# Remembering Basil Gordon, 1931–2012

*Krishnaswami Alladi, Coordinating Editor*

Basil Gordon, who made major research contributions to number theory, combinatorics, and algebra and who enhanced our understanding of Ramanujan's remarkable identities, passed away on January 12, 2012, at the age of eighty. Gordon, who was a professor of mathematics at UCLA, was an outstanding teacher at all levels. His legacy will continue, owing to the impact of his fundamental work and to the many students he groomed, as well as several other mathematicians he influenced. In this article, five noted mathematicians describe various contributions of Gordon and include personal reflections as well. Krishnaswami Alladi discusses Gordon's research on partitions and extensions of Ramanujan's identities. George Andrews describes Gordon's work on plane partitions. Ken Ono's article is on Gordon's work on modular forms, whereas Robert Guralnick talks about Gordon's contributions to algebra. Finally, Bruce Rothschild recalls how he and Basil Gordon ran the *Journal of Combinatorial Theory, Ser. A*, as managing editors.

Keith Kendig, a Ph.D. student of Professor Gordon in the 1960s, conducted a detailed interview of Gordon in 2011 dealing with his life and mathematical career. The text of this interview with pictures of Gordon from his childhood will appear in *Fascinating Mathematical People*, to be published by Princeton University Press in 2014. Some pictures of Gordon, courtesy of Keith Kendig, are included here.

*Krishnaswami Alladi is professor of mathematics at the University of Florida, Gainesville. His email address is* alladik@ufl.edu.

DOI: http://dx.doi.org/10.1090/noti1017

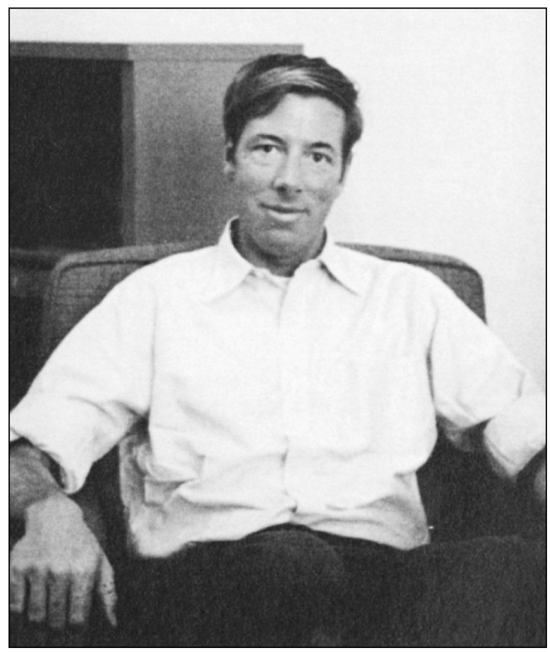

Photo courtesy of Keith Kendig.

**Basil Gordon as a young faculty member at UCLA. Photo taken by Paul Halmos.**

## Krishnaswami Alladi

### The Great Guru

Professor Basil Gordon was a towering figure in combinatorics and number theory. He made fundamental contributions to several areas, such as the theory of partitions, modular forms, mock theta functions, and coding theory. He was one of the very few who was at home with both combinatorial and modular form techniques. He was one of the

Photo courtesy of Krishnaswami Alladi.

**The five mathematicians associated with the Capparelli Conjecture, its resolution, and generalizations all got together in Gainesville, Florida, in 2004. Clockwise from bottom right: Basil Gordon, Jim Lepowsky, Stefano Capparelli, George Andrews, and Krishnaswami Alladi.**

leaders in the world of Ramanujan's mathematics. As managing editors for over two decades, he and his UCLA colleague Bruce Rothschild developed the *Journal of Combinatorial Theory-A* into a premier journal. A few years ago, the JCT-A came up with a special issue in honor of Gordon and Rothschild on their retirement from that editorial board. George Andrews, Ken Ono, Richard McIntosh, and I wrote a paper [3] about Gordon's work for that volume. The current article will be very different. I will give samples of some of Gordon's most appealing theorems and provide some personal reflections.

