents are playing and ensures that those who would otherwise try to game the system cannot take advantage of other agents who may always (naively) report their true preferences. While it is necessary to weaken either fairness or nonwastefulness in the presence of minimum quotas, it is still desirable from a normative perspective to satisfy the weakened property as much as possible. We study this question for our mechanisms both theoretically and using simulations, showing that our mechanisms outperform more ad hoc approaches. ESDA is preferable to ACDA because the latter algorithm fails to consider agents' preferences when imposing its caps. MSDA yields far fewer blocking pairs than the modified serial dictatorship because the former effectively allows DA to be run on subsets of individuals, while the latter effectively runs DA sequentially on single individuals.

From a practical perspective, we expect our mechanisms to improve performance in markets that currently use ad hoc approaches to ensure minimum quotas are satisfied. We hope that the addition of minimum quota mechanisms to the market designer's toolkit will lead to the explicit appearance of minimum quota constraints in markets that thus far have only dealt with minimum quotas implicitly. More explicit expression of institutions' desiderata can allow the community of market designers to better address these concerns and ultimately help organizations produce allocations that are as desirable as possible.

A. PROOFS OF THEOREMS

A.1. Proofs of Theorems 3.1 and 3.3

For all parts of this proof, we let $\chi^{ESDA}(\cdot)$ be the ESDA mechanism and let $\tilde{\chi}(\cdot)$ be the output in the extended market before mapping back to the original market. Let $\mu := \chi^{ESDA}(\succ_S)$ be the outcome of the ESDA mechanism when the submitted profile of student preferences is $\succ_S$, and let $\tilde{\mu} := \tilde{\chi}(\tilde{\succ}_S)$ be the outcome in the extended market when the extended market preferences (created from the submitted preferences) are $\tilde{\succ}_S$. Let $\tilde{\mu}_k$ denote the tentative matching in the extended market at the end of round k. Let $\bar{k}$ denote the earliest round for which $\sum_{c^* \in C^*} |\tilde{\mu}_{\bar{k}}(c^*)| = e$. Note that $\sum_{c^* \in C^*} |\tilde{\mu}_k(c^*)| = e$ for all $k \geq \bar{k}$.

LEMMA A.1. *If a student s is rejected from an extended school c^* in some round k, then $|\tilde{\mu}_{k'}(c^*)| \leq |\tilde{\mu}_k(c^*)|$ for all $k' > k$ (in particular, $|\tilde{\mu}(c^*)| \leq |\tilde{\mu}_k(c^*)|$), and $s' \tilde{\succ}_{c^*} s$ for all $s' \in \tilde{\mu}(c^*)$.*

PROOF. If $|\tilde{\mu}_k(c^*)| = \tilde{q}_{c^*}$, then the statement is obvious. If $|\tilde{\mu}_k(c^*)| < \tilde{q}_{c^*}$, then student s was rejected because the extended school cap e was reached before c^* was able to admit s. Consider round $k+1$, and let s' be the student who applies in $k+1$. If s' applies to a regular school or an extended school other than c^*, then it is obvious that $|\tilde{\mu}_{k+1}(c^*)| \leq |\tilde{\mu}_k(c^*)|$.[32] So, assume that s' applies to c^*. Since every student who was tentatively accepted to an extended school in round k is still being tentatively held at that school at the beginning of round $k+1$, by the construction of step 4 of the algorithm, the extended seat cap e will be reached before school c^* is able to admit its $(|\tilde{\mu}_k(c^*)|+1)th$ student, and thus $|\tilde{\mu}_{k+1}(c^*)| \leq |\tilde{\mu}_k(c^*)|$. Induction then implies that the same holds for all $k' > k$.

To show the second part, first note that since s is rejected in round k and school c^* admits students according to their ranking on $\tilde{\succ}_{c^*}$, it is clear that $s' \tilde{\succ}_{c^*} s$ for all $s' \in \tilde{\mu}_k(c^*)$. Now, consider round $k+1$. Since all students tentatively assigned to c^* in round k are still tentatively held by c^* at the beginning of round $k+1$ and we have already shown that $|\tilde{\mu}_{k+1}(c^*)| \leq |\tilde{\mu}_k(c^*)|$, the lowest-ranked student in $\tilde{\mu}_{k+1}(c^*)$ according to $\tilde{\succ}_{c^*}$ must be ranked weakly higher than the lowest-ranked student in $\tilde{\mu}_k(c^*)$. This, together with the fact that $s' \tilde{\succ}_{c^*} s$ for all $s' \in \tilde{\mu}_k(c^*)$, implies that $s' \tilde{\succ}_{c^*} s$ for all $s' \in \tilde{\mu}_{k+1}(c^*)$. Then, by induction, $s' \tilde{\succ}_{c^*} s$ for all $s' \in \tilde{\mu}(c^*)$. □

Proof of Theorem 3.1

Group strategyproofness

We first show that if the extended market mechanism $\tilde{\chi}$ is group strategyproof (allowing the students to report any possible preference relation among the full set of standard and extended schools $\tilde{C}$), then the ESDA mechanism χ^{ESDA} is also group strategy proof.

[32]Note that in this case, the inequality may be strict: $|\tilde{\mu}_{k+1}(c^*)| < |\tilde{\mu}_k(c^*)|$. This can happen when s' applies to a school earlier in the picking order than c^* and fills an extended seat that will no longer be assigned to c^*.

We do this by showing the contrapositive: if χ^{ESDA} is not group strategyproof, then $\tilde{\chi}$ is also not group strategyproof. To this end, let $\succ'_{S'}$ be a profitable manipulation in the original market, that is, $\chi_s^{ESDA}(\succ'_{S'}, \succ_{S\setminus S'}) \succ_s \chi_s^{ESDA}(\succ_S)$ for all $s \in S'$. Consider some student $s \in S'$, and let $\chi_s^{ESDA}(\succ_S) = c$ and $\chi_s^{ESDA}(\succ'_{S'}, \succ_{S\setminus S'}) = c'$, so that $c' \succ_s c$. Then, consider the associated profile of preferences in the extended market $\tilde{\succ}'_{S'}$, constructed as in the description of the ESDA algorithm (where c^* is placed immediately after c). It must be that $\tilde{\chi}_s(\tilde{\succ}_S) = c$ or c^* and $\tilde{\chi}_s(\tilde{\succ}'_{S'}, \tilde{\succ}_{S\setminus S'}) = c'$ or c'^*. In either case, the fact that $c' \succ_s c$ implies that $\tilde{\chi}_s(\tilde{\succ}'_{S'}, \tilde{\succ}_{S\setminus S'}) \tilde{\succ}_s \tilde{\chi}_s(\tilde{\succ}_S)$. Since this same argument holds for all $s \in S'$, we have shown that if the ESDA mechanism is not group strategyproof, then the extended market mechanism (where students are allowed to submit preferences over both extended and regular schools) is not group strategyproof. What remains to be shown is that $\tilde{\chi}$ is in fact group strategyproof.

