TAMAGAWA PRODUCTS OF ELLIPTIC CURVES OVER $\mathbb{Q}$

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Abstract. We explicitly construct the Dirichlet series

$$L_{\mathrm{Tam}}(s) := \sum_{m=1}^{\infty} \frac{P_{\mathrm{Tam}}(m)}{m^s},$$

where $P_{\mathrm{Tam}}(m)$ is the proportion of elliptic curves $E/\mathbb{Q}$ in short Weierstrass form with Tamagawa product m. Although there are no $E/\mathbb{Q}$ with everywhere good reduction, we prove that the proportion with trivial Tamagawa product is $P_{\mathrm{Tam}}(1) = 0.5053\ldots$. As a corollary, we find that $L_{\mathrm{Tam}}(-1) = 1.8193\ldots$ is the *average* Tamagawa product for elliptic curves over $\mathbb{Q}$. We give an application of these results to canonical and Weil heights.

1. Introduction and statement of results

It is well known that there are no elliptic curves $E/\mathbb{Q}$ with everywhere good reduction (for example, see Ch. VII-XIII of [13]). In spite of this fact, there are many $E/\mathbb{Q}$, such as

$$E/\mathbb{Q}: \quad y^2 = x^3 - 3x - 4,$$

(i.e. 5184.m1 in [10]) with the weaker property that $[E(\mathbb{Q}_p) : E_0(\mathbb{Q}_p)] = 1$ for all primes p, where $E_0(\mathbb{Q}_p)$ is the open subgroup of $E(\mathbb{Q}_p)$ consisting of nonsingular points. These *Tamagawa trivial curves* have

$$\mathrm{Tam}(E) := \prod_{p \ prime} c_p = 1, \tag{1.1}$$

where $c_p := [E(\mathbb{Q}_p) : E_0(\mathbb{Q}_p)]$ is the usual Tamagawa number at p.

Tamagawa trivial curves enjoy properties that motivate this note. For example, if $E/\mathbb{Q}$ is a Tamagawa trivial curve for which $E(\mathbb{Q})$ has rank r, then the Birch and Swinnerton-Dyer Conjecture predicts that

$$\frac{L^{(r)}(E,1)}{r!} = \frac{|\text{Ш}(E)| \cdot \Omega_E R_E}{|E_{\mathrm{tor}}(\mathbb{Q})|^2}.$$

Here $L(E,s)$ is the Hasse-Weil L-function for $E/\mathbb{Q}$, $\text{Ш}(E)$ is the Shafarevich-Tate group, Ω_E is the real period, R_E is the regulator, and $E_{\mathrm{tor}}(\mathbb{Q})$ is the $\mathbb{Q}$-rational torsion subgroup. Tamagawa trivial elliptic curves also play a prominent role in the work of Balakrishnan, Kedlaya, and Kim [3] that offers the first explicit positive genus examples of nonabelian Chabauty: the case of quadratic Chabauty for determining integral points on rank 1 elliptic curves [3, 8]. The main result of [3] is formulated for these curves. As a final example, we consider *convenient* elliptic curves, a subclass of Tamagawa trivial curves with the guaranteed property (i.e. without computing $E(\mathbb{Q})$) that $\widehat{h}(P) \geq \frac{1}{2} h_W(P)$ for all $P \in E(\mathbb{Q})$, where $\widehat{h}(P)$ (resp. $h_W(P)$) is the canonical (resp. Weil) height of P.

It is also natural to consider curves with any fixed Tamagawa product m. As motivation, we recall that algorithms of Mazur, Stein, and Tate [11] for computing the global p-adic heights of rational points on elliptic curves assume that the reductions of points at primes of bad reduction are non-singular. In follow-up work by Balakrishnan, Çiperiani, and Stein [2], and Balakrishnan, Çiperiani, Lang, Mirza, and Newton [1], where points can be defined over a more general number field, such assumptions can be computationally expensive. The knowledge of the proportion of curves with arbitrary Tamagawa product m gives an indication of the cost of such algorithms.

Motivated by these applications, we compute the proportion of short Weierstrass elliptic curves

$$E = E(a_4, a_6): \quad y^2 = x^3 + a_4 x + a_6, \tag{1.2}$$

with $a_4, a_6 \in \mathbb{Z}$ and $\Delta(a_4, a_6) := -16(4a_4^3 + 27a_6^2) \neq 0$, with $\mathrm{Tam}(E) = m$. To this end, we recall that E has *height*

$$\mathrm{ht}(E) := \max\{4|a_4|^3, 27a_6^2\}, \tag{1.3}$$

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and we employ the counting function

$$\mathcal{N}(X) := \#\{E = E(a_4, a_6) \ : \ \mathrm{ht}(E) \leq X\}, \tag{1.4}$$

which is the number of $E(a_4, a_6)$ with height $\leq X$. The number with Tamagawa product m is

$$\mathcal{N}_m(X) := \#\{E := E(a_4, a_6) \ : \ \mathrm{ht}(E) \leq X \text{ with } \mathrm{Tam}(E) = m\}, \tag{1.5}$$

where $\mathrm{Tam}(E)$ is the product for the global minimal model[1] of E. Our aim is to compute

$$P_{\mathrm{Tam}}(m) := \lim_{X \to +\infty} \frac{\mathcal{N}_m(X)}{\mathcal{N}(X)}. \tag{1.6}$$

We compute the Dirichlet series generating function for these proportions.

Theorem 1.1. *The $P_{\mathrm{Tam}}(m)$ are well-defined, and are the Dirichlet coefficients of*

$$L_{\mathrm{Tam}}(s) := \sum_{m=1}^{\infty} \frac{P_{\mathrm{Tam}}(m)}{m^s} = \prod_{p \text{ prime}} \left(\frac{\delta_p(1)}{1^s} + \frac{\delta_p(2)}{2^s} + \frac{\delta_p(3)}{3^s} + \dots \right),$$

where $\delta_p(n)$ are rational numbers defined in Lemma 3.1.

Remark. *The number $\delta_p(n)$ is the proportion of short Weierstrass curves whose minimal model has $c_p = n$. Cremona and Sadek [7] compute such proportions for long Weierstrass models, which are different for $p \in \{2, 3\}$. Our choice is motivated by an application to heights (see Corollary 1.5).*

Corollary 1.2. *Assuming the notation above, the following are true.*

(1) We have that

$$P_{\mathrm{Tam}}(1) = \prod_{p \text{ prime}} \delta_p(1) = 0.5053\dots,$$

where $\delta_2(1) = 241/396$, $\delta_3(1) = 1924625/2125728$, and for primes $p \geq 5$ we have

$$\delta_p(1) = 1 - \frac{p(6p^7 + 9p^6 + 9p^5 + 7p^4 + 8p^3 + 7p^2 + 9p + 6)}{6(p+1)^2(p^8 + p^6 + p^4 + p^2 + 1)}.$$

(2) For primes ℓ, we have $P_{\mathrm{Tam}}(\ell) = \sum_p \delta_p(\ell) \prod_{\substack{q \neq p \\ \text{prime}}} \delta_q(1)$.

Example. *These tables give $P_{\mathrm{Tam}}(1), \dots, P_{\mathrm{Tam}}(12)$, and show the convergence to $P_{\mathrm{Tam}}(1)$, $P_{\mathrm{Tam}}(2)$, and $P_{\mathrm{Tam}}(3)$.*

m	1	2	3	4	5	6
$P_{\mathrm{Tam}}(m)$	$0.5053\dots$	$0.3391\dots$	$0.0683\dots$	$0.0622\dots$	$7.98\dots \times 10^{-5}$	$0.0158\dots$
m	7	8	9	10	11	12
$P_{\mathrm{Tam}}(m)$	$5.56\dots \times 10^{-6}$	$0.0056\dots$	$0.0011\dots$	$4.56\dots \times 10^{-5}$	$2.01\dots \times 10^{-7}$	$0.0015\dots$

TABLE 1. Proportions $P_{\mathrm{Tam}}(1), \dots, P_{\mathrm{Tam}}(12)$

X	$\mathcal{N}_1(X)/\mathcal{N}(X)$	$\mathcal{N}_2(X)/\mathcal{N}(X)$	$\mathcal{N}_3(X)/\mathcal{N}(X)$
10^6	$0.5072\dots$	$0.3384\dots$	$0.0672\dots$
10^8	$0.5056\dots$	$0.3389\dots$	$0.0685\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	$0.5053\dots$	$0.3391\dots$	$0.0683\dots$

TABLE 2. Convergence to $P_{\mathrm{Tam}}(1), P_{\mathrm{Tam}}(2)$ and $P_{\mathrm{Tam}}(3)$

[1] By Lemma 3.2, the proportion of $E = E(a_4, a_6)$ with $\mathrm{ht}(E) \leq X$ that are already minimal is $\rho = 0.9960\dots$.

To place Theorem 1.1 in context, we recall that work by Klagsbrun and Lemke-Oliver [9], and of Chan, Hanselman and Li [6] shows that $\mathrm{Tam}(E)$ is unbounded in families of elliptic curves with prescribed $\mathbb{Q}$-rational 2-torsion. Moreover, Figure A.14 of [4] suggests an "average Tamagawa product" of $\approx 1.82\ldots$, which we confirm using

$$S_{\mathrm{Tam}(X)} := \sum_{\mathrm{ht}(E(a_4,a_6))\leq X} \mathrm{Tam}(E(a_4,a_6)), \tag{1.7}$$

and the convergent special value of $L_{\mathrm{Tam}}(-1)$.

Theorem 1.3. *We have*

$$L_{\mathrm{Tam}}(-1) = \lim_{X\to+\infty} \frac{S_{\mathrm{Tam}}(X)}{\mathcal{N}(X)} = 1.8193\ldots$$

Example. *Table 3 illustrates Theorem 1.3.*

X	10^4	10^6	$10^8\ldots$	$\ldots$	∞
$S_{\mathrm{Tam}}(X)/\mathcal{N}(X)$	$1.8358\ldots$	$1.8291\ldots$	$1.8240\ldots$	$\ldots$	$1.8193\ldots$

TABLE 3. Convergence to average Tamagawa product

Finally, we identify a subclass of elliptic curves that are convenient for height calculations. For $E = E(a_4, a_6)$, each $P \in E(\mathbb{Q})$ has the form $P = (\frac{A}{C^2}, \frac{B}{C^3})$, with $A, B, C \in \mathbb{Z}$, with $\gcd(A,C) = \gcd(B,C) = 1$. The *naive height* of P is $H(P) := \max(|A|, |C^2|)$. The Weil height is $h_W(P) := \log H(P)$, and the *canonical height* is

$$\widehat{h}(P) := \tfrac{1}{2} \lim_{n\to\infty} \frac{h_W(nP)}{n^2}. \tag{1.8}$$

Logarithmic and canonical heights are generally close. Generalizing an observation of Buhler, Gross and Zagier [5], we identify a natural subset of Tamagawa trivial curves that automatically (i.e. without computing $E(\mathbb{Q})$) have the property that $\widehat{h}(P) \geq \frac{1}{2}h_W(P)$ for every $P \in E(\mathbb{Q})$.

Definition. *A Tamagawa trivial $E(a_4,a_6)/\mathbb{Q}$ is* **convenient** *if it satisfies one of the following:*

1. *We have that $E(a_4,a_6)$ is a minimal model, and that $E(\mathbb{R})$ has one connected component with*
$$a_4 \leq 0 \quad \text{and} \quad (\alpha,\infty) \subset \left\{x \in \mathbb{R} : 2a_4x^2 + 8a_6x - a_4^2 < 0\right\},$$
where α is the real root of $x^3 + a_4x + a_6$.
2. *We have that $E(a_4,a_6)$ is a minimal model, and that $E(\mathbb{R})$ has two connected components with*
$$a_4 \leq 0 \quad \text{and} \quad (\gamma,\beta)\cup(\alpha,\infty) \subset \left\{x \in \mathbb{R} : 2a_4x^2 + 8a_6x - a_4^2 < 0\right\},$$
where $\gamma < \beta < \alpha$ are the real roots of $x^3 + a_4x + a_6$.