One of the finest examples of Gordon's fundamental research is his generalization of the Rogers-Ramanujan identities to odd moduli [11]. In analytic form, the Rogers-Ramanujan (R-R) identities provide product representations to two $q$-hypergeometric series (see [7]). The combinatorial interpretation of these identities is:

**Theorem R-R.** *For $i = 1, 2$, the number of partitions of an integer n into parts that differ by at least 2, with least part $\geq i$, equals the number of partitions of n into parts $\equiv \pm i (mod\, 5)$.*

In the 1960s Gordon [11] obtained the following beautiful generalization of the Rogers-Ramanujan partition theorem to all odd moduli $\geq 5$:

**Theorem 1.** *For any pair of integers i and k satisfying $1 \leq i \leq k$ and $k \geq 2$, the number of partitions of an integer n of the form $b_1 + b_2 + \cdots + b_\nu$, where $b_j - b_{j+k-1} \geq 2$, and with at most $i - 1$ ones, is equal to the number of partitions of n into parts $\not\equiv 0, \pm i \ (mod\, 2k+1)$.*

In a partition we always write the parts $b_i$ in descending order.

This result opened up a new direction of research (see [7]) on R-R type identities, namely identities which connect partitions with parts satisfying difference conditions to partitions with parts satisfying congruence conditions. George Andrews has been the leader in this field, and he was inspired by Gordon's generalization of the R-R identities.

One reason that the classical Rogers-Ramanujan identities are so important is because the ratio of the series in the two identities admits a continued fraction expansion, and this continued fraction has a lovely product representation:

$$R(q) = 1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{\cdots}}} = \prod_{m=1}^{\infty} \frac{(1-q^{5m-2})(1-q^{5m-3})}{(1-q^{5m-1})(1-q^{5m-4})}. \tag{1}$$

In view of the product, the continued fraction plays an important role in the theory of modular forms in relation to the congruence subgroup $\Gamma_0(5)$ of the modular group.

Another gem that Gordon found was an analogue of (1) to the modulus 8. More precisely, by working with two $q$-hypergeometric identities that were analogous to the Rogers-Ramanujan identities but for the modulus 8 instead of the modulus 5 and by considering their ratio, Gordon [12] showed that

$$G(q) = 1 + q + \cfrac{q^2}{1 + q^3 + \cfrac{q^4}{1 + q^5 + \cfrac{q^6}{\cdots}}} = \prod_{m=1}^{\infty} \frac{(1-q^{8m-3})(1-q^{8m-5})}{(1-q^{8m-1})(1-q^{8m-7})}. \tag{2}$$

Like $R(q)$, the fraction $G(q)$ also has an important role in the theory of modular forms but to the congruence subgroup $\Gamma_0(8)$. This continued fraction identity was independently discovered by H. Göllnitz [10], and so this is now called the Göllnitz-Gordon continued fraction. The partition theorem that underlies the continued fraction in (2) is:

**Theorem 2** (Göllnitz-Gordon)**.** *For $i = 1, 2$, the number of partitions of an integer n into parts that differ by at least 2, with least part $\geq 2i - 1$, and with no consecutive even numbers as parts, equals the number of partitions of n into parts $\equiv 4$ or $\pm (2i-1) (mod\, 8)$.*

Gordon told me that he was led to this fraction and Theorem 2 by his *meta theorem*: "What works for 5 works also for 8."

I always called Gordon "the great guru". His knowledge of mathematics was vast and deep and he shared his ideas generously. His contributions can not only be seen from his seminal papers but also in the work of his students whose careers he molded. During the Rademacher Centenary Conference at Penn State University in 1992 there were four of Gordon's Ph.D. students—Doug Bowman, Ken Ono, Richard McIntosh, and Sinai Robins—each presenting significant work on very different topics in number theory. That showed the breadth of Gordon's expertise.