To show that the extended market mechanism $\tilde{\chi}$ is group strategyproof, we relate it to the model of Kamada and Kojima [2015]. Kamada and Kojima use the notation of a set of doctors D and hospitals H with capacity q_h for each $h \in H$. Each hospital belongs to exactly one of several regions R, with regional caps q_r. To relate our extended model to their model, in the Kamada and Kojima notation, we set $D = S$, $H = \tilde{C}$, where note that we use our extended market. The capacity of each agent in $\tilde{C}$ is then $\tilde{q}_c = p_c$ (for regular schools) and $\tilde{q}_{c^*} = q_c - p_c$ for extended schools. Preference relations are defined using the extended market, as in the main text. Note that Kamada and Kojima require the hospital (i.e., school) preference relations to be responsive, which is satisfied in our model.

For the set R, we divide the schools into $M + 1$ regions: each regular school c_j belongs to its own region r_j, with regional cap equal to $q_{r_j} = \tilde{q}_{c_j} (= p_{c_j})$. For the extended schools, they all belong to one region, labeled r^*, with regional cap $q_{r^*} = e$. The set R is thus $R = \{r_1, \ldots, r_m, r^*\}$. The target capacities in their notation are $\bar{q}_{c_j^*}$ for each extended school (the regular schools are each in their own unique region, and so the target capacities are irrelevant). Then, note that $\tilde{\chi}$ is equivalent to the flexible deferred acceptance mechanism of Kamada and Kojima applied to the market $(S, \tilde{C}, R, (\tilde{q}_{\tilde{c}})_{\tilde{c}\in\tilde{C}}, (q_r)_{r\in R}, \tilde{\succ}_S, \tilde{\succ}_{\tilde{C}})$ with target capacities $(\bar{q}_{c^*})_{c^*\in C^*}$ and school ordering, without loss of generality, equal to $c_1 > c_2 > \cdots > c_m$. Theorem 2 of Kamada and Kojima then implies that $\tilde{\chi}$ is group strategyproof, which, by the previous argument, implies that χ^{ESDA} is also group strategyproof.[33]

Fairness

Consider any student–school pair (s, c) such that $c \succ_s \mu(s)$. The construction of the extended market mechanism implies that s must have been rejected from both c and c^*. Note that for any $s' \in \tilde{\mu}(c)$, we have $s' \tilde{\succ}_c s$. This is true because in the extended market, s must have been rejected from the regular school c at some point in favor of $\tilde{q}_c$ students with a higher priority than s. Because c only rejects a student when it is full and a higher-ranked student applies to it, the rank of the lowest-ranked student according to $\tilde{\succ}_c$ only increases throughout the algorithm, and so at the final matching $\tilde{\mu}$, we have $s' \tilde{\succ}_c s$ for all $s' \in \tilde{\mu}(c)$. On the other hand, Lemma A.1. shows that $s' \tilde{\succ}_{c^*} s$ for all $s' \in \tilde{\mu}(c^*)$. The fact that $\tilde{\succ}_{c^*} = \tilde{\succ}_c = \succ_c$ and $\mu(c) = \tilde{\mu}(c) \cup \tilde{\mu}(c^*)$ implies that $s' \succ_c s$ for all $s' \in \mu(c)$; that is, s cannot form a blocking pair with c. Therefore, the ESDA algorithm is fair.

[33]Aygün and Sönmez [2013] and Aygün and Sönmez [2012] point out some technical ambiguities in the original matching with contracts model and show that a condition called irrelevance of rejected contracts (IRC) is necessary for this result. They show that the choice functions of the schools satisfying substitutes and the law of aggregate demand imply IRC. As noted in Kamada and Kojima [2015], these conditions are satisfied in their model (and thus also in ours), and so the result does indeed hold.

Proof of Theorem 3.3
That ACDA is strongly wasteful is shown by the example in Section 3 (this example can be easily embedded in larger markets). To show that ESDA is weakly nonwasteful, assume toward a contradiction that $Z(\mu) \neq \emptyset$ and $|\mu(\mu(s))| > p_{\mu(s)}$ for all $s \in Z(\mu)$. Let $s \in Z(\mu)$ be the student who is assigned to his or her final match last in the running of the algorithm among all of the students in $Z(\mu)$. Let $\mu(s) = c$, and let the round at which s was assigned to his or her final match in the extended market (which can be either c or c^*) be k.

Case (i): *s* (directly) claims an empty seat at a school *c*′. Note that $k \geq \bar{k}$.[34] First, assume that $\tilde{\mu}(s) = c$. If $|\tilde{\mu}_{k-1}(c)| = \tilde{q}_c$, then when s applies to c in round k, some student must be rejected. This student will then apply to c^* at some later stage $\hat{k} > \bar{k}$. Since $\hat{k} > \bar{k}$, some student $\hat{s}$ is rejected from some extended school $\hat{c}^*$. If $\hat{c}^* = c^*$, then $\hat{s}$ at least weakly claims an empty seat at c through the chain $(c', s, c, \hat{s})$.[35] If $\hat{c}^* \neq c^*$, then $|\tilde{\mu}_{\hat{k}}(\hat{c}^*)| < \tilde{q}_{\hat{c}^*}$. By Lemma A.1, $|\tilde{\mu}(\hat{c}^*)| < \tilde{q}_{\hat{c}^*}$ as well, and so $\hat{s}$ claims an empty seat at $\hat{c}$. In either case, $\hat{s} \in Z(\mu)$, which contradicts that s was the student who was assigned to his or her final match last in the running of the algorithm.

If $|\tilde{\mu}_{k-1}(c)| < \tilde{q}_c$, then some student s' must apply to c^* at some step $k' > k$ (otherwise, $|\mu(c)| = p_c$ and the matching is weakly nonwasteful). We can then apply the same argument as earlier to reach a contradiction.

A similar argument applies when $\tilde{\mu}(s) = c^*$.