Lemma 1.4. *If $E(a_4,a_6)$ is convenient, then for every $P \in E(\mathbb{Q})$ we have $\widehat{h}(P) \geq \frac{1}{2}h_W(P)$.*

As a corollary to Theorem 1.1, we show that convenient curves have a natural density using

$$\mathcal{N}_c(X) := \#\{E(a_4,a_6) \text{ convenient } : \ \mathrm{ht}(E(a_4,a_6)) \leq X\}. \tag{1.9}$$

Corollary 1.5. *We have*

$$\lim_{X\to+\infty} \frac{\mathcal{N}_c(X)}{\mathcal{N}(X)} = \frac{12805748865(2+\sqrt{6})}{1830488\pi^{10}} \cdot P_{\mathrm{Tam}}(1) = 0.1679\ldots.$$

Example. *Table 4 illustrates Corollary 1.5.*

X	10^5	10^6	$10^7\ldots$	$\ldots$	∞
$\mathcal{N}_c(X)/\mathcal{N}(X)$	$0.1741\ldots$	$0.1687\ldots$	$0.1678\ldots$	$\ldots$	$0.1679\ldots$

TABLE 4. Proportions of convenient curves

The results obtained here are not difficult to derive. They follow from an analysis of Tate's algorithm. In Section 2 we provide a slight reformulation of the algorithm that is amenable for arithmetic statistics. In Section 3.1 we prove Theorems 1.1 and 1.3 using this analysis. In Section 3.2 we prove Lemma 1.4 by making use of standard facts about local height functions, and by adapting a clever device of Buhler, Gross and Zagier [5]. Finally, in Section 3.3 we derive Corollary 1.5 from Theorem 1.1 and Lemma 1.4.

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2. Tate's algorithm over $\mathbb{Z}$

Given a prime p, Tate's algorithm [14, 15] is a recursive procedure that determines the minimal model, conductor, Kodaira type, and the Tamagawa number c_p of an elliptic curve. There are eleven steps, often involving changes of variable that produce simpler p-adic models. The algorithm can terminate at any of the first ten steps. Curves that reach the eleventh step are not p-minimal. This eleventh step then applies the substitution $(x, y) \to (p^2x, p^3y)$, giving a p-integral model with discriminant that is reduced by a factor of p^{12}. The resulting curve is inserted into the algorithm at step one, and the algorithm eventually terminates due to the nonvanishing of discriminants.

We offer a reformulation of the algorithm that is suited for arithmetic statistics. Our goal is to compute $\delta'_p(K, n)$, the proportion of curves $E(a_4, a_6)$ that are p-minimal, have Kodaira type K, and Tamagawa number $c_p = n$. We consider short Weierstrass curves $E = E(a_4, a_6)$, where $a_4, a_6 \in \mathbb{Z}$, with non-zero discriminant $\Delta = \Delta(a_4, a_6) := -16(4a_4^3 + 27a_6^2)$. A model is *p-minimal* if the p-adic valuation of Δ is minimal among p-integral models. The desired proportions, as well as some others, are summarized in tables in the Appendix.

2.1. Classification for primes $p \geq 5$.

For primes $p \geq 5$, the algorithm uses five or seven numbers, which we refer to as the *Tate data for E at p* (i.e. $\alpha_4, \alpha_6, A_4, A_6, d \in \mathbb{Z}$ and $t, s \in \mathbb{Z}_p^\times$). The first five quantities are easily defined. We let $d := v_p(\Delta)$, and we let (note: $v_p(0) := +\infty$)

$$\alpha_4 := v_p(a_4) \quad \text{and} \quad \alpha_6 := v_p(a_6). \tag{2.1}$$

Moreover, if $a_4 \neq 0$ or $a_6 \neq 0$, then A_4 and A_6 are defined by

$$a_4 = p^{\alpha_4} A_4 \quad \text{and} \quad a_6 = p^{\alpha_6} A_6. \tag{2.2}$$

We require the following p-minimality criterion, whose proof defines the invariants s and t.

Lemma 2.1. *For primes $p \geq 5$, $E(a_4, a_6)$ is non-minimal at p if and only if $\alpha_4 \geq 4$ and $\alpha_6 \geq 6$.*

Proof. Obviously, if $\alpha_6 \geq 6$ and $\alpha_4 \geq 4$, then the curve is non-minimal at p. To justify the converse, we note that any counterexamples would satisfy $v_p(\Delta) \geq 12$ and

$$v_p(4a_4^3) = v_p(27a_6^2) < v_p(\Delta), \tag{2.3}$$

due to the required cancellation of p-adic valuations. Furthermore, (2.3) implies that $2\alpha_6 = 3\alpha_4$, which in turn implies that $3 \mid \alpha_6$ and $2 \mid \alpha_4$.

For curves satisfying (2.3), we define invariants s and t which are often required for Tate's algorithm. As $v_p(\Delta) > 0$, we have that $-3A_4$ is a quadratic residue modulo p, and so there is a p-adic unit $t \in \mathbb{Z}_p^\times$ with $a_4 = -3p^{\alpha_4}t^2$. After a short calculation, which requires the correct choice of sign for t, we obtain a p-adic unit $s \in \mathbb{Z}_p^\times$ for which $a_6 = 2p^{\alpha_6}t^3 + p^{d-\alpha_6}s$. The utility of s and t arises from the singular point $(p^{\alpha_6/3}t, 0)$ on

$$y^2 \equiv x^3 - 3p^{\alpha_4}t^2x + 2p^{\alpha_6}t^3 \pmod{p^{d-\alpha_6}}.$$

Under the substitution $x \to x + p^{\alpha_6/3}t$, it is mapped conveniently to $(0, 0)$ on

$$y^2 = x^3 + 3p^{\alpha_6/3}tx^2 + p^{d-\alpha_6}s. \tag{2.4}$$

This substitution does not change the discriminant, and one can apply Tate's algorithm to this simpler model. For $\alpha_6 \in \{0, 3\}$, the algorithm terminates at one of the first ten steps, implying the p-minimality of the original model. The remaining cases satisfying (2.3) have $\alpha_6 \geq 6$ and $\alpha_4 \geq 4$, thereby completing the proof. ∎

We now reformulate the algorithm (for $p \geq 5$) by identifying its steps with one of eleven disjoint possibilities for (α_4, α_6, d). The first ten cases, which correspond to p-minimal models, are in one-to-one correspondence with the possible Kodaira types. We compute $\delta'_p(K, n)$ in each case. By Lemma 2.1, the eleventh case, where $\alpha_4 \geq 4$ and $\alpha_6 \geq 6$, correspond to non-minimal short Weierstrass models. We follow the steps as ordered in Section IV.9 of [14], and so it would be convenient for the reader to have this reference readily available.

Case 1 ($d = 0$). This case is Kodaira type I_0, which has good reduction at p. As $d = v_p(\Delta) = 0$, we have $(a_4, a_6) \not\equiv (-3w^2, 2w^3) \pmod{p}$, for any $w \in \mathbb{F}_p$. Therefore, there are $p^2 - p$ choices of (a_4, a_6) out of p^2 total possible pairs modulo p, and so Tate's algorithm gives $\delta'_p(I_0, 1) = (p-1)/p$.

Case 2 ($\alpha_4 = \alpha_6 = 0 < d$). This case is Kodaira type $I_{n\geq 1}$, where $0 < n = d = v_p(\Delta)$. We require (2.3) (i.e. cancellation of p-adic valuations in the discriminant), and we employ the discussion in the proof of Lemma 2.1, where $(a_4, a_6) = (-3t^2, 2t^3 + p^d s)$. If we let $\varepsilon(n) := ((-1)^n + 3)/2$, then the algorithm gives

$$c_p := \frac{n\left(1 + \left(\frac{3t}{p}\right)\right)}{2} + \frac{\varepsilon(n)\left(1 - \left(\frac{3t}{p}\right)\right)}{2}.$$

For each $d \geq 1$, we consider $(a_4, a_6) \pmod{p^{d+1}}$. Since $p \nmid t$, there are $p^{d+1} - p^d$ choices of t. We also have $p-1$ choices of $s \not\equiv 0 \pmod{p}$. This gives us $p^d(p-1)^2$ choices of (a_4, a_6) out of p^{2d+2} total possible pairs modulo p^{d+1}. If $n > 2$, then half of these choices will have $c_p = n$, and the other half will have $c_p = 1$ or 2. Therefore, we have that $\delta'_p(I_1, 1) = (p-1)^2/p^3$, $\delta'_p(I_2, 2) = (p-1)^2/p^4$, and $\delta'_p(I_{n\geq 3}, n) = \delta'_p(I_{n\geq 3}, \varepsilon(n)) = (p-1)^2/2p^{n+2}$.

Case 3 ($\alpha_4 \geq 1$ and $\alpha_6 = 1$). This is Kodaira type II, with $c_p = 1$. There are p choices of $a_4 \pmod{p^2}$ and $p-1$ choices of $a_6 \pmod{p^2}$, giving $p(p-1)$ many options from p^4 possibilities. Therefore, we have $\delta'_p(II, 1) = (p-1)/p^3$.

Case 4 ($\alpha_4 = 1$ and $\alpha_6 \geq 2$). This case is Kodaira type III, where $c_p = 2$. There are $p - 1$ choices of $a_4 \pmod{p^2}$, and 1 choice of $a_6 \pmod{p^2}$. As there are p^4 many possible pairs, we have $\delta'_p(III, 2) = (p-1)/p^4$.

Case 5 ($\alpha_4 \geq 2$ and $\alpha_6 = 2$). This case is Kodaira type IV, where $c_p \in \{1, 3\}$. As $a_6 = p^2 A_6$, Tate's algorithm gives $c_p := 2 + \left(\frac{A_6}{p}\right)$. There are p choices of $a_4 \pmod{p^3}$ and $p-1$ choices of $a_6 \pmod{p^3}$, and p^6 possible pairs modulo p^3. Half of these pairs have $c_p = 1$ (resp. $c_p = 3$). Therefore, we have $\delta'_p(IV, 1) = \delta'_p(IV, 3) = (p-1)/2p^5$.

Case 6 ($\alpha_4 \geq 2, \alpha_6 \geq 3$ and $d = 6$). This case is Kodaira type I_0^*, where $c_p \in \{1, 2, 4\}$. This step of Tate's algorithm requires an auxiliary polynomial $P(T)$, which we now define. Given a long Weierstrass model

$$y^2 + \underline{a}_1 xy + \underline{a}_3 y = x^3 + \underline{a}_2 x^2 + \underline{a}_4 x + \underline{a}_6, \tag{2.5}$$

(possibly after a change of variable) so that $p \mid \underline{a}_2$, $p^2 \mid \underline{a}_4$, and $p^3 \mid \underline{a}_6$, we define

$$P(T) := T^3 + p^{-1}a_2 T^2 + p^{-2}a_4 T + p^{-3}a_6. \tag{2.6}$$

As we are working with short Weierstrass models, we have $P(T) = T^3 + a_4 p^{-2} T + a_6 p^{-3}$, and its discriminant is $2^{-4}p^{-6}\Delta$. Since $d = 6$, we see that $P(T)$ has distinct roots modulo p. In this case, the algorithm states that the Tamagawa number is 1 more than the number of roots of $P(T)$ in $\mathbb{F}_p$ (the finite field with p elements).