Besides his Ph.D. students, there were several others like me who were influenced by his mathematical ideas and philosophy. In particular, it was due to his guidance that I was able to make the transition in the early nineties from classical analytic number theory to the theory of partitions and $q$-hypergeometric series, an area in which I continue to work today.

In December 1987 the Ramanujan Centennial was being celebrated, and several conferences were being conducted in India. I was asked to organize one such conference in Madras, and I invited Gordon to give a plenary talk. At that time I was working on classical analytic number theory, but I was charmed by the lectures of Gordon, Andrews, and others on partitions and $q$-hypergeometric series which I heard during the centennial. But I was in awe of the tantalizing $q$-hypergeometric identities and transformations being presented and a bit scared to enter this domain.

In 1989 I received a message from Gordon saying that he had a fully paid sabbatical and that he would like to spend a good part of it at the University of Florida. This was like a gift from heaven for me, because I realized that I could benefit from his visit by getting introduced into the world of partitions and $q$-hypergeometric series. And that is exactly what happened, and for this I am most grateful.

During that visit Gordon and I investigated a general continued fraction of Ramanujan, and through its study we obtained several results related to a number of classical identities in the theory of partitions and $q$-series from a unified perspective. This was my first paper [4] on partitions and $q$-series, and it appeared in the JCT-A. It was also during this visit that we started considering a very general approach to the celebrated 1926 partition theorem of Schur, which is:

**Theorem S.** *The number of partitions A(n) of an integer n into parts* $\equiv \pm 1 (mod\, 6)$ *is equal to the number of partitions B(n) of n into distinct parts* $\equiv \pm 1\ (mod\, 3)$*, and this is equal to the number of partitions C(n) of n into parts that differ by at least* 3 *but without consecutive multiples of* 3 *as parts.*

The condition "no consecutive multiples of 3 as parts" in Theorem S is analogous to the condition "no consecutive even numbers as parts" in Theorem 2. The equality $A(n) = C(n)$ in Schur's theorem can be considered as the next level result beyond the Rogers-Ramanujan partition theorem, because the gap 2 in Theorem R-R is replaced by 3 in Theorem S, and the modulus 5 in Theorem R-R is replaced by 6 in Theorem S. But Gordon told me that it is the equality $B(n) = C(n)$ that is more fundamental and capable of a significant generalization. He thus initiated me into his philosophy of "the method of weighted words," which is described in [3]. Guided by this philosophy of his, we found a generalization of Theorem S that involved words formed by colored integers satisfying certain gap conditions, and we were able to encapsulate it in the form of an elegant analytic *key identity* in two free parameters, $a, b$ (see [5]).

Gordon's visit to the University of Florida in 1989 started our substantial collaboration. I visited him in Los Angeles over the next few years and worked on extensions of the method of weighted words to the deep partition theorem of Göllnitz [10]. We found a remarkable "key identity" in three free parameters, $a, b, c$, which contained our two-parameter identity for Schur's theorem as a special case. But we could not prove this three-parameter identity. In 1990, when George Andrews visited the University of Florida, I showed him the three-parameter identity Gordon and I had found. During that stay Andrews proved our identity, and that resulted in our triple joint paper [1]. Andrews told me that, in some sense as a pure partition result, Schur's theorem was more fundamental than the Rogers-Ramanujan partition theorem. Gordon and I worked out several ramifications of Schur's theorem [6] using the method of weighted words, and this confirmed Andrews's view of its importance.

I have already mentioned the Rademacher Centenary Conference of 1992 at Penn State and Gordon's presence at the conference with his students. I now state an interesting development that took place during that conference.