Case (ii): *s* weakly claims an empty seat at a school *c*′. Let $(c^0, s^0, \ldots, c', s)$ be the chain through which s weakly claims an empty seat. In this case, we have $k > \bar{k}$.[36]

First, assume that $\tilde{\mu}(s) = c$. If $|\tilde{\mu}_{k-1}(c)| = \tilde{q}_c$, then when s applies in round k, some student must be rejected from c. This student then applies to c^* in some later round $\hat{k} > k$. Since $\hat{k} > \bar{k}$, some student $\hat{s}$ is rejected from some extended school $\hat{c}^*$. If $\hat{c}^* = c^*$, then $\hat{s}$ at least weakly claims an empty seat through the chain $(c^0, s^0, \ldots, c', s, c, \hat{s})$.[37] If $\hat{c}^* \neq c^*$, then $|\tilde{\mu}_{\hat{k}}(\hat{c}^*)| < \tilde{q}_{\hat{c}^*}$. By Lemma A.1, $|\tilde{\mu}(\hat{c}^*)| < \tilde{q}_{\hat{c}^*}$ as well, and so $\hat{s}$ claims an empty seat at $\hat{c}$. In either case, $\hat{s} \in Z(\mu)$, which contradicts that s was the student who was assigned to his or her final match last in the running of the algorithm.

If $|\tilde{\mu}_{k-1}(c)| < \tilde{q}_c$, then some student must apply to c^* at some step $\hat{k} > k$ (otherwise, $|\mu(c)| = p_c$, and the matching is weakly nonwasteful). In either case, we can apply the same argument used earlier to reach a contradiction that s was the last student in $Z(\mu)$ assigned his or her final match.

A similar argument applies when $\tilde{\mu}(s) = c^*$.

A.2. Proof of Theorem 4.2

Proof of group strategyproofness
To show group strategyproofness, note that in each stage of MSDA, the standard DA algorithm is used, the latter being group strategyproof [Hatfield and Kojima 2009]. Assume the theorem is false, and consider a set of students S' that collectively do strictly better by misreporting $\succ'_{S'}$ than by reporting the truth, $\succ_{S'}$. Let $S'' \subseteq S'$ be the set of students that are matched in the earliest stage under the true preferences $\succ_{S'}$,

[34]Student s must have been rejected from c'^* at some round $k' < k$. If $k < \bar{k}$, then s is rejected because $|\tilde{\mu}_{k'}(c'^*)| = \tilde{q}_{c'^*}$. Because s directly claims an empty seat, we know that $|\tilde{\mu}(c'^*)| < \tilde{q}_{c'^*}$, and so some student s' is rejected from c'^* in some step $k'' \geq \bar{k}$. Then, s' also claims an empty seat at school c', which contradicts that s is the last student in $Z(\mu)$ to be assigned his or her final match.
[35]If $|\mu(c)| < q_c$, then $\hat{s}$ directly claims an empty seat at c.
[36]Student s^0 must have already been rejected from c^{0*} prior to round k (if not, then s is not the last student in $Z(\mu)$ to be assigned his or her final match). Similar logic implies that $|\tilde{\mu}_{k-1}(c^{0*})| < \tilde{q}_{c^{0*}}$. The previous two statements then imply that $\sum_{c^* \in C^*} |\tilde{\mu}_{k-1}(c^*)| = e$, which, by the definition of $\bar{k}$, gives $\bar{k} < k$.
[37]If $|\mu(c)| < q_c$, then $\hat{s}$ directly claims an empty seat at c.

and denote this stage as stage k. Note that there is no way in which S' can misreport so that a member of S' is matched earlier than stage k. Thus, under any misreport, the students in S'' will be assigned in stage k. However, all of the students in S'' becoming strictly better off by misreporting $\succ'_{S''}$ in stage k is a contradiction to the fact that the standard DA algorithm is group strategyproof. Thus, MSDA is group strategyproof.

Proof of nonwastefulness

Next, we show that MSDA is nonwasteful. Consider a student s who claims an empty seat at some school c. Let k be the stage of the mechanism at which student s participates and q_c^k be the quotas at school c during that stage. There are two cases.

(i) In stage k, step 1(a) of the algorithm is executed. In this case, the quotas for the standard DA algorithm run in stage k are $q_c^k = q_c^{k-1} - |\mu^{k-1}(c)| = q_c - \sum_{k'<k} |\mu^{k'}(c)|$. Since s is not assigned a seat at c in k, then, since we are just using DA, it must be that q_c^k students were assigned at stage k to school c. However, adding this to the number of students assigned in stages $k' < k$, we see that $|\mu(c)| = q_c$, and thus there are no empty seats at school c.

(ii) In stage k, step 1(b) of the algorithm is executed. In this case, the quotas are $\max\{0, p_c - \sum_{k'<k} |\mu^{k'}(c)|\}$. In particular, we have $|\mu(\mu(s))| = p_{\mu(s)}$ for all s assigned in stage k, and so the matching is nonwasteful (since s cannot be moved without violating the minimum quotas at his or her current school $\mu(s)$).

Proof of *PL*-fairness

Consider some s such that $c \succ_s \mu(s)$. We will show that s cannot form a *PL*-blocking pair with school c. Let k be the stage at which s participates in the MSDA algorithm. If school c has no seats available at the beginning of stage k, then every student $s' \in \mu(c)$ is such that $s' \succ_{PL} s$, and so s cannot form a blocking pair with c. On the other hand, if school c does have seats available in round k, then, since within round k we just run the standard DA algorithm, s must have applied to c and been rejected because c was filled to its maximum quota with students ranked higher than s, that is, $s' \succ_c s$ for all s' assigned to c in round k. On the other hand, $s' \succ_{PL} s$ for all s' assigned to c in rounds $k < k'$. Thus, s cannot form a *PL*-blocking pair with c, since all $s' \in \mu(c)$ are ranked higher than s on either $\succ_c$ or $\succ_{PL}$.

A.3. Proof of Theorem 4.3

Proof of group strategyproofness

Assume the contrary, and let S' be a set of students that collectively were all made strictly better off by misreporting $\succ'_{S'}$ than by reporting the truth $\succ_{S'}$. Let s_k be the student in S' who is ranked highest according to $\succ_{PL}$. Note that since all students $s_1, \dots, s_{k-1}$ are not in S', the reports of these students do not change under the misreport. Thus, when it is s_k's turn to choose, the set of schools available to him or her is the same whether the group S' reports $\succ_{S'}$ or $\succ'_{S'}$, and he or she will be assigned the most preferred school according to his or her submitted preferences among those that are available to him or her. This is clearly maximized by reporting truthfully, and so s_k cannot be made strictly better off under any misreport $\succ'_{S'}$.