To calculate $\delta'_p(I_0^*, 1)$, we count the number of trace 0 separable cubics over $\mathbb{F}_p$. We first consider the number of choices of $P(T)$ which are irreducible, corresponding to $c_p = 1$. There are $p^3 - p$ elements of $\mathbb{F}_{p^3}$ not in $\mathbb{F}_p$, and $1/p$ of those have trace 0. Thus there are $(p^2 - 1)/3$ possible choices for $P(T)$ to be irreducible. We next consider the possibility that $P(T)$ factors as $Q(T)(T - \alpha)$, with $Q(T)$ an irreducible quadratic, corresponding to $c_p = 2$. In this case α is uniquely determined by $Q(T)$ so that the trace of $P(T)$ is 0. There are $(p^2 - p)/2$ irreducible quadratics modulo p, and therefore $(p^2 - p)/2$ possible choices of $P(T)$ with a single root in $\mathbb{F}_p$. Finally we consider the case that $P(T)$ factors completely in $\mathbb{F}_p$ with distinct roots, corresponding to the case $c_p = 4$. There are $\binom{p}{3}$ ways of choosing 3 distinct roots in $\mathbb{F}_p$, and $1/p$ of those have trace 0. Thus, there are $(p-1)(p-2)/6$ choices for $P(T)$ in this case. There are p^3 total possible choices of $a_4 \pmod{p^3}$ and p^4 total possible choices of $a_6 \pmod{p^4}$. Therefore, we find that $\delta'_p(I_0^*, 1) = (p^2 - 1)/3p^7$, $\delta'_p(I_0^*, 2) = (p-1)/2p^6$, and $\delta'_p(I_0^*, 4) = (p-1)(p-2)/6p^7$.

Case 7 ($\alpha_4 = 2,\ \alpha_6 = 3$ and $d > 6$). This case is Kodaira type $I_{n\geq 1}^*$ with $c_p \in \{2, 4\}$, where $0 < n := d - 6$. We employ model (2.4), where $a_4 = -3p^2 t^2$ and $a_6 = 2p^3 t^3 + p^{d-3} s$. Tate's algorithm requires the polynomial $P(T)$ defined in (2.6), as well as an additional auxiliary polynomial which we denote by $R(Y)$. Given a long Weierstrass model (2.5) (possibly after a change of variable) so that $p^2 \mid \underline{a}_3$, and $p^4 \mid \underline{a}_6$, we define

$$R(Y) := Y^2 + p^{-2}a_3 Y - p^{-4}a_6. \tag{2.7}$$

As our curves are in short form, we have $P(T) = T^3 + 3tT^2 + p^n s$, and $R(Y) = Y^2 - p^{n-1}s$. The Tamagawa number c_p depends on $P(T)$ and $R(Y)$. If n is odd, then we consider the roots of $R(Y)$. We make the substitution $p^{-n+1}R(p^{\frac{n-1}{2}}Y) = Y^2 - s$ which corresponds to the application of the sub-procedure of Step 7 of Tate's algorithm n times. The number of roots depends on whether or not s is a quadratic residue. The algorithm gives $c_p := 3 + \left(\frac{s}{p}\right)$. If n is even, then we consider the number of roots of $P(T)$. We make the substitution $p^{-n}P(p^{\frac{n}{2}}T) \equiv 3tT^2 + s \pmod{p}$ which corresponds to n steps through the sub-procedure of Step 7 of Tate's algorithm. The number of roots depends on whether or not $-s/3t$ is a quadratic residue. The algorithm gives that $c_p := 3 + \left(\frac{-s/3t}{p}\right)$.

To calculate the proportion of curves satisfying this condition for a given $n \geq 1$, we note that $(a_4, a_6) \pmod{p^{n+4}}$ is determined by any choice of $t \pmod{p^{n+2}}$ and $s \pmod{p}$, where $p \nmid st$. Half of the possible choices of s correspond to each c_p. Therefore, we obtain $\delta'_p(I_n^*, 2) = \delta'_p(I_n^*, 4) = (p-1)^2/2p^{7+n}$.

Case 8 ($\boldsymbol{\alpha_4 \geq 3}$ and $\boldsymbol{\alpha_6 = 4}$). This is Kodaira type IV^*, with $c_p \in \{1, 3\}$. As $a_6 = p^4 A_6$, the algorithm gives $c_p := 2 + \left(\frac{A_6}{p}\right)$. There are p^2 choices of $a_4 \pmod{p^5}$ and $p-1$ choices of $a_6 \pmod{p^5}$, and there are p^{10} possible pairs. Half of these pairs have $c_p = 1$ (resp. $c_p = 3$), and so we obtain $\delta'_p(IV^*, 1) = \delta'_p(IV^*, 3) = (p-1)/2p^8$.

Case 9 ($\boldsymbol{\alpha_4 = 3}$ and $\boldsymbol{\alpha_6 \geq 5}$). This case is Kodaira type III^*, where $c_p = 2$. There are $p(p-1)$ many choices of $a_4 \pmod{p^5}$, and 1 choice of $a_6 \pmod{p^5}$. As there are p^{10} possible pairs modulo p^5, we obtain $\delta'_p(III^*, 2) = (p-1)/p^9$.

Case 10 ($\boldsymbol{\alpha_4 \geq 4}$ and $\boldsymbol{\alpha_6 = 5}$). This case is Kodaira type II^*, where $c_p = 1$. This case depends on $(a_4, a_6) \pmod{p^6}$. There are p^2 choices of $a_4 \pmod{p^6}$ and $p-1$ choices of $a_6 \pmod{p^6}$. As there are p^{12} possible pairs modulo p^6, we obtain $\delta'_p(II^*, 1) = (p-1)/p^{10}$.

Case 11 ($\boldsymbol{\alpha_4 \geq 4}$ and $\boldsymbol{\alpha_6 \geq 6}$). As the model is not minimal, the algorithm replaces a_4 and a_6 with a_4/p^4 and a_6/p^6 respectively. One repeats these substitutions until one obtains a model which is one of the ten cases above.

2.2. **Classification for $\boldsymbol{p = 3}$.** The Tate data for $p = 3$ also consists of five or seven numbers. The first five (i.e. $\alpha_4, \alpha_6, A_4, A_6$, and d) are defined by (2.1) and (2.2). The next lemma classifies those $E(a_4, a_6)$ that are not 3-minimal, and its proof defines invariants s and t in some of the cases where further invariants are required.

Lemma 2.2. *The curve $E(a_4, a_6)$ is not 3-minimal if and only if (α_4, α_6, d) satisfies one of the following conditions:*

(1) We have that $\alpha_4 \geq 4$ and $\alpha_6 \geq 6$.
(2) We have that $\alpha_4 = \alpha_6 = 3$ and $d \geq 12$.

Proof. Obviously, $E(a_4, a_6)$ is not minimal at 3 when (1) holds. Therefore, to prove the lemma, suppose that $E(a_4, a_6)$ does not satisfy (1) and is not 3-minimal. Then, we have $v_3(\Delta) \geq 12$ and

$$v_3(4a_4^3) = v_3(27a_6^2) < v_3(\Delta), \tag{2.8}$$

due to the necessary cancellation of 3-adic valuations. Furthermore, (2.8) implies that $2\alpha_6 + 3 = 3\alpha_4$, which in turn implies that $3 \mid \alpha_6$ and α_4 is odd. Arguing precisely as in the previous subsection, we find that for any curve satisfying (2.8) there are 3-adic units $s, t \in \mathbb{Z}_3^\times$ for which

$$a_4 = -3^{\alpha_4}t^2 \quad \text{and} \quad a_6 = 2 \cdot 3^{\alpha_6}t^3 + 3^{d-\alpha_6-3}s.$$

The substitution $x \to x + 3^{\alpha_6/3}t$ returns the model

$$y^2 = x^3 + 3^{1+\alpha_6/3}t \cdot x^2 + 3^{d-\alpha_6-3}s, \tag{2.9}$$

which has the same discriminant. The assumption that $E(a_4, a_6)$ does not satisfy (1) implies that $\alpha_6 \in \{0, 3\}$. For $\alpha_6 = 0$, the algorithm applied to this model terminates at one of the first 7 steps, and so the original model is 3-minimal. However, if $\alpha_4 = \alpha_6 = 3$, and $d \geq 12$, we see that (2.9) is not minimal. The additional substitution $(x, y) \to (9x, 27y)$ returns the reduced model

$$y^2 = x^3 + tx^2 + 3^{d-12}s, \tag{2.10}$$

with smaller discriminant $3^{-12}\Delta$. This completes the proof of the lemma. ∎

We also require two further invariants s and t when $\alpha_6 \in \{0, 3\}$, and $v_p(4a_4^3) \geq v_p(27a_6^2) = v_3(\Delta)$. We note that if $v_p(4a_4^3) = v_p(27a_6^2) = d$, then $A_4 \equiv 1 \pmod{3}$. A straightforward calculation with Hensel's lemma shows that there is a 3-adic unit $t \in \mathbb{Z}_3^\times$, and $s \in \{0, \pm 1\}$ for which $a_6 = 3^{\alpha_6}t^3 + 3^{\alpha_6/3}a_4 t + 3^{\alpha_6+1}s$. The substitution $x \to x - 3^\ell t$ gives the new model

$$y^2 = x^3 - 3^{\alpha_6/3+1}t \cdot x^2 + (a_4 + 3^{\frac{2}{3}\alpha_6+1}t^2) \cdot x + 3^{\alpha_6+1}s. \tag{2.11}$$

In this situation we shall apply the algorithm to this model.

As in the previous subsection, we reformulate the algorithm for $p = 3$ by identifying its steps with one of a number of disjoint possibilities for (α_4, α_6, d). Unlike the situation for primes $p \geq 5$, a few further cases arise due to the fact that 3−minimal models in short Weierstrass form do not always exist for $E(a_4, a_6)$. These extra cases are designated below with an $*$, and we let $\widehat{\delta}_3(K, n)$ be the proportion of curves which fall into these cases.

Case 1 ($\boldsymbol{\alpha_4 = 0}$). This case is Kodaira type I_0, when we have good reduction at $p = 3$. We have $\alpha_4 = 0$ and $v_3(\Delta) = 0$. There are 2 choices for a_4 modulo 3, and so $\delta_3'(I_0, 1) = 2/3$.

Case 1* ($\boldsymbol{\alpha_4 = \alpha_6 = 3,}$ **and** $\boldsymbol{d = 12}$). This case is also for Kodaira type I_0, but only arises in situations where the original E is not 3-minimal. Therefore, we employ model (2.10) per the discussion above. Namely, we have $a_4 = -3^3t^2$, and $a_6 = 2 \cdot 3^3t^3 + 3^6s$, with $3 \nmid st$, and so the change of variable reduces E to the 3-minimal model $y^2 = x^3 + tx^2 + s$, which has discriminant $-16(4t^3s + 27s^2) \not\equiv 0 \pmod 3$. There are 18 choices for t modulo 27 with $t \not\equiv 0 \pmod 3$, and for each choice of t, there are 2 choices of s modulo 3. Together, these determine a_4 modulo 3^6 and a_6 modulo 3^7 Combining these observations, we obtain $\widehat{\delta}_3(I_0, 1) = 4/3^{11}$.

Case 2 (**None**). This case is for Kodaira types $I_{n\geq 1}$. Since $(x+c)^3 = x^3 + 3cx^2 + \ldots$, the substitutions $x \to x + c$, with $c \in \mathbb{Z}$, always produces models with $3 \mid b_2$, where $b_2 := \underline{a}_1^2 + 4\underline{a}_2$ for long models as in (2.5). Tate's algorithm, which employs b_2, bypasses these cases for short models when $p = 3$, and so we have $\delta_3'(I_{n\geq 1}, c_3) = 0$.