At the start of the Rademacher conference, Jim Lepowsky gave a talk on the connections between Lie algebras and partitions and stated a partition theorem as a conjecture made by his student Capparelli from a study of vertex operators in Lie algebras. Andrews went into hiding for the remainder of the conference to work on this conjecture, emerging from his hideout just to attend the talks. On the last day of the conference, Andrews outlined a generating function proof of Capparelli's conjecture and published it in the *Proceedings of the Rademacher Centenary Conference* [8]. Basil Gordon, who heard

Lepowsky's lecture, had noticed right away that the Capparelli conjecture could be generalized in the framework of the method of weighted words. He informed me about this a few weeks later as I was about to visit Penn State for my sabbatical in 1992–93 to work with Andrews. Gordon and I found a key identity for a generalized Capparelli theorem and provided a combinatorial bijective proof as well. This resulted in another triple paper [2], which we submitted to the *Journal of Algebra*, since Capparelli's work appeared there [9]. Neither Capparelli nor I were at the Rademacher Centenary Conference, but in fall 2004, Gordon, Andrews, Lepowsky, Capparelli, and I were all at a conference at the University of Florida. At that Florida conference, Gordon gave a beautiful lecture entitled "The return of the mock theta functions", in which he described among other things the work with his former student Richard McIntosh [14] on some new mock theta functions of order 8. Ramanujan, as is well known, had communicated his discovery of the mock theta functions in his last letter to Hardy in January 1920 just weeks before he died and in that letter gave examples of mock theta functions of orders 3, 5, and 7. Gordon and McIntosh investigated mock theta functions and their asymptotics in great detail. Many of their important results can be found in the survey paper [15]. There is also a good description of the Gordon-McIntosh work in [3], and all of this relates to the classical theory of mock theta functions. In the last few years dramatic advances have been made in a modern approach to mock theta functions which connects them to harmonic Maass forms, but I will not discuss that here.

Professor Gordon was very supportive of my effort to launch the *Ramanujan Journal* devoted to all areas of mathematics influenced by Ramanujan, and served on the editorial board since its inception in 1997. He contributed a fine paper to the very first issue, and I will describe this briefly.

The celebrated Euler's Pentagonal Numbers theorem has the combinatorial interpretation that if the set of partitions on an integer $n$ into distinct parts is split into two subsets based on the parity of the number of parts, then the two subsets are equal in size except at the pentagonal numbers, in which case the difference is 1 between the sizes of the two subsets. Thus $Q(n)$, the number of partitions of $n$ into distinct parts, is odd precisely at the pentagonal numbers. In 1995 I proved that

$$(3) \qquad Q(n) = \sum_{k \geq 0} p_3(n;k) 2^k,$$

where $p_3(n;k)$ is the number of partitions $b_1 + b_2 + \cdots + b_\nu$ of $n$ into parts that differ by at least 3 and have precisely k gaps $b_i - b_{i+1}$ which are $> 3$, with $b_{\nu+1} = -1$. I told Gordon that this identity seems to indicate that, for each integer $k$ ,

$$(4) \qquad Q(n) \equiv 0 (mod\, 2^k) \quad \text{almost always},$$

and one could prove (4) for small $k$ using (3). But I did not know how to establish (4) for all $k \geq 1$. Gordon told me that such a congruence result is best approached through the theory of modular forms. Indeed he and Ken Ono established this conjecture using modular forms and contributed an important paper to the first issue of *The Ramanujan Journal* [13].

I had the pleasure of hosting Gordon at the University of Florida as well as in India. Whenever I visited UCLA to work with him, he would have me stay at his home or at least have me spend a substantial part of my stay working with him at his stately home in Santa Monica. In our ancient Hindu culture, we have the practice of *gurukula*. That is, the student lives with the guru and observes the guru in close quarters as he is practicing his art. By being so close to the guru, the student learns the nuances of whatever art form is being taught. My stay in Gordon's home was in some sense like a gurukula. I am sure his Ph.D. students must have had similar gurukula experiences.

Owing to his visits to Florida and to India, Gordon became close to my family. More than once when my parents passed through Los Angeles, he hosted them magnificently, and we were all touched by his gracious hospitality.

Gordon had great knowledge of Western classical music and knew much about art and world history. During his visits to India, when we took him around for sightseeing, he knew more about the history than most of us and would educate us by making comparisons with similar things in Europe. In 2004, before my visit to Italy with my wife and daughters, he instructed me that in Florence, in addition to the well-known sights such as Michelangelo's *David*, we should see the four statues at the Capelle