Proof of nonwastefulness

Assume the contrary, and let s_k be a student who claims an empty seat at some school c. Let $\bar{k}$ be the earliest stage of the mechanism such that $r^{\bar{k}}$ is exactly equal to the number of students left to be assigned (including student $s_{\bar{k}}$). Note that $k \geq \bar{k}$; otherwise, since s_k was not assigned a seat at c, it must be that the maximum quota at c had already been reached before stage k, and hence s_k cannot claim an empty seat at c. Then, note that for all students assigned in stages $k' \geq \bar{k}$, we have $|\mu(\mu(s_{k'}))| = p_{\mu(s_{k'})}$. Therefore, student s_k cannot be moved without violating the minimum quotas at his or her current school $\mu(s_k)$. Since this holds for any student s_k who claims an empty seat, the matching is nonwasteful.

Proof of PL-fairness

Assume the contrary, and let (s_k, c) form a *PL*-blocking pair, and let $\mu(c)$ be the final matching produced by the serial dictatorship with minimum quotas. Since $\mu(s_k) \neq c$, it must be that at the beginning of stage k, either (1) school c has already been assigned q_c students or (2) the number of students assigned is strictly less than q_c but (weakly) greater than p_c, and hence, s_k cannot be assigned to c because he or she must fill a minimum quota at another school. Note that in either case, none of the students $s_{k+1}, \ldots, s_n$ will be assigned to c. In case (1), this is because c is filled to capacity before stage k; in case (2), all remaining students must be assigned to a school that has not yet reached its minimum quota, which does not include c. Thus, at the end of the algorithm, $s' \succ_{PL} s_k$ for all $s' \in \mu(c)$, and so s_k cannot form a *PL*-blocking pair.

A.4. Proof of Theorem 4.4

We start by stating the following lemma, where recall that $\underline{q} = \min_{c \in C} q_c$ and the value of r^1 used is the optimal value as determined by the procedure described in Appendix C (see also the discussion following the definition of MSDA in Section 4). In addition, let $\underline{c} \in \arg\min_{c \in C} q_c$ (so $q_{\underline{c}} = \underline{q}$). The proof of the lemma is provided after the proof of the main theorem.

LEMMA A.2. *The following holds:* $r^1 \leq n - \underline{q}$.

Given this lemma, we will show the theorem by showing (1) a worst-case lower bound for SD is $n - \underline{q}$ and (2) a worst-case upper bound for MSDA is r^1.

Part (i): $W^{SD} \geq n - \underline{q}$.

PROOF. We exhibit a profile $(\hat{\succ}_S, \hat{\succ}_C)$ that produces at least $n - \underline{q}$ students with justified envy. Define $\hat{\succ}_S$ as follows: for all s, $\hat{\succ}_s$ ranks school $\underline{c}$ first. The rankings of all other schools can be arbitrary. For school $\underline{c}$, define

$$\hat{\succ}_{\underline{c}} : s_n \hat{\succ}_{\underline{c}} s_{n-1} \cdots \hat{\succ}_{\underline{c}} s_2 \hat{\succ}_{\underline{c}} s_1$$

(recall that $\succ_{PL}$ is such that $s_1 \succ_{PL} s_2 \cdots \succ_{PL} s_n$). The priorities of the other schools $c \neq \underline{c}$ can be arbitrary. Then, SD assigns some set of students $\{s_1, s_2, \ldots, s_k\}$ to school $\underline{c}$ for some $k \leq \underline{q}$.[38] All students $\{s_{k+1}, \ldots, s_n\}$ will be assigned to a school other than $\underline{c}$ and so will have justified envy toward student s_1. Since $k \leq \underline{q}$, there are at least $n - k \geq n - \underline{q}$ such students.

Part (ii): $W^{MSDA} \leq r^1$.

PROOF. By definition of r^1, the students who participate in stage 1 of MSDA are those in the set $S^1 = \{s_1, \ldots, s_{n-r^1}\}$. Because DA eliminates all justified envy, no student $s \in S^1$ can have any justified envy toward any other $s' \in S^1$. Further, no $s \in S^1$ can have justified envy toward any $s' \in S \setminus S^1$ either. To see this, assume that $\mu(s') P_s \mu(s)$ for some $s \in S^1$ and $s' \in S \setminus S^1$. The fact that $\mu(s') P_s \mu(s)$ implies that student s was rejected from school $\mu(s')$. However, the fact that s' was assigned to school $\mu(s')$ in stage $k > 1$ implies that school $\mu(s')$ was not filled to capacity at the end of stage 1, which

[38] Since we assume that $n > \sum_{c \in C} p_c$, student 1 will surely be assigned to $\underline{c}$. Not all students $\{s_1, s_2, \ldots, s_{\underline{q}-1}, s_{\underline{q}}\}$ will necessarily be assigned to $\underline{c}$ (this will depend on the point in the running of SD at which the minimum quota constraints become binding), but it is certain that no students outside of this set will be assigned to $\underline{c}$ (by definition of $\underline{q}$).

contradicts the fact that s was rejected from $\mu(s')$. Therefore, no students in S^1 have any justified envy toward any other student in S. Thus, the number of students with justified envy is bounded above by $|S \setminus S^1| = r^1$, that is, $W^{MSDA} \leq r^1$.

Then, we can combine part (1), part (2), and Lemma A.2 to get the following inequalities:

$$W^{MSDA} \leq r^1 \leq n - \underline{q} \leq W^{SD}.$$

Therefore, $W^{MSDA} \leq W^{SD}$ always, and if, in addition, $r^1 < n - \underline{q}$, the inequality is strict: $W^{MSDA} < W^{SD}$.