Case 2* ($\boldsymbol{\alpha_4 = \alpha_6 = 3}$ **and** $\boldsymbol{d > 12}$). This case concerns Kodaira types $I_{n\geq 1}$, when E is not 3−minimal. Following the discussion above, we employ (2.10), where $a_4 = -3^3t^2$ and $a_6 = 2 \cdot 3^3t^3 + 3^{d-6}s$, with $3 \nmid st$. The new model $y^2 = x^3 + tx^2 + 3^{d-12}s$ is 3-minimal, and has discriminant $-16(4t^33^{d-12}s + 3^{2d-24}s^2) \equiv 0 \pmod{3^{d-12}}$. Tate's algorithm gives type I_n, with $n = d - 12$, the 3-adic valuation of this discriminant. Moreover, the Tamagawa number c_3 depends on t modulo 3, and we find that

$$c_3 := \frac{n\left(1 + \left(\frac{t}{3}\right)\right)}{2} + \frac{\varepsilon(n)\left(1 - \left(\frac{t}{3}\right)\right)}{2}.$$

Here $\varepsilon(n)$ is as in Case 2 of the previous subsection. If we fix $n \geq 1$, and $t \pmod 3$, then $a_4 \pmod{3^4}$ is uniquely determined. Furthermore, for each choice of a_4, there are 2 choices of $a_6 \pmod{3^{n+7}}$. Combining these facts, we obtain $\widehat{\delta}_3(I_1, 1) = 4/3^{12}$, $\widehat{\delta}_3(I_2, 2) = 4/3^{13}$, and $\widehat{\delta}_3(I_{n\geq 3}, c_3) = 2/3^{n+11}$.

Case 3. This case is Kodaira type II, where $c_3 = 1$. The following three possibilities for this case are:

(i) We have $\alpha_4 \geq 1$ and $\alpha_6 = 1$.
(ii) We have $\alpha_4 \geq 1, \alpha_6 = 0$, $d = 3$, and $s \neq 0$. Note that s is from model (2.11).
(iii) We have $\alpha_4 = 1, \alpha_6 = 0$, and $d = 4$.

There are 6 pairs (a_4, a_6) modulo 9 satisfying (i), and so their proportion is 6/81. For curves satisfying (ii), we use model (2.11). In this situation we have $a_6 = 3^{\alpha_6}t^3 + 3^{\alpha_6/3}a_4t + 3^{\alpha_6+1}s$. We have 2 choices for $a_4 \equiv 0, 3 \pmod 9$, 2 choices for $t \equiv \pm 1 \pmod 3$, and 2 choices for $s = \pm 1$, which together determine a_6 modulo 9. Therefore their proportion is 8/81. For curves satisfying (iii), we use model (2.9). We have $a_4 = -3t^2$ and $a_6 = 2t^3 + 3s$. There are 2 choices each for $t, s \equiv \pm 1 \pmod 3$, which together determine (a_4, a_6) modulo 9. Therefore, their proportion is 4/81. Combining these observations, we obtain $\delta_3'(II, 1) = \frac{2}{9}$.

Case 4. This case is Kodaira type III, where $c_3 = 2$. The following two possibilities for this case are:

(i) We have $\alpha_4 = 1$ and $\alpha_6 \geq 2$.
(ii) We have $\alpha_4 \geq 1, \alpha_6 = 0$, $d = 3$, and $s = 0$. Note that s is from model (2.11).

There are 2 pairs (a_4, a_6) modulo 9 satisfying (i). For curves satisfying (ii), we use model (2.11). In this situation, we have 2 choices for $a_4 \equiv 0$ or $3 \pmod 9$, and 2 choices for $t \equiv \pm 1 \pmod 3$, which together determine a_6 modulo 9. Hence, there are 4 pairs (a_4, a_6) modulo 9 satisfying (ii). Overall, the 6 pairs $(a_4, a_6) \in \{(3, 0), (6, 0), (0, \pm 1), (3, \pm 4) \pmod 9\}$, chosen from 81 possibilities, gives $\delta_3'(III, 2) = 2/27$.

Case 5. This case is Kodaira type IV, where $c_3 \in \{1, 3\}$. The following two possibilities for this case are:

(i) We have $\alpha_4 \geq 2$ and $\alpha_6 = 2$.
(ii) We have $\alpha_4 = 1$, $\alpha_6 = 0$, and $d = 5$.

For condition (i), Tate's algorithm gives $c_3 := 2 + \left(\frac{A_6}{3}\right)$. For condition (ii), the algorithm gives $c_3 := 2 + \left(\frac{s}{3}\right)$, where s corresponds to model (2.9).

It is simple to determine the proportions of curves in these cases. In case of (i), we have $9 \mid a_4$, and 1 choice of a_6 modulo 27 for each c_3, giving a proportion of 3^{-5} each of the two possible Tamagawa numbers. Curves satisfying (ii) are cases of (2.9), and so we have $a_4 = -3t^2$ and $a_6 = 2t^3 + 9s$. There are 6 choices t modulo 9, and one choice of s modulo 3 for each c_3. These together determine the 6 choices of a_4 and a_6 modulo 27 which fall under this set of conditions for each c_3. Combining these observations, we obtain $\delta_3'(IV, 1) = \delta_3'(IV, 3) = 1/81$.

Case 6. This case is Kodaira type I_0^*, where $c_3 \in \{1, 2, 4\}$. The following two possibilities for this case are:

(i) We have $\alpha_4 = 2$ and $\alpha_6 \geq 3$.
(ii) We have $\alpha_4 = 1$, $\alpha_6 = 0$, and $d = 6$.

In each case, c_3 is 1 more than the number of roots of the polynomial $P(T)$ defined in (2.6), possibly after a change of variable. Assuming (i), we have $P(T) = T^3 + A_4T + p^{\alpha_6 - 3}A_6$, This polynomial has 1 root modulo 3 if $A_4 \equiv 1 \pmod 3$, 3 distinct roots if $A_4 \equiv -1 \pmod 3$ and $\alpha_6 > 3$, and 0 roots modulo 3 if $A_4 \equiv -1 \pmod 3$ and $\alpha_6 = 3$. For curves satisfying (ii), we define $P(T)$ using model (2.9). We have $a_4 = -3t^2$ and $a_6 = 2t^3 + 27s$, where s and t are 3-adic units, and $P(T) = T^3 + tT^2 + s$. This polynomial has 1 root modulo 3 if $t \equiv s \pmod 3$, and 0 roots modulo 3 if $t \not\equiv s \pmod 3$. Each $(a_4, a_6) \pmod{81}$ corresponds to a pair t modulo 27 and s modulo 3. Therefore, we have that $c_p = 1$ for 24 pairs of (a_4, a_6) modulo 81 (6 pairs satisfying (i), and 18 satisfying (ii)) $c_p = 2$ for 27 pairs of (a_4, a_6) modulo 81 (9 pairs satisfying (i), and 18 satisfying (ii)), and $c_p = 4$ for 3 pairs of (a_4, a_6) modulo 81 (all satisfying condition (i)). Hence, we have $\delta_3'(I_0^*, 1) = 8/3^7$, $\delta_3'(I_0^*, 2) = 1/3^5$, and $\delta_3'(I_0^*, 4) = 1/3^7$.

Case 7 ($\boldsymbol{\alpha_4 \geq 1, \alpha_6 = 0}$ **and** $\boldsymbol{d \geq 7}$). This case is Kodaira type $I_{n\geq 1}^*$, with $c_3 \in \{2, 4\}$, where $0 < n := d - 6$. We use model (2.9), and define the auxiliary polynomials $P(T)$ and $R(Y)$ as in (2.6) and (2.7). We have that $P(T) = T^3 + tT^2 + 3^n s$ and $R(Y) = Y^2 - 3^{n-1}s$.

If d is odd, then we have

$$3^{-n+1}R(3^{\frac{n-1}{2}}Y) \equiv Y^2 - s \pmod 3,$$

which follows from the application of the sub-procedure in Step 7 of the algorithm n times. We have $c_3 := 4$ when $s \equiv 1 \pmod 3$ (so that $R(Y)$ factors over $\mathbb{Z}_3$), and $c_3 := 2$ when $s = -1 \pmod 3$ (so that $R(Y)$ does not factor).

If d is even, then we have that $3^{-n}P(3^{\frac{n}{2}}T) \equiv tT^2 + s \pmod 3$, which corresponds to n steps through the sub-procedure of Step 7 of Tate's Algorithm. We have $c_3 := 4$ if $s \equiv -t \pmod 3$ (so that $P(T)$ factors completely), and $c_3 := 2$ if $s \equiv t \pmod 3$ (so that $P(T)$ does not factor completely).

For each $n \geq 1$ and each $c_3 \in \{2, 4\}$, we have $2 \cdot 3^{n+2}$ choices for t modulo 3^{n+3} and 1 choice of s modulo 3 (depending on c_3 and t). This determines $2 \cdot 3^{n+2}$ pairs (a_4, a_6) modulo 3^{n+4} corresponding to Kodaira type I_n^* with the chosen Tamagawa number c_3, out of a total of 3^{2n+8} pairs modulo 3^{n+4}. Hence, Tate's algorithm gives $\delta_3'(I_n^*, 2) = 2/3^{n+6}$, and $\delta_3'(I_n^*, 4) = 2/3^{n+6}$.

Case 8. This case is Kodaira type IV^*, where $c_3 \in \{1, 3\}$. The following two possibilities for this case are:

(i) We have $\alpha_4 \geq 3$ and $\alpha_6 = 4$.
(ii) We have $\alpha_4 \geq 3, \alpha_6 = 3$, $d = 9$, and $s \neq 0$. Note that s is from model (2.11).

Tate's algorithm gives $c_3 := 2 + \left(\frac{A_6}{3}\right)$ under condition (i), and $c_3 := 2 + \left(\frac{s}{3}\right)$ under condition (ii). For condition (i), we have 1 choice for a_4 modulo 27, and one choice for a_6 modulo 3^5 for each of the two possible values of c_3 (i.e. a proportion of $1/3^8$). For condition (ii), there are 2 choices for $a_4 \equiv 0$ or $27 \pmod{3^4}$, 2 choices for $t \not\equiv 0 \pmod 3$, and 1 choice of $s \neq 0$ for each c_3. Together, these determines a_6 modulo 3^5. These pairs (a_4, a_6) occur with proportion $4/3^9$ for each of the two possible values for c_3. Therefore, we obtain $\delta_3'(IV^*, 1) = \delta_3'(IV^*, 3) = 7/3^9$.

Case 9. This case is Kodaira type III^*, where $c_3 = 2$. The following three possibilities for this case are:

(i) We have $\alpha_4 = 3$ and $\alpha_6 \geq 5$.
(ii) We have $\alpha_4 \geq 3, \alpha_6 = 3$, $d = 9$, and $s = 0$. Note that s is from model (2.11).
(iii) We have $\alpha_4 = \alpha_6 = 3$ and $d = 10$.

The proportion of curves satisfying (i) is $2/3^9$. For curves satisfying (ii), we use (2.11). In this situation, we find that $a_4 \equiv 0, 27 \pmod{3^4}$. Thus, there are 2 choices of a_4 modulo 3^4, and 2 choices for $t \equiv \pm 1 \pmod 3$. These choices determine a_6 modulo 3^5. Therefore, the proportion of curves satisfying this set of conditions is $4/3^9$. For curves satisfying (iii), we use (2.9). In this situation, we have $a_4 = -27t^2$ and $a_6 = 54t^3 + 81s$. There are 2 choices each for $t, s \equiv \pm 1 \pmod 3$. These choices together determine a_4 modulo 3^4 and a_6 modulo 3^5. Therefore, the proportion of these curves is $4/3^9$. Combining these observations, we obtain $\delta_3'(III^*, 2) = 10/3^9$.

Case 10. This case is Kodaira type II^*, where $c_3 = 1$. The following two possibilities for this case are:

(i) We have $\alpha_4 \geq 4$ and $\alpha_6 = 5$.
(ii) We have $\alpha_4 = \alpha_6 = 3$, and $d = 11$.

The proportion of curves satisfying (i) is $2/3^{10}$. For curves satisfying (ii), we use model (2.9). In this situation, there are 6 choices for t modulo 9 so that $t \not\equiv 0 \pmod 3$, and 2 choices for $s \equiv \pm 1 \pmod 3$. These choices together determine a_4 modulo 3^5 and a_6 modulo 3^6. Therefore, the proportion of pairs (a_4, a_6) satisfying these conditions is $4/3^{10}$ Combining these observations, we obtain $\delta_3'(II^*, 1) = 2/3^9$.

Case 11 ($\boldsymbol{\alpha_4 \geq 4}$ **and** $\boldsymbol{\alpha_6 \geq 6}$). As the model is not minimal, the algorithm replaces a_4 and a_6 with $a_4/3^4$ and $a_6/3^6$ respectively. One repeats these substitutions until one obtains a model which is one of the ten cases above.

2.3. **Classification for** $p = 2$**.** The Tate data for $p = 2$ also consists of five or seven numbers. The first five (i.e. $\alpha_4, \alpha_6, A_4, A_6$, and d) are defined by (2.1) and (2.2). The next lemma classifies those $E(a_4, a_6)$ that are not 2-minimal, and its proof defines invariants s and t in some of the cases where these invariants are required.

Lemma 2.3. *A curve $E(a_4, a_6)$ is not 2-minimal if and only if (a_4, a_6, d) satisfies one of the following conditions:*

(1) We have that $\alpha_4 \geq 4$ and $\alpha_6 \geq 6$.
(2) We have that $\alpha_4 \geq 4$, $a_6 \equiv 16 \pmod{64}$.
(3) We have $(a_4, a_6) \equiv (5, 6) \pmod 8$, where $d \geq 12$

Proof. Obviously, $E(a_4, a_6)$ is not minimal at 2 when (1) holds. Now, suppose that $E(a_4, a_6)$ does not satisfy (1) and is not 2-minimal. Then, we have $v_2(\Delta) \geq 12$. This implies that either $a_4 \geq 2$ and $a_6 \geq 4$, or

$$v_2(4a_4^3) = v_2(27a_6^2) < v_2(\Delta), \tag{2.12}$$

due to the necessary cancellation of 2-adic valuations. If $\alpha_4 = 2$ (resp. 3) and $\alpha_6 \geq 4$, then we find that Tate's algorithm terminates in Step 7 (resp. Step 8). If $\alpha_4 \geq 4$ and $a_6 \equiv 48$ or $32 \pmod{64}$, then Tate's algorithm terminates at Step 10. However, if $a_6 \equiv 16 \pmod{64}$, then we have $a_4 = 4A_4$ $a_6 = 16 + 2^k s$, where either $s = 0$ or s odd, and $k \geq 6$. The substitution $(x, y) \to (4x, 8y + 4)$, reduces the equation of the curve to

$$y^2 + y = x^3 + A_4 x + 2^{k-6} s, \tag{2.13}$$

which has discriminant $2^{-12}\Delta$. Since $\alpha_6 = 4$, we have $v_2(\Delta) = 12$, and so this model is 2-minimal, giving (2).

If $E(a_4, a_6)$ satisfies (2.12), then $2\alpha_6 = 3\alpha_4 + 2$, which in turn implies that α_4 is even. We find that $\alpha_4 \in \{0, 2\}$, however the possible case that $\alpha_4 = 2$ and $\alpha_6 = 4$ was already considered above. If $\alpha_4 = 0$, then $\alpha_6 = 1$. For any curve of this type satisfying (2.12) with $v_2(\Delta) \geq 12$, we find $-2^{-6}\Delta = A_4^3 + 27A_6^2 \equiv A_4 + 3 \equiv 0 \pmod 8$. Thus, $a_4 \equiv -3 \pmod 8$, and there is some $t \in Z_2$ so that $a_4 = -3t^2$. Moreover, there is a choice of sign of t so that $a_6 = 2t^3 + 2^{d-6}s$, for some 2-adic unit s. If $t \equiv 1 \pmod 4$ (so that $a_6 \equiv 2 \pmod 8$), then we find that Tate's algorithm terminates in either step 6 or 7. However if $t \equiv 3 \pmod 4$ (so that $a_6 \equiv 6 \pmod 8$), then the substitution $(x, y) \to (4x + t, 8y + 1)$ reduces the equation of the curve.

$$y^2 + xy = x^3 + \frac{3t-1}{4}x^2 + 2^{d-12}s \tag{2.14}$$

This completes the proof of the lemma, as this situation is case (3). □

Our analysis of Tate's algorithm requires invariants k, s, t, and v whenever (2.12) is satisfied with $\alpha_4 = 0$ and $v_2(\Delta) \geq 8$. As in the proof above, this implies that $(a_4, a_6) \equiv (1, 2) \pmod 4$. If d is even, we set $v = 2^{\frac{d-6}{2}}$, and otherwise we set $v = 0$. Then there is a unique $t \in \mathbb{Z}_2$ with $t \equiv a_6/2 \pmod 4$, so that $a_4 = -3t^2 + 2v$. This uses the fact that $d = 8$ if and only if $a_4 \equiv 1 \pmod 8$. A short calculation gives numbers s and k, where $a_6 = 2t^3 - 2vt + v^2 + 2^k s$, with either $s = 0$, or s is odd and $k \geq d - 6$. We find that $k > d - 6$ if d is even (so that $v \neq 0$), and $k = d - 6$ if d is odd (so that $v = 0$). After the substitution $(x, y) \to (x + t, y + 1 + v)$, we obtain

$$y^2 + 2xy + 2vy = x^3 + (3t - 1)x^2 + 2^k s. \tag{2.15}$$

As in the previous subsections, we reformulate the algorithm for $p = 2$ by identifying its steps with suitable disjoint possibilities for the Tate data $a_4, a_6, \alpha_4, \alpha_6, d$, and k. Unlike the case where $p \geq 5$, a further cases arise due to the fact that $2-$minimal models in short Weierstrass form do not always exist for E. These special cases are designated below with an $*$, and we define $\widehat{\delta}_2(K, n)$ to be the proportion of curves which fall into these cases.

Case 1 (**None**). This case is Kodaira type I_0, which has good reduction at $p = 2$. This case does not occur in a first pass through the algorithm since $16 \mid \Delta$. Therefore, we have $\delta_2'(I_0, 1) = 0$.

Case 1*. This is Kodaira type I_0 with $c_2 = 1$, where E is not 2-minimal. The two possibilities for this case are:

(i) We have $\alpha_4 \geq 4$ and $a_6 \equiv 16 \pmod{64}$.

(ii) We have $(a_4, a_6) \equiv (5, 6) \pmod 8$, and $d = 12$.

Under the condition (i), we use the model (2.13). After making the substitution $(x, y) \to (4x, 8y + 4)$, we obtain

$$y^2 + y = x^3 + A_4x + 2^{k-6}s.$$

The new discriminant $\Delta/64 = -64A_4^3 - 27(1 + 2^{k-2}s)^2$ is odd. The proportion of these curves is $1/2^{10}$.

Under the condition (ii), we use the model (2.14). We have $a_4 = -3t^2$, with $t \equiv 3 \pmod 4$, and $a_6 \equiv 2t^3 + 64s$. After making the substitution $(x, y) \to (4x + t, 8y + 1)$, we obtain

$$y^2 + xy = x^3 + \frac{3t-1}{4}x^2 + s.$$

The discriminant of this model is $\Delta/64 = -27(s\,t^3 + 16s^2)$, which is odd. We may take any choice of $a_4 \equiv 5 \pmod 8$. This choice determines t, and therefore determines $a_6 \pmod{128}$. The proportion of curves satisfying this situation is $1/2^{10}$. Therefore, Tate's algorithm gives $\widehat{\delta}_2(I_0, 1) = 1/1024 + 1/1024 = 1/512$.

Case 2 (**None**). This case is for Kodaira types I_n. There are no short form curves in this case as in Case 2 for $p = 3$ (i.e. $2 \mid b_2$ because $b_2 \equiv \underline{a}_1^2 \pmod 4$). Therefore, we have $\delta_2'(I_{n\geq1}, c_2) = 0$.

Case 2* ($\boldsymbol{(a_4, a_6) \equiv (5, 6) \pmod 8, d > 12}$). This case is for the Kodaira type $I_{n\geq1}$, with $n = d - 12$, where E is not 2-minimal. We use model (2.14). We have $a_4 = -3t^2$ and $a_6 = 2t^3 + 2^{d-6}s$, where $t \equiv 3 \pmod 4$. After making the substitution $(x, y) \to (4x + t, 8y + 1)$, we obtain

$$y^2 + xy = x^3 + \frac{3t-1}{4}x^2 + 2^{d-12}s.$$

The discriminant is $\Delta/64 = -27(2^{d-12}s\,t^3 + 2^{2d-20}s^2)$. Tate's algorithm gives $c_2 \in \{1, 2, n\}$, depending on the polynomial $T^2 + T + \frac{3t-1}{4}$ modulo 2. We have $c_2 := n$ if it has roots modulo 2 (i.e if $t \equiv 3 \pmod 8$). If it does not have roots modulo 2 (i.e. if $t \equiv 5 \pmod 8$), then $c_2 := 1$ (resp. $c_2 := 2$) if n is odd (resp. even).

Therefore to compute the proportions, for any n we may take any $a_4 \equiv 5 \pmod 8$, which determines $t \in \mathbb{Z}_2$, with $t \equiv 3 \pmod 4$. This then determines $a_6 \equiv 2t^3 + 2^{n+6} \pmod{2^{n+7}}$. Using $\varepsilon(n)$ as in Case 2 of Subsection 2.1, the algorithm gives that $\delta_2'(I_1, 1) = 1/2^{11}$, $\delta_2'(I_2, 2) = 1/2^{12}$, and $\delta_2'(I_{n\geq3}, n) = \delta_2'(I_{n\geq3}, \varepsilon(n)) = 1/2^{11+n}$.

Case 3 ($\boldsymbol{(a_4, a_6) \in \{(0, 2), (0, 3), (1, 0), (1, 1), (2, 2), (2, 3), (3, 2), (3, 3) \pmod 4\}}$). This case is Kodaira type II, where $c_2 = 1$. A brute force analysis shows that these cases correspond only to the indicated congruence conditions. As these account for 8 out of the 16 possible pairs (a_4, a_6) modulo 4, we obtain $\delta_2'(II, 1) = 1/2$.

Case 4 ($\boldsymbol{(a_4, a_6) \in \{(1, 3), (2, 1), (2, 0), (3, 0) \pmod 4\}}$). This case is Kodaira type III, where $c_2 = 2$. A brute force analysis shows that these cases correspond only to the indicated congruence conditions. As these account for 4 out of the 16 possible pairs (a_4, a_6) modulo 4, we obtain $\delta_2'(III, 2) = 1/4$.

Case 5 ($\boldsymbol{(a_4, a_6) \in \{(0, 1), (3, 1) \pmod 4\}}$). This case is for Kodaira type IV, where $c_2 \in \{1, 3\}$, depending on the parity of $(a_6 + a_4t + t^3 - v^2)/4$. The algorithm gives $c_2 := 3$ (resp. $c_2 := 1$) if this number is even (resp. odd). By brute force, we find that $c_2 = 1$ for

$$(a_4, a_6) \pmod 8 \in \{(0, 5), (3, 1), (4, 5), (7, 5) \pmod 8\},$$

and $c_2 = 3$ for

$$(a_4, a_6) \pmod 8 \in \{(0, 1), (3, 5), (4, 1), (7, 1) \pmod 8\}.$$

Therefore, we obtain $\delta_2'(IV, 1) = 1/16$, and $\delta_2'(IV, 3) = 1/16$.

Case 6. This case is for Kodaira type I_0^*, where $c_2 \in \{1, 2\}$. The following two possibilities for this case are:

(i) We have $\alpha_4 \geq 2$ and $a_6 \equiv 8, 12 \pmod{16}$.

(ii) We have $(a_4, a_6) \equiv (1, 2) \pmod 4$, and $k = 3$, where k is defined as in (2.15).