Proof of Lemma A.2. The number r^1 is the fewest number of students who can be reserved in stage 1 such that, no matter how the remaining $n-r^1$ students are allocated, the total number of minimum quota seats remaining to be filled at the end of stage 1 is less than r^1. The value of r^1 can be determined by considering an adversary who, given n' students to allocate in stage 1, allocates them in such a way as to maximize the number of minimum quota seats that are not satisfied. For a given n', let $v(n')$ be the number of minimum quota seats that are filled in such an adversary's assignment. Then, we let n^* be the largest value such that $n - n^* \geq \sum_{c \in C} p_c - v(n^*)$, and set $r^1 = n - n^*$.[39]

Now, consider an adversary who is given $n' = \underline{q}$ students to allocate in stage 1. Let $\hat{c}$ be a school with the lowest minimum quota, $p_{\hat{c}} = \min_{c \in C} p_c$. Clearly, one way for the adversary to maximize the number of minimum quota seats left unfilled is to assign all $\underline{q}$ students to school $\hat{c}$, and under this assignment, all of the minimum quota seats at $\hat{c}$ are filled, which implies $v(\underline{q}) = p_{\hat{c}}$.[40] Now, recall that $\underline{q} \leq q_{\hat{c}} \leq n - \sum_{c' \neq \hat{c}} p_{c'}$.[41] Rearranging this inequality and using $v(\underline{q}) = p_{\hat{c}}$, we can write

$$\begin{aligned} n - \underline{q} &\geq \sum_{c' \neq \hat{c}} p_{c'} \\ &= \sum_{c' \neq \hat{c}} p_{c'} + (p_{\hat{c}} - v(\underline{q})) \\ &= \sum_{c' \in C} p_{c'} - v(\underline{q}). \end{aligned}$$

So, $n - \underline{q} \geq \sum_{c' \in C} p_{c'} - v(\underline{q})$; that is, if we assign $\underline{q}$ students in stage 1, we can be certain that the number of students remaining after stage 1 $(n - \underline{q})$ will be at least as large as the number of minimum quota seats remaining $(\sum_{c' \in C} p_{c'} - v(\underline{q}))$. By definition of n^*, we have $\underline{q} \leq n^*$, which implies $n - \underline{q} \geq n - n^* = r^1$.

B. MSDA IS MORE FAIR THAN ANY OTHER NONWASTEFUL MECHANISM

In Section 4, we introduced a weakening of the standard fairness definition to *PL*-fairness, in which the school district declared some standard blocking pairs as invalid. In this section, we generalize this concept to what we call σ-fairness. σ-fairness captures *how many* such blocking pairs the school district must declare as invalid, and thus will allow us to rank mechanisms in terms of fairness. Intuitively, if the school district must declare *more* blocking pairs as invalid, the matching is *less* fair.

[39] See Appendix C for further details on this procedure.

[40] Note that $\hat{c}$, the school with the lowest minimum quota, may or may not be the same as $\underline{c}$, the school with the lowest *maximum* quota, $q_{\underline{c}} = \underline{q}$. However, by definition of $\underline{q}$, we have $q_{\hat{c}} \geq \underline{q} \geq p_{\hat{c}}$. Therefore, it is possible to assign all $\underline{q}$ students to school $\hat{c}$ without assigning any students to fill minimum quota seats at any other school. Doing so also satisfies the minimum quota at school $\hat{c}$.

[41] The second inequality is a primitive assumption of the model. See footnote 7.

Formally, let σ_c be some integer such that $p_c \leq \sigma_c \leq q_c + 1$.[42] We say that (s, c) form a σ_c-**blocking pair** if the following three conditions are met for some $s' \in \mu(c)$:

(i) $c \succ_s \mu(s)$
(ii) $s \succ_c s'$
(iii) If $|\mu(c)| \geq \sigma_c$, then $s \succ_{PL} s'$.

We can then generalize the definition of fairness to one that eliminates all σ_c-blocking pairs for some $\sigma = (\sigma_{c_1}, \ldots, \sigma_{c_m})$. Let $\Sigma(\mu) = \{\sigma : \mu \text{ contains no } \sigma_c\text{-blocking pairs for any } c \in C\}$.

Definition B.1. If $\sigma = \max\{\sigma' : \sigma' \in \Sigma(\mu)\}$ exists, then we say that μ is σ-**fair**.

In the definition, we take the maximum σ vector because if (s, c) is a σ_c-blocking pair for some σ_c, then it is a σ'_c-blocking pair for all $\sigma'_c \geq \sigma_c$ (but not vice versa). Thus, if a matching contains no σ-blocking pairs and no σ'-blocking pairs, it also contains no σ''-blocking pairs, where $\sigma'' = \sigma \vee \sigma'$ is the join of σ and σ'. More formally, the set $\Sigma(\mu)$ is a lattice, and (provided it is nonempty) the maximum is unique.[43] An important special case of σ-fairness is obtained when $\sigma = p$. The definition of p-fairness is very similar, though not technically equivalent to the definition of *PL*-fairness given in Section 4. The reason is due to the max operator in the definition of σ-fairness. To see the distinction, note that a matching μ that is *PL*-fair in the sense of Section 4 may not be p-fair in the sense of Definition B.1 (the converse is true, however: a matching that is p-fair is also *PL*-fair). This will happen, for example, when μ eliminates all q-blocking pairs as well, in which case it would be labeled q-fair rather than p-fair. Thus, to show that a matching is σ-fair, two conditions must be checked: (1) the matching admits no σ-blocking pairs and (2) the matching **does** admit a σ'-blocking pair for all $\sigma'\sigma$. A similar remark applies to mechanisms. Similarly, for a mechanism χ, we can define a set $\Sigma(\chi) = \{\sigma' : \chi \text{ produces a } \sigma'\text{-fair matching for all } \succ_S \text{ and } \succ_C\}$. Then, we have the following definition.

Definition B.2. If $\sigma = \max\{\sigma' : \sigma' \in \Sigma(\chi)\}$ exists, then we say that χ is σ-**fair**.

Using these ideas, we can formally rank mechanisms according to fairness. Note that a higher value of σ means that the mechanism itself is eliminating more potential blocking pairs. Since, all else equal, a school district would desire to use a mechanism that is more fair (i.e., a higher σ) for both positive and normative reasons, we are interested in finding the most fair mechanisms we can achieve without sacrificing other goals (strategyproofness and nonwastefulness). It should be noted that while in general, mechanisms such as standard DA and the new mechanisms we introduce can be applied to markets of any size, it is only meaningful to compare two mechanisms with respect to fairness for fixed minimum and maximum quotas p and q.

Definition B.3. Fix the set of schools C and minimum and maximum quota vectors p and q, and consider two mechanisms χ and ψ, which are σ^{χ}- and σ^{ψ}-fair, respectively. If $\sigma^{\chi} \geq \sigma^{\psi}$, then we say that χ is **more fair** than ψ.

Again, the comparison in the previous definition is taken with respect to the product order. Because we use the product order, the relation is not complete: it may be that given two mechanisms, neither is more fair than the other. While this is allowed a priori, the next theorem shows that, under a small additional assumption that is likely satisfied in practice, MSDA is in fact more fair than any other nonwasteful mechanism.

[42]For $\sigma_c > q_c + 1$, a σ_c-blocking pair is equivalent to a $(q_c + 1)$—blocking pair, and so we restrict σ_c to be weakly less than $q_c + 1$.
[43]Note $\Sigma(\mu)$ may be empty if there is some $c \in C$ such that μ contains a σ_c-blocking pair for all σ_c. In this case, the matching is not σ-fair for any σ. A similar statement can be applied to mechanisms.

THEOREM B.4. *Assume that p and q are such that* $0 < p_c < q_c$ *for all* $c \in C$.[44] *Then:*

(i) *The MSDA mechanism is p-fair.*
(ii) *If a mechanism* χ *is nonwasteful and* σ*-fair, then* $\sigma = p$*. Alternatively, MSDA is more fair than any other nonwasteful mechanism.*

PROOF.
Part (i)
To prove part (i), we must show (1) that MSDA never produces any p_c-blocking pairs, and (2) for any σp, there are preferences and priorities such that MSDA will produce a σ_c-blocking pair for some c (see Remark B). Part (1) follows immediately from Theorem 4.2 and the fact that a *PL*-blocking pair is equivalent to a p_c-blocking pair at any school $c \in C$.

To show part (2), consider some school c_1 and let $\sigma_{c_1} = p_{c_1} + 1$. Consider a market of size $n = 1 + \sum_{c \in C} p_c$. Let the priorities at c_1 be $s_n \succ_{c_1} s_1 \succ_{c_1} s_2 \cdots \succ_{c_1} s_{n-1}$, and let the priorities at all other schools be $\succ_{c_j} = \succ_{PL}$ for all $j \neq 1$. Let the preferences of the students be as follows. Let the first p_{c_1} students on $\succ_{PL}$ rank c_1 first (the rest of their preferences will be irrelevant). Let the next p_{c_2} students on $\succ_{PL}$ rank c_2 first. Continue in this manner until we reach student s_γ, where $\gamma = \sum_{j=1}^{m-1} p_{c_j}$. Let the next $p_{c_m} - 1$ students rank c_m first. The only students thus remaining are the last students on $\succ_{PL}$, students s_{n-1} and s_n. Let s_{n-1} rank c_2 first, and s_n rank c_1 first. Under these preferences and priorities, the assignment of MSDA is such that every student gets his or her first choice, except for s_n, who is assigned $\chi_{s_n}(\succ_S) = c_m$. Note that $|\chi_{c_1}(\succ_S)| = p_{c_1}$, and so s_n forms a σ_{c_1}-blocking pair with c_1.

Part (ii)
To show part (ii), we reduce the market of n students and m schools to a smaller market of three students and three schools. More specifically, we construct preferences and priorities such that the assignments of students $s_1, \ldots, s_{n-3}$ are pinned down by σ-fairness. These students can then be removed from the problem, leaving students s_{n-2}, s_{n-1}, and s_n to be assigned. We then examine all possible feasible allocations for these students and show that none of these allocations are simultaneously σ-fair and nonwasteful for any σ such that $\sigma_c > p_c$ for at least one $c \in C$.

To this end, let χ be nonwasteful and σ-fair, and assume that $\sigma_c > p_c$ for at least one $c \in C$. Without loss of generality, denote this school c_2, and let $\sigma_{c_2} = p_{c_2} + 1$.[45] Consider a market with $n = 1 + \sum_{c \in C} p_c$ students, and recall that $s_1 \succ_{PL} \cdots \succ_{PL} s_n$.[46] For the remainder of the proof, we consider the following priority profile: $\succ_c = \succ_{PL}$ for all $c \neq c_2$, while $\succ_{c_2}$ is defined as $s_1 \succ_{c_2} s_2 \succ_{c_2} \cdots \succ_{c_2} s_{n-3} \succ_{c_2} s_n \succ_{c_2} s_{n-1} \succ_{c_2} s_{n-2}$ (i.e., the priority of the last three students is reversed).

Let $S' = S \setminus \{s_{n-2}, s_{n-1}, s_n\}$. We partition S' into m sets, $S'_1, \ldots, S'_m$ (one for each school) as follows: S'_1 consists of the $p_{c_1} - 1$ highest-ranked students according to $\succ_{c_1}$ $(= \succ_{PL})$ and S'_2 consists of the $p_{c_2} - 1$ highest-ranked students in $S' \setminus S'_1$ according to $\succ_{c_2}$. For the remaining schools, let S'_k consist of the p_{c_k} highest-ranked students in the set $S' \setminus \cup_{i=1}^{k-1} S'_k$ according to $\succ_{c_k}$.

For the remainder of the proof, we consider preference profiles such that for all $k = 1, \ldots, m$, if $s \in S'_k$, then s ranks school c_k first. Since S'_1 contains the highest-ranked

[44] We impose this assumption for technical convenience. If $p_c = 0$ for some c, a 0-blocking pair is equivalent to a 1-blocking pair. Because according to the definition of σ-fairness we look for the *largest* σ such that all σ_c-blocking pairs are eliminated, in the general case the MSDA mechanism is actually $\bar{p}$-fair, where $\bar{p}_c = \max\{p_c, 1\}$. When $p_c > 0$ for all $c \in C$, $\bar{p} = p$.
[45] Since σ-fairness implies σ'-fairness for all $\sigma' \leq \sigma$, this is sufficient to show the impossibility.
[46] The argument can be easily adapted for any number of students such that $\sum_{c \in C} p_c < n < \sum_{c \in C} q_c$.

students in S according to $\succ_{c_1}$ and all student in S'_1 rank c_1 first, σ-fairness implies that all students in S'_1 are assigned to c_1. If this were not the case, then some student $s \in S'_1$ is assigned to a school worse than c_1 for him or her, and some $s' \notin S'_1$ is assigned c_1 (since we know at least p_{c_1} students are assigned to c_1). But then (s, c_1) forms a σ_{c_1}-blocking pair (for any σ_{c_1}), which contradicts that the mechanism was σ-fair. Now, S'_2 contains the highest-ranked students in $S' \setminus S'_1$ according to $\succ_{c_2}$, and so, by similar logic, all students in S'_2 must be assigned to c_2. Continuing in this manner, all students in S'_k must be assigned to school c_k for all k, regardless of the preferences of $\{s_{n-2}, s_{n-1}, s_n\}$.