Assuming (i), there are numbers s and v for which $a_6 = 8s + v^2$, where s is odd and $v = 0$ (resp. $v = 2$) when $a_6 \equiv 8 \pmod{16}$ (resp. when $a_6 \equiv 12 \pmod{16}$). After making the substitution $(x, y) \to (x, y + v)$, we obtain $y^2 + 2vy = x^3 + a_4x + 8s$. Using this model, we define (as in (2.6))

$$P(T) = T^3 + \frac{1}{4}a_4T + s.$$

If $a_4 \equiv 4 \pmod 8$, then $P(T)$ is irreducible modulo 2 and so $c_2 := 1$. If $a_4 \equiv 0 \pmod 8$, then $P(T)$ has one root modulo 2, and so $c_2 := 2$. Therefore the contributions from condition (i) to $\delta_2'(I_0^*, 1)$ and $\delta_2'(I_0^*, 2)$ are both $1/64$.

For curves satisfying (ii), we use (2.15) to define

$$P(T) = T^3 + \frac{1}{2}(3t-1)T^2 + s,$$

and Tate's algorithm gives $c_2 := 1$ when $t \equiv 1 \pmod 4$, and $c_2 := 2$ when $t \equiv 3 \pmod 4$. Since $k = 3$, we have $v = 0$ or 2. To calculate the proportion of curves satisfying (ii), we may pick any $a_4 \equiv 1 \pmod 4$. Then we have $v = 0$ when $a_4 \equiv 1 \pmod 8$, and $v = 2$ otherwise. The choice of a_4 and c_2 fixes t uniquely, which then determines $a_6 \equiv 2t^3 - 2vt + v^2 + 8 \pmod{16}$. Thus, for each choice of c_2, condition (ii) contributes a proportion of 1/64 to $\delta_2'(I_0^*, c_2)$. Therefore Tate's algorithm gives $\delta_2'(I_0^*, 1) = \delta_2'(I_0^*, 2) = 1/64 + 1/64 = 1/32$.

Case 7. This case is for Kodaira type $I_{n\geq 1}^*$, where $c_2 \in \{2, 4\}$. The only possibilities for (α_4, α_6) are:

(i) We have $\alpha_4 = 2$ and $a_6 \equiv 0, 4 \pmod{16}$.
(ii) We have $a_4 \equiv 1 \pmod 4$, $a_6 \equiv 2 \pmod 8$, and $k \geq 4$, where k is defined in (2.15).

Instead of proceeding as in the previous cases, we determine the conditions which result in any given choice of n and $c_2 \in \{2, 4\}$. We are essentially working backwards through iterations of the sub-procedure in Step 7 of the algorithm. In particular, n is the number of iterations required. As illustrated in the two previous subsections, this step of the algorithm makes use of two auxiliary polynomials, $P(T)$ and $R(Y)$, which are defined by (2.6) and (2.7), from a long model (2.5) with $2 \mid \underline{a}_1$, $\underline{a}_2 \equiv 2 \pmod 4$, $4 \mid \underline{a}_3$, $8 \mid \underline{a}_4$ and $16 \mid \underline{a}_6$.

Suppose that $n = 2a + 1$ is odd. Then the sub-procedure finds a model for which

$$2^{-2a} R(2^a Y) \equiv Y^2 + Y \text{ or } Y^2 + Y + 1 \pmod 2,$$

and also satisfies

$$2^{-2a} P(2^a T) \equiv \begin{cases} T^2 \pmod 2 & \text{if } a > 0, \\ T^3 + T^2 \pmod 2 & \text{if } a = 0. \end{cases}$$

The point here is the $2^{-2a} P(2^a T) \pmod 2$ has a double root at $T = 0$.

These conditions are equivalent to the existence of $A, B, C \in \mathbb{Z}$, with A odd, so that $\underline{a}_3 = 2^{a+2} A$, $\underline{a}_4 = 2^{a+3} B$, and $\underline{a}_6 = 2^{2a+4} C$. If C is even, then we see that $R(Y)$ factors over $\mathbb{Z}_2$, and so the algorithm gives $c_2 := 4$. Otherwise, we have $c_2 := 2$. The substitutions $y \to y + 2x$ and $y \to y + 2^{a+2}$ do not alter any of the required conditions on $P(T)$ and $R(Y)$. Therefore, we may assume without loss of generality $\underline{a}_1 = 2u$ with $u \in \{0, 1\}$, and $A = 1$. Similarly, the substitution $x \to x + 2^{a+2}$ does not alter the required conditions, and so we may assume that $\underline{a}_2 = 3t_0 - u^2$ for some $t_0 \equiv 1 \pmod 4$ (if $u = 1$) or $t_0 \equiv 2 \pmod 4$ (if $u = 0$), and $0 < t_0 < 2^{a+2}$. Then the substitution $(x, y) \to (x - t_0, y - ux - 2^{a+2})$ returns the equation of the curve to Weierstrass short form, $y^2 = x^3 + a_4 x + a_6$, where we see that $a_4 = -3t_0^2 + 2^{a+2} u + 2^{a+2} B$, and $a_6 = -t_0^3 - t_0 a_4 + 2^{2a+2} + 2^{2a+4} C$. We therefore have 2 choices for u and 2^a choices for t depending on u. This determines $a_4 \pmod{2^{a+3}}$. Together with c_2, this determines $a_6 \pmod{2^{2a+5}}$. Thus for a fixed odd n and choice of c_2, we see that $\delta_2'(I_{n\geq 1}, c_2) = 2^{a+1} \cdot 1/2^{a+3} \cdot 1/2^{2a+5} = 1/2^{n+6}$.

Now suppose that $n = 2a$ is even. Then the sub-procedure finds a model for which

$$2^{-2a} P(2^a T) \equiv T^2 + T \text{ or } T^2 + T + 1 \pmod 2,$$

and satisfies $2^{-2a+2} R(2^{a-1} Y) \equiv Y^2 \pmod 2$. This is equivalent to the existence of integers A, B, and C, with B odd, so that $\underline{a}_3 = 2^{a+2} A$, $\underline{a}_4 = 2^{a+2} B$, and $\underline{a}_6 = 2^{2a+3} C$. If C is even, then we see that $R(T)$ factors over $\mathbb{Z}_2$, and the algorithm gives $c_2 := 4$. Otherwise, we have $c_2 := 2$. As before, we note that the substitutions $y \to y + 2x$ and $y \to y + 2^{a+2}$ do not alter any of the required conditions. Therefore, we may assume without loss of generality $\underline{a}_1 = 2u$ with $u \in \{0, 1\}$, and $A = 0$. Similarly, the substitution $x \to x + 2^{a+2}$ does not alter the conditions, so we may assume that $\underline{a}_2 = 3t_0 - u^2$ for some $t_0 \equiv 1, 2 \pmod 4$ (depending on u), and $0 < t_0 < 2^{a+2}$. After making the substitution $(x, y) \to (x - t_0, y - ux)$, we obtain the short Weierstrass model, $y^2 = x^3 + a_4 x + a_6$, where $a_4 = -3t_0^2 + 2^{a+2} B$, and $a_6 = -t_0^3 - t_0 a_4 + 2^{2a+3} C$. We therefore have 2 choices for u and 2^a choices for t_0 depending on u. This determines $a_4 \pmod{2^{a+3}}$. Together with the choice of c_2, this determines $a_6 \pmod{2^{2a+4}}$. Hence, for a fixed odd n and choice of c_2, we see that $\delta_2'(I_{n\geq 1}, c_2) = 2^{a+1} \cdot 1/2^{a+3} \cdot 1/2^{2a+4} = 1/2^{n+6}$.

A short calculation shows that the case $u = 0$ implies that (a_4, a_6) satisfies (i), where as $u = 1$ implies that (a_4, a_6) satisfies (ii). In summary, for each $n \geq 1$ we have $\delta_2'(I_{n\geq 1}, 2) = \delta_2'(I_{n\geq 1}, 4) = 1/2^{n+6}$.

Case 8. This case is for Kodaira type IV^*, where $c_2 \in \{1, 3\}$. The following two possibilities for this case are:

(i) We have $\alpha_4 \geq 3$ and $a_6 \equiv 4 \pmod{16}$.
(ii) We have $(a_4, a_6) \equiv (1, 6) \pmod 8$, and $k \geq 4$. Note that k is from model (2.15).

For (i), the algorithm implies that $c_2 := 1$ if $a_6 \equiv 20 \pmod{32}$, and $c_2 := 3$ when $a_6 \equiv 4 \pmod{32}$. Therefore, (i) contributes a proportion of $1/256$ for both $c_2 = 1$ and $c_2 = 3$. For condition (ii), the algorithm implies that $c_2 := 1$ if $k = 4$ (resp. $c_2 := 3$ if $k \geq 5$). In this situation, we have $t \equiv 3 \pmod 4$, and $v = 2$. The condition that $k \geq 4$ then implies that $(a_4, a_6) \equiv (1, 6)$ or $(9, 14) \pmod{16}$. Half of each set of possible pairs (a_4, a_6) will correspond to each possible c_2, thereby contributing another proportion of $1/256$, and so we obtain $\delta_2'(IV^*, 1) = \delta_2'(IV^*, 3) = 1/128$.

Case 9. This is for Kodaira type III^*, where $c_2 = 2$. The following two possibilities for this case are:

(i) We have $\alpha_4 = 3$ and $\alpha_6 \geq 4$.
(ii) We have $(a_4, a_6) \equiv (5, 6) \pmod 8$, and $k = 4$. Note that k is from model (2.15).

For condition (i), we have $(a_4, a_6) \equiv (8, 0) \pmod{16}$, and so the proportion of curves in this case is $1/256$. For condition (ii), we have $a_4 = -3t^2$, and $a_6 \equiv 16 - a_4 t - t^3 \pmod{32}$, which implies that $t \equiv 3 \pmod 4$. Therefore, by brute force we find that $(a_4, a_6) \pmod{32} \in \{(5, 6), (13, 30), (21, 22), (29, 14) \pmod{32}\}$, representing a proportion $1/256$, and so we obtain $\delta_2'(III^*, 2) = 1/256 + 1/256 = 1/128$.

Case 10. This case is Kodaira type II^*, where $c_2 = 1$. The following two possibilities for this case are:

(i) We have $\alpha_4 \geq 4$ and $a_6 \equiv 32, 48 \pmod{64}$.
(ii) We have $(a_4, a_6) \equiv (5, 6) \pmod 8$, and $k = 5$. Note that k is from model (2.15).

Clearly, the proportion of curves satisfying (i) is $1/512$. For (ii), we note that $v = 0$, and that t is uniquely determined by $a_4 \equiv 5 \pmod 8$, which determines $a_6 \equiv -a_4 t - t^3 + 32 \pmod{64}$. Therefore, the proportion of curves in this case is also $1/512$, and so $\delta_2'(II^*, 1) = 1/512 + 1/512 = 1/256$.

Case 11 ($\boldsymbol{\alpha_4 \geq 4}$ **and** $\boldsymbol{\alpha_6 \geq 6}$). As the model is not minimal, the algorithm replaces a_4 and a_6 with $a_4/16$ and $a_6/64$ respectively. One repeats these substitutions until one obtains a model which is one of the ten cases above.

3. Proofs

3.1. **Tamagawa Numbers and the proof of Theorems 1.1 and 1.3.** Using the results from the previous section, we now compute each $\delta_p(n)$, the proportion of curves $E(a_4, a_6)$ whose p-minimal models have $c_p = n$.