Last, we consider the assignments of $\{s_{n-2}, s_{n-1}, s_n\}$. Now, we know that schools c_1, c_2, and c_3 satisfy the following (where μ' denotes the tentative matching, before $\{s_{n-2}, s_{n-1}, s_n\}$ are assigned, as described in the previous paragraph): $|\mu'(c_1)| = p_{c_1} - 1 < q_{c_1}$, $|\mu'(c_2)| = p_{c_2} - 1 < q_{c_2}$, and $p_{c_3} = |\mu'(c_3)| < q_{c_3}$. So, we have effectively reduced the problem to a subproblem with three students and three schools in which c_1 and c_2 have one minimum quota seat remaining (and at least two seats total) and c_3 has at least one empty seat. Consider the following preferences for $\{s_{n-2}, s_{n-1}, s_n\}$ and the priorities for the schools, as given earlier:

$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
$\vdots$	$\vdots$	$\vdots$
s_{n-2}	s_n	s_{n-2}
s_{n-1}	s_{n-1}	s_{n-1}
s_n	s_{n-2}	s_n

$\succ_{s_{n-2}}$	$\succ_{s_{n-1}}$	$\succ_{s_n}$
c_2	c_3	c_3
c_3	c_2	c_2
c_1	c_1	c_1
$\vdots$	$\vdots$	$\vdots$

Clearly, of the three students, at least one must be assigned to c_1 and at least one must be assigned to c_2. There are three possible ways to choose these two students:

Case (a): s_{n-2} and s_{n-1} are assigned to c_1 and c_2. Nonwastefulness then requires that s_n be assigned c_3, and so there are two possible allocations here. One is shown by the boxes, the other by the circles.

$\succ_{s_{n-2}}$	$\succ_{s_{n-1}}$	$\succ_{s_n}$
$\boxed{c_2}$	c_3	$\boxed{\enclose{circle}{c_3}}$
c_3	$\enclose{circle}{c_2}$	c_2
$\enclose{circle}{c_1}$	$\boxed{c_1}$	c_1
$\vdots$	$\vdots$	$\vdots$

Note, however, that under the circled allocation, both s_{n-2} and s_{n-1} can form a σ_{c_3}-blocking pair with c_3 for any σ_{c_3}, since they are both ranked higher than s_n according to $\succ_{PL}$ and $\succ_{c_3}$. Likewise, under the boxed allocation, s_{n-1} can again form a σ_{c_3}-blocking pair with c_3. Thus, neither matching is σ-fair.

Case (b): s_{n-2} and s_n are assigned to c_1 and c_2. In this case, s_{n-1} must be assigned to c_3 by nonwastefulness, and there are again two possible allocations:

$\succ_{s_{n-2}}$	$\succ_{s_{n-1}}$	$\succ_{s_n}$
$\boxed{c_2}$	$\boxed{\enclose{circle}{c_3}}$	c_3
c_3	c_2	$\enclose{circle}{c_2}$
$\enclose{circle}{c_1}$	c_1	$\boxed{c_1}$
$\vdots$	$\vdots$	$\vdots$

Here, under the circled allocation, s_{n-2} forms a σ_{c_3}-blocking pair with c_3. Under the boxed allocation, s_n forms a σ_{c_2}-blocking pair with c_2, because $s_n \succ_{c_2} s_{n-2}$ and $|\mu(c_2)| = p_{c_2} < \sigma_{c_2}$, and so s_n need not be higher on $\succ_{PL}$ to form a blocking pair. Thus, neither matching is σ-fair.

Case (c): s_{n-1} and s_n are assigned to c_1 and c_2. In this case, nonwastefulness requires that s_{n-2} be assigned c_2, and two possible allocations are boxed/circled.

$\succ_{s_{n-2}}$	$\succ_{s_{n-1}}$	$\succ_{s_n}$
$\boxed{\textcircled{c_2}}$	c_3	c_3
c_3	$\boxed{c_2}$	$\textcircled{c_2}$
c_1	$\textcircled{c_1}$	$\boxed{c_1}$
$\vdots$	$\vdots$	$\vdots$

It is simple to see that under either allocation, $|\mu(c_2)| = p_{c_2} + 1$, and thus both matchings are not nonwasteful, as either s_{n-1} or s_n can be feasibly moved to the more preferred c_3.

Thus, we see that no matching is simultaneously nonwasteful and σ-fair for any σ such that $\sigma_c > p_c$ for at least one $c \in C$.

C. PROCEDURE FOR MINIMIZING R^K IN THE MULTISTAGE MECHANISMS

We describe a procedure to obtain the smallest r^k for the current minimum/maximum quotas $p^k = (p^k_{c_1}, \ldots, p^k_{c_m})$, $q^k = (q^k_{c_1}, \ldots, q^k_{c_m})$, where the number of remaining students is $n^k = n - \sum_c |\mu(c)|$. Our goal is to identify the smallest r^k such that no matter how $n^k - r^k$ students are allocated, by allocating the remaining r^k students appropriately, we can fill all of the remaining minimum quotas. If we set $r^k = \sum_c p^k_c$, it is clear that we can satisfy all of the minimum quotas. Our procedure checks whether we can set r^k to be smaller than $\sum_c p^k_c$. Let us assume that when we decide to allocate n' students, an adversary chooses the worst outcome; that is, the adversary selects an assignment of the n' students such that minimum quotas are the least satisfied. For a given n', let us denote $v(n')$ as the total number of students that are effective to reduce the minimum quotas in the assignment of the adversary. Once we know $v(n')$ for each n', where $n^k - \sum_c p^k_c \le n' \le n^k$, we can select the largest n' such that $\sum_c p^k_c - v(n') \le n^k - n'$ holds. Then, r^k is chosen as $n^k - n'$. The remaining question is how to obtain $v(\cdot)$. Let us assume the adversary first solves the following optimization problem. For a given p', which is the total number of students that are effective to reduce the minimum quotas, find $u(p')$, which is the largest number of students that can be assigned without further reducing the minimum quotas. If we know $u(\cdot)$, we can define $v(n')$ as $v(n') = p'$, where p' is the smallest number such that $u(p') \ge n'$. We can formalize the optimization problem of finding $u(\cdot)$ as the well-known knapsack problem (see, e.g., Kellerer et al. [2004]). In this formalization, we assume that p' represents the capacity of a knapsack. Further, we assume that each school c is an item with capacity p_c and value q_c. Since we can partially assign students to a school, we assume there exist enough additional items, where the capacity/value of an additional item is 1. The goal of the knapsack problem is to select items such that the total value is maximized under the capacity constraint. When the capacity of the knapsack is bounded (which is true in our case), a pseudo-polynomial time dynamic programming algorithm exists. Let us show an example. Assume there are 15 students and 10 schools. For each school c, $p^k_c = 1$ and $q^k_c = 2$. Here, $\sum_c p^k_c = 10$. Then, we obtain $u(0) = 0, u(1) = 2, u(2) = 4, \ldots, u(5) = 10, u(6) = 12, \ldots, u(10) = 20$. Thus, we obtain $v(1) = 1, v(2) = 1, v(3) = 2, v(4) = 2, \ldots, v(11) = 6, v(12) = 6, \ldots$. Thus, if we set $n' = 11$, $\sum_c p^k_c - v(n') = 10 - 6 = 4$. This is equal to $n^k - n' = 15 - 11 = 4$.