Lemma 3.1. *If p is prime and $n \geq 1$, then the following are true.*

(1) For $p = 2$, we have $\delta_2(1) = 241/396$, $\delta_2(2) = 7495/24552$, $\delta_2(3) = 1153/16368$, and $\delta_2(4) = 171/10912$. Moreover, if $n \geq 5$, then

$$\delta_2(n) = \frac{1}{2^{n+1} \cdot 1023}.$$

(2) For $p = 3$, we have $\delta_3(1) = 1924625/2125728$, $\delta_3(2) = 510641/6377184$, $\delta_3(3) = 7594/597861$, and $\delta_3(4) = 1193/652212$. Moreover, if $n \geq 5$, then

$$\delta_3(n) = \frac{1}{3^{n+1} \cdot 29524}.$$

(3) If $p \geq 5$ is prime, then we have

$$\delta_p(n) = \begin{cases} 1 - \dfrac{p(6p^7 + 9p^6 + 9p^5 + 7p^4 + 8p^3 + 7p^2 + 9p + 6)}{6(p+1)^2(p^8 + p^6 + p^4 + p^2 + 1)} & \text{if } n = 1, \\[2ex] \dfrac{p(2p^7 + 2p^6 + p^5 + p^4 + 2p^3 + p^2 + 2p + 2)}{2(p+1)^2(p^8 + p^6 + p^4 + p^2 + 1)} & \text{if } n = 2, \\[2ex] \dfrac{p^2(p^4 + 1)}{2(p+1)(p^8 + p^6 + p^4 + p^2 + 1)} & \text{if } n = 3, \\[2ex] \dfrac{p^3(3p^2 - 2p - 1)}{6(p+1)(p^8 + p^6 + p^4 + p^2 + 1)} & \text{if } n = 4, \\[2ex] \dfrac{p^{10} - 2p^9 + p^8}{2p^n(p^{10} - 1)} & \text{if } n \geq 5. \end{cases}$$

Proof. We first prove (3), the formulas for $\delta_p(n)$, where $p \geq 5$ is prime. In the previous section, we computed the numbers $\delta_p'(K, n)$, the proportion of p-minimal short Weierstrass models with Kodaira type K and Tamagawa number $c_p = n$. We determine $\delta_p(n)$ from the $\delta_p'(K, n)$ by keeping track of the distribution of all short Weierstrass models onto the p-minimal models as dictated by Tate's algorithm. Thanks to Lemma 2.1, we only need to consider the iterations of substitution in case eleven, which takes into account the divisibility of a_4 (resp. a_6) by powers of p^4 (resp. p^6). Moreover, $1/p^{10^n}$ represents the proportion of curves that pass through at least n additional iterations before satisfying one of the first ten cases. Therefore, we obtain the formula

$$\delta_p(n) := \sum_K \delta_p'(K,n) \cdot \left(1 + \frac{1}{p^{10}} + \frac{1}{p^{20}} + \dots\right) = \frac{p^{10}}{p^{10}-1} \sum_K \delta_p'(K,n). \tag{3.1}$$

The formulas are obtained using the entries in Table 5 in the Appendix. For example, if $n = 1$, then we have

$$\delta_p(1) = \frac{p^{10}}{p^{10}-1} \cdot \left(\frac{p-1}{p} + \frac{(p-1)^2}{p^3} + \frac{(p-1)}{p^3} + \frac{(p-1)}{2p^5} + \frac{(p^2-1)}{3p^7} + \frac{(p-1)}{2p^8} + \frac{(p-1)}{p^{10}}\right) + \frac{p^{10}}{p^{10}-1} \sum_{\substack{m=3 \\ m \text{ odd}}}^{\infty} \frac{(p-1)^2}{2p^{m+2}}$$

$$= 1 - \frac{p(6p^7 + 9p^6 + 9p^5 + 7p^4 + 8p^3 + 7p^2 + 9p + 6)}{6(p+1)^2(p^8 + p^6 + p^4 + p^2 + 1)}.$$

The infinite sum on $m \geq 3$ corresponds to the Kodaira types $I_{m\geq 3}$, where m is odd.

We now turn to the proof of (2). Curves satisfying condition (1) or (2) of Lemma 2.2 are not included in the proportions $\delta_p'(K,n)$. We replace the curves satisfying (2) with the models given in (2.10). Since t is a 3-adic unit, these curves cannot be transformed into short Weierstrass form without either introducing a denominator of 3 or increasing the discriminant. After a second pass through the algorithm, we find that these curves terminate in either case 1 or 2. We designated these situations in the previous subsection with an asterisk, and we denote the proportion of such curves satisfying (2) with minimal model with Kodaira type K and Tamagawa number $c_3 = n$ by $\widehat{\delta}_3(K,n)$. As above, we have that $1/3^{10}$ is the proportion of curves satisfying condition (1), and which then pass through the algorithm again. More generally, $1/3^{10^n}$ represents the proportion of curves that pass through at least n additional iterations before satisfying one of the first ten cases, or condition (2) of Lemma 2.2. We may therefore use a geometric series to count these curves. This leads to the formula

$$\delta_3(n) := \sum_K (\delta_3'(K,n) + \widehat{\delta}_3(K,n)) \cdot \left(1 + \frac{1}{3^{10}} + \frac{1}{3^{20}} + \dots\right) = \frac{3^{10}}{3^{10}-1} \cdot \sum_K (\delta_3'(K,n) + \widehat{\delta}_3(K,n)). \tag{3.2}$$

By brute force calculation using the entries listed in Tables 6 and 7 in the Appendix, we obtain (2).

Finally, we prove (1). Curves satisfying condition (1), (2), or (3) of Lemma 2.3 are not included in the proportions $\delta_p'(K,n)$. We replace the curves satisfying (2) with the models given in (2.13), and the curves satisfying (3) with the models given in (2.14). Since either the coefficient of y or of xy in the reduced model is odd, these curves cannot be transformed into short Weierstrass form without either introducing a denominator of 2 or increasing the discriminant. After a second pass through the algorithm, we find that these curves terminate in either case 1 or 2. We designated these situations in the previous subsection with an asterisk, and we denote the proportion of such curves satisfying (2) with minimal model with Kodaira type K and Tamagawa number $c_2 = n$ by $\widehat{\delta}_2(K,n)$. Moreover as above, $1/2^{10}$ is the proportion of curves that satisfy condition (1), and pass through the algorithm again. More generally, $1/2^{10^n}$ is the proportion of curves that pass through at least n additional iterations of the algorithm before satisfying one of the first ten cases, or satisfies (2) or (3) of Lemma 2.3. We may therefore use a geometric series to count these curves. Therefore, it follows that the analog of (3.2), again using Tables 6 and 7, is

$$\delta_2(n) := \frac{2^{10}}{2^{10}-1} \sum_{\text{Type } K} (\delta_2'(K,n) + \widehat{\delta}_2(K,n))$$

□

Proof of Theorem 1.1. By Lemma 3.1 and the Chinese Remainder Theorem, we have

$$P_{\text{Tam}}(1) := \lim_{X \to +\infty} \frac{\mathcal{N}_1(X)}{\mathcal{N}(X)} = \prod_{p \text{ prime}} \delta_p(1) = 0.5053\dots.$$

More generally, by multiplicativity, we formally find that

$$L_{\text{Tam}}(s) := \sum_{m=1}^{\infty} \frac{P_{\text{Tam}}(m)}{m^s} = \prod_{p\ prime} \left(\frac{\delta_p(1)}{1^s} + \frac{\delta_p(2)}{2^s} + \frac{\delta_p(3)}{3^s} + \dots\right).$$

To complete the proof it suffices to verify the convergence of the Dirichlet coefficients defined by this infinite product. To this end, we note that Lemma 3.1 (3) establishes, for primes $p \geq 5$, that $1 - \frac{1}{p^2} < \delta_p(1) < 1$. Therefore, convergence follows by comparison with $1/\zeta(2) = \prod_p \left(1 - \frac{1}{p^2}\right) = 6/\pi^2$. ☐

Proof of Theorem 1.3. Using Lemma 3.1, we find that the "average value" of $\mathrm{Tam}(E(a_4, a_6))$ is

$$L_{\mathrm{Tam}}(-1) = \sum_{m=1}^{\infty} P_{\mathrm{Tam}}(m)m = \prod_{p\ prime} (\delta_p(1) + 2\delta_p(2) + 3\delta_p(3) + \dots) \tag{3.3}$$

provided that this expression is convergent. To this end, we apply Lemma 3.1. For $p \geq 5$, Lemma 3.1 gives $\delta_p(1) = 1 - \frac{1}{p^2} + o(1/p^2)$, and for $n \geq 2$ gives $0 < \delta_p(n)n < \frac{n}{p^n}$. Therefore, since $\sum_{n=1}^{\infty} \frac{n}{p^n} = p/(p-1)^2$ we have

$$\sum_{n=1}^{\infty} \delta_p(n)n = 1 - \frac{1}{p^2} + O\left(\frac{1}{p^2}\right).$$

Similarly, we have convergence for $p \in \{2, 3\}$, and we have $\sum_{n\geq1} \delta_2(n)n = 1.4941\dots$ and $\sum_{n\geq1} \delta_3(n)n = 1.1109\dots$. The convergence of (3.3) follows by multiplicativity, and with a computer one finds $L_{\mathrm{Tam}}(-1) = 1.8193\dots$. ☐

3.2. **Proof of Lemma 1.4.** We expand on an example of Buhler, Gross and Zagier [5], which is based on a method of Tate (for example, see [12]). For $E = E(a_4, a_6)$, we define

$$F_E(x) := \frac{1}{2}\log|x|_\infty + \frac{1}{8}\sum_{n=0}^{\infty} \frac{\log|z_n|_\infty}{4^n}, \tag{3.4}$$

where $|\cdot|_\infty$ is the usual archimedean valuation of $\mathbb{R}$, and where for $n \geq 0$ we let

$$x_0 = x, \quad z_n = 1 - \frac{2a_4}{x_n^2} - \frac{8a_6}{x_n^3} + \frac{a_4^2}{x_n^4}, \quad \text{and} \quad x_{n+1} = \frac{x_n^4 - 2a_4x_n^2 - 8a_6x_n + a_4^2}{4(x_n^3 + a_4x_n + a_6)}. \tag{3.5}$$

For $P \in E(\mathbb{Q})$, if $x := x(P)$, then we have $x_n = x(2^nP)$. Moreover, as $n \to +\infty$, $F_E(x_n) - \log|x_n|_\infty/2$ converges.

For brevity, we consider the case where $E(\mathbb{R})$ has one component, as a similar argument applies to convenient curves with two real components. Let $\mathcal{O} \neq P \in E(\mathbb{Q})$ be a rational point. The canonical height $\widehat{h}(P)$ can be computed as a sum of local heights (for example, see Ch. VI of [14]) $\widehat{h}(P) = \sum_v \widehat{h}_v(P)$, where the sum is over all the places of $\mathbb{Q}$ (including ∞).

Since E is Tamagawa trivial, Theorem 5.2 b) of [12] with $N = 1$ and $n = 0$ shows that places of bad reduction make no contribution to this summation. Furthermore, thanks to the calculation in [5, p. 475], we have

$$\widehat{h}(P) = \widehat{h}_\infty(P) = \frac{1}{2}h_W(P) + F_E(x(P)) - \frac{1}{2}\max(0, \log|x(P)|_\infty). \tag{3.6}$$

Finally, the hypothesis that $a_4 \leq 0$ and $(\alpha, \infty) \subset \left\{x \in \mathbb{R} : 2a_4x^2 + 8a_6x - a_4^2 < 0\right\}$ implies that $x(P) \geq 1$ for all (if any) non-torsion points. Therefore, $\log|x(P)|_\infty \geq 0$ and $F_E(x(P)) - \frac{1}{2}\log|x(P)|_\infty \geq 0$. Hence, by (3.6), the first case follows immediately.