On the other hand, if we set $n' = 12$, $\sum_c p_c^k - v(n') = 10 - 6 = 4 > 3 = 15 - 12 = n^k - n'$. Thus, the largest n' is 11 and the smallest r^k is 4. In other words, if the adversary allocates 11 students, the adversary ends up filling at least six seats that are effective to reduce minimum quotas. Then, the total number of remaining minimum quotas is four. Thus, we can assign remaining four students to fill these remaining minimum quotas.

D. MARKETS IN WHICH ALL STUDENTS FIND ALL SCHOOLS ACCEPTABLE

If agents on one side of the market can find agents on the other side unacceptable, it is obviously impossible to guarantee the existence of an individually rational matching that satisfies all minimum quotas. There are in fact many markets where the assumption of complete preference listings is satisfied. For example, in the military cadet matching studied by Sönmez [2013] and Sönmez and Switzer [2013], cadets cannot refuse an assignment to a branch. For many new medical residents, the first year is often an "intern year" in which they rotate through various departments in a hospital. Since doctors are employees of the hospital, they cannot refuse any assignment. The assumption also holds in the Kyushu University computer science problem mentioned in the introduction, where the students must complete a laboratory requirement to finish their thesis, and so cannot refuse an assignment. Even in settings in which the assumption of a complete preference ranking is not strictly satisfied, it is not an unreasonable approximation. In public school choice, every student must be offered a seat at some public school. For example, in New York City in 2002, over 30,000 students were assigned "administratively" to a school they had not indicated a preference for, with the vast majority of these students actually enrolling at their assigned school (see, e.g., Abdulkadiroğlu et al. [2005]). The assignment plan in Jefferson County, Kentucky, described in Echenique and Yenmez [2013], also suggests that the school district has a strong degree of control over enforcing an assignment. In labor market settings such as the hospital residency market, while doctors may only submit a short preference list, doctors who are unmatched by the algorithm find a job through a secondary market (the "Scramble"), suggesting that they actually find many more hospitals acceptable than those on their submitted preference list. We also refer the reader to Section 5 of this article, where we relax this assumption by allowing students to declare schools as unacceptable, and show that even though we can no longer guarantee that the minimum quotas will be satisfied, our mechanisms can still be used and will produce good outcomes.

E. SPECIAL CASES: $M = 2$ OR $P_C < Q_C$ FOR AT MOST ONE SCHOOL

This appendix shows that when $m = 2$ or $p_c < q_c$ for at most one school, there are in fact simple mechanisms that are strategyproof, fair, and nonwasteful.

First, consider the case of $m = 2$. When there are only two schools, we can simply impose artificial caps of $\tilde{q}_{c_1} = \min\{n - p_{c_2}, q_{c_1}\}$ and $\tilde{q}_{c_2} = \min\{n - p_{c_1}, q_{c_2}\}$ and run the standard DA algorithm with these upper quotas.[47] Again, fairness and strategyproofness are immediate from the properties of DA. In this special case, we also get nonwastefulness, because if a student s is rejected from his or her first choice c_i, it is because c_i is filled with $\tilde{q}_{c_i}$ students in the first round of DA. If $\tilde{q}_{c_i} = q_{c_i}$, there are no empty seats at c_i and so s cannot claim an empty seat; on the other hand, if $\tilde{q}_{c_i} = n - p_2$, then it must be that $|\mu(c_j)| = p_2$, and so s cannot be moved to c_i without violating the

[47] Note that $\tilde{q}$ ensures a feasible match, and so DA run with quotas $\tilde{q}$ will always produce a feasible matching. When $m > 2$, the natural extension of these caps, namely, $\tilde{q}_{c_j} = \min\{q_{c_k}, n - \sum_{k \neq j} p_{c_k}\}$, will in general not ensure a feasible match.

minimum quota at c_j. It is only in the special case of $m = 2$ that we can find artificial caps that will always deliver a nonwasteful assignment: when $m \geq 3$, it is in general not obvious which school should be ex ante capped to ensure the minimum quotas at other schools are satisfied, and if a popular school is capped, the assignment may be very wasteful.

Next, consider the case $p_{c_i} < q_{c_i}$ for only c_i (and $p_{c_j} = q_{c_j}$ for all $i \neq j$). Here, any feasible matching will be such that $|\mu(c_j)| = q_{c_j}$ for $j \neq i$ and $|\mu(c_i)| = n - \sum_{j \neq i} p_{c_j}$. Since we know exactly how many seats will be assigned at every school in any feasible matching, standard DA with maximum quotas $\tilde{q}_{c_j} = q_{c_j} (= p_{c_j})$ for $j \neq i$ and $\tilde{q}_{c_i} = n - \sum_{j \neq i} p_{c_j}$ will produce a feasible matching for any preference profile. This will clearly be strategyproof and fair; it is nonwasteful as well because the only "empty" seats will be at school c_i. If a student s claims an empty seat, he or she must be at a school $c_j \neq c_i$ for which $|\mu(c_j)| = \tilde{q}_{c_j} = p_{c_j}$; that is, s cannot be moved without violating the minimum quota at c_j.

ACKNOWLEDGMENTS

We are grateful to Brian Baisa, John Hatfield, Fuhito Kojima, Scott Kominers, Muriel Niederle, Al Roth, members of the Stanford market design reading group, and an editor and two anonymous referees for helpful comments. We thank Naoyuki Hashimoto and Masahiro Goto for research assistance.

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Received July 2014; revised December 2014; accepted February 2015