3.3. **Proof of Corollary 1.5.** Before we prove Corollary 1.5, we begin with an auxiliary lemma which establishes that the vast proportion of curves $E(a_4, a_6)$ with bounded height are already minimal models. Namely, we let

$$\mathcal{N}_{\min}(X) := \#\{E(a_4, a_6) \text{ a minimal model } : \ \mathrm{ht}(E(a_4, a_6)) \leq X\}. \tag{3.7}$$

Lemma 3.2. *As $X \to +\infty$, we have*

$$\rho := \lim_{X\to+\infty} \frac{\mathcal{N}_{\min}(X)}{\mathcal{N}(X)} = \frac{21342914775}{228811\pi^{10}} = 0.9960\dots.$$

Proof. For primes $p \geq 5$, Lemma 2.1, shows that the only short Weierstrass models which are not p-minimal have $v_p(a_4) \geq 4$ and $v_p(a_6) \geq 6$. Therefore, the multiplicative contribution to ρ for such primes is $1 - 1/p^{10}$. Similarly, for $p = 2$ (resp. $p = 3$), Lemma 2.3 (resp. Lemma 2.2) determines those short Weierstrass models which are 2-minimal (resp. 3-minimal). Using the tables in the Appendix, we find that the proportion of 2-minimal curves is $\sum_{\text{Type } K} \delta_2'(K, n) = 255/256 = 1 - 1/2^8$ (resp. 3-minimal curves is $\sum_K \delta_3'(K, n) = 19682/19683 = 1 - 1/3^9$). The formula for ρ follows by multiplicativity and the fact that $\zeta(10) = \prod_p \left(1 - \frac{1}{p^{10}}\right)^{-1} = \pi^{10}/93555$. ☐

Proof of Corollary 1.5. Thanks to Theorem 1.1, we find that

$$\lim_{X\to+\infty} \frac{\mathcal{N}_c(X)}{\mathcal{N}(X)} = \kappa \cdot P_{\mathrm{Tam}}(1) = \kappa \prod_{p\ prime} \delta_p(1). \tag{3.8}$$

where κ is the proportion of $E(a_4, a_6)$ that are minimal models that also satisfy one of the following two conditions.

(1) We have that $E(\mathbb{R})$ has one connected component, and

$$a_4 \leq 0 \quad \text{and} \quad (\alpha, \infty) \subset \left\{x \in \mathbb{R} : 2a_4x^2 + 8a_6x - a_4^2 < 0\right\},$$

where α is the real root of $x^3 + a_4x + a_6$.

(2) We have that $E(\mathbb{R})$ has two connected components, and

$$a_4 \leq 0 \quad \text{and} \quad (\gamma, \beta) \cup (\alpha, \infty) \subset \left\{x \in \mathbb{R} : 2a_4x^2 + 8a_6x - a_4^2 < 0\right\},$$

where $\gamma < \beta < \alpha$ are the real roots of $x^3 + a_4x + a_6$.

Therefore, we have that $\kappa := \rho \cdot (\kappa_1 + \kappa_2)$, where ρ is given in Lemma 3.2, and κ_1 (resp. κ_2) denotes the proportion of $E = E(a_4, a_6)$ with $\mathrm{ht}(E) \leq X$ that satisfy condition (1) (resp. (2)).

It is convenient to first reformulate these two cases in terms of models over $\mathbb{R}$ given by a single parameter T. To this end, we make use of the change of variable

$$(x, y) \to (\sqrt{|a_4|}x, |a_4|^{3/4}y). \tag{3.9}$$

By letting $T := a_6/|a_4|^{3/2}$, we then obtain

$$y^2 = x^3 - x + T.$$

If we set $F(x,T) = x^3 - x + T$ and $G(x,T) = -2x^2 + 8Tx - 1$, then both (1) and (2) are reformulated as

$$G(x,T) < 0 \ \text{ for all } \ x \in \mathbb{R} \ \text{ such that } \ F(x,T) > 0. \tag{3.10}$$

The convenient curves with $a_4 = 0$ have density 0 as $X \to +\infty$. Therefore, it suffices to consider (3.10).

It is straightforward to determine when (3.10) holds using the discriminant of $F(x,T)$. Indeed, the discriminant is positive (resp. negative) when the curve has 2 real components (resp. 1 real component). Hence, the two cases are determined by the location of T in $\mathbb{R}$, with respect to the points satisfying one of the following possibilities:

- the discriminant of $F(x,T)$ (with respect to x) is 0
- the discriminant of $G(x,T)$ (with respect to x) is 0
- points T where $F(x,T)$ and $G(x,T)$ share a root.

These conditions are dictated by the common zeros of $F(x,T)$ and $G(x,T)$.

The discriminant of $F(x,T)$ with respect to x is $4 - 27T^2$, which is zero when $T = \pm\frac{2}{\sqrt{27}}$. The discriminant of $G(x,T)$ with respect to x is $64T^2 - 8$, which is zero when $T = \pm\frac{1}{\sqrt{8}}$. To determine when F and G share a root, set

$$r_{\pm}(T) = \frac{-8T \pm \sqrt{64T^2 - 8}}{-4} = 2T \mp \sqrt{4T^2 - \tfrac{1}{2}},$$

which are the two roots in x of $G(x,T)$. Then a straightforward calculation reveals that $F(x,T)$ and $G(x,T)$ share a root in x if and only if T is a root of the polynomial

$$F(r_+(T)) \cdot F(r_-(T)) \;=\; 64T^4 - 17T^2 + \tfrac{9}{8} \;=\; \tfrac{1}{8}(8T^2 - 1)(8T + 3)(8T - 3).$$

Hence, we have the two additional critical values $T = \pm\frac{3}{8}$. By calculating the functions $F(x,T)$ and $G(x,T)$ for T in the various intervals between these critical values, we see that (3.10) is satisfied only when $T \in \left(-\infty, -2/\sqrt{27}\right)$ or $T \in \left(-1/\sqrt{8}, 1/\sqrt{8}\right)$. The first interval corresponds to case (1), while the second is case (2).

We now analyze these cases separately taking into account (3.9). In the first case, $T < -\frac{2}{\sqrt{27}}$ implies that $a_6 < 0$ and $-3\,(a_6/2)^{2/3} < a_4 < 0$. If $\mathrm{ht}(E(a_4, a_6)) \leq X$, then we have that $|a_4| \leq \sqrt[3]{X/4}$, and $|a_6| \leq \sqrt{X/27}$. As X approaches infinity, the proportion of such curves satisfying $-3\,(a_6/2)^{2/3} < a_4 < 0$, with $a_6 < 0$, satisfies

$$\kappa_1 \;:=\; \lim_{X\to+\infty} \frac{3\displaystyle\int_{-\sqrt{X/27}}^{0} (s/2)^{2/3}\, ds}{4 \cdot \sqrt[3]{X/4} \cdot \sqrt{X/27}} \;=\; \frac{3}{20}.$$

For (2), we note that $|T| < \frac{1}{\sqrt{8}}$ implies $|a_6| < \frac{1}{\sqrt{8}}|a_4|^{3/2}$, and so

$$\kappa_2 \;:=\; \lim_{X\to+\infty} \frac{2\displaystyle\int_0^{\sqrt[3]{X/4}} \frac{1}{\sqrt{8}} s^{3/2} ds}{4 \cdot \sqrt[3]{X/4} \cdot \sqrt{X/27}} \;=\; \frac{3\sqrt{6}}{40}.$$

Therefore, Lemma 3.7 shows that (3.8) is

$$\kappa \cdot P_{\mathrm{Tam}}(1) = \rho \cdot (\kappa_1 + \kappa_2) \cdot P_{\mathrm{Tam}}(1) = \frac{21342914775}{228811\pi^{10}} \cdot \left(\frac{3}{20} + \frac{3\sqrt{6}}{40}\right) \cdot P_{\mathrm{Tam}}(1) = 0.1679\ldots.$$

□

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4. Appendix

Type	c_p	$\delta'_p(K,n)$	Type	c_p	$\delta'_p(K,n)$	Type	c_p	$\delta'_p(K,n)$
I_0	1	$\frac{p-1}{p}$	I_0^*	1	$\frac{1}{3}\frac{(p^2-1)}{p^7}$	III	2	$\frac{(p-1)}{p^4}$
I_1	1	$\frac{(p-1)^2}{p^3}$	I_0^*	2	$\frac{1}{2}\frac{(p-1)}{p^6}$	III^*	2	$\frac{(p-1)}{p^9}$
I_2	2	$\frac{(p-1)^2}{p^4}$	I_0^*	4	$\frac{1}{6}\frac{(p-1)(p-2)}{p^7}$	IV	1	$\frac{1}{2}\frac{(p-1)}{p^5}$
$I_{n\geq 3}$	$\varepsilon(n)$	$\frac{1}{2}\frac{(p-1)^2}{p^{n+2}}$	$I_{n\geq 1}^*$	2	$\frac{1}{2}\frac{(p-1)^2}{p^{7+n}}$	IV	3	$\frac{1}{2}\frac{(p-1)}{p^5}$
$I_{n\geq 3}$	n	$\frac{1}{2}\frac{(p-1)^2}{p^{n+2}}$	$I_{n\geq 1}^*$	4	$\frac{1}{2}\frac{(p-1)^2}{p^{7+n}}$	IV^*	1	$\frac{1}{2}\frac{(p-1)}{p^8}$
II	1	$\frac{(p-1)}{p^3}$	II^*	1	$\frac{(p-1)}{p^{10}}$	IV^*	3	$\frac{1}{2}\frac{(p-1)}{p^8}$

Table 5. The $\delta'_p(K,n)$ for $p \geq 5$ (Note. $\varepsilon(n) := ((-1)^n + 3)/2$.)

Type	c_2	$\widehat{\delta}_2(K,n)$	Type	c_2	$\widehat{\delta}_2(K,n)$
I_0	1	$\frac{1}{512}$	$I_{n\geq 3,\text{odd}}$	1	$\frac{1}{2^{n+11}}$
I_1	1	$\frac{1}{2^{11}}$	$I_{n\geq 3,\text{even}}$	2	$\frac{1}{2^{n+11}}$
I_2	2	$\frac{1}{2^{12}}$	$I_{n\geq 3}$	n	$\frac{1}{2^{n+11}}$

Type	c_3	$\widehat{\delta}_3(K,n)$	Type	c_3	$\widehat{\delta}_3(K,n)$
I_0	1	$\frac{4}{3^{11}}$	$I_{n\geq 3,\text{odd}}$	1	$\frac{2}{3^{n+11}}$
I_1	1	$\frac{4}{3^{12}}$	$I_{n\geq 3,\text{even}}$	2	$\frac{2}{3^{n+11}}$
I_2	2	$\frac{4}{3^{13}}$	$I_{n\geq 3}$	n	$\frac{2}{3^{n+11}}$

TABLE 6. The $\widehat{\delta}_2(K,n)$ and $\widehat{\delta}_3(K,n)$

Type	c_2	$\delta_2'(K,n)$	Type	c_2	$\delta_2'(K,n)$
I_0	1	0	I_0^*	1	$\frac{1}{32}$
I_1	1	0	I_0^*	2	$\frac{1}{32}$
I_2	2	0	I_0^*	4	0
$I_{n\geq 3}$	$\varepsilon(n)$	0	$I_{n\geq 1}^*$	2	$\frac{1}{2^{n+6}}$
$I_{n\geq 3}$	n	0	$I_{n\geq 1}^*$	4	$\frac{1}{2^{n+6}}$
II	1	$\frac{1}{2}$	II^*	1	$\frac{1}{256}$
III	2	$\frac{1}{4}$	III^*	2	$\frac{1}{128}$
IV	1	$\frac{1}{16}$	IV^*	1	$\frac{1}{128}$
IV	3	$\frac{1}{16}$	IV^*	3	$\frac{1}{128}$

Type	c_3	$\delta_3'(K,n)$	Type	c_3	$\delta_3'(K,n)$
I_0	1	$\frac{2}{3}$	I_0^*	1	$\frac{8}{3^7}$
I_1	1	0	I_0^*	2	$\frac{1}{3^5}$
I_2	2	0	I_0^*	4	$\frac{1}{3^7}$
$I_{n\geq 3}$	$\varepsilon(n)$	0	$I_{n\geq 1}^*$	2	$\frac{2}{3^{n+6}}$
$I_{n\geq 3}$	n	0	$I_{n\geq 1}^*$	4	$\frac{2}{3^{n+6}}$
II	1	$\frac{2}{9}$	II^*	1	$\frac{2}{3^9}$
III	2	$\frac{2}{27}$	III^*	2	$\frac{10}{3^9}$
IV	1	$\frac{1}{81}$	IV^*	1	$\frac{7}{3^9}$
IV	3	$\frac{1}{81}$	IV^*	3	$\frac{7}{3^9}$

TABLE 7. The $\delta_2'(K,n)$ and $\delta_3'(K,n)$

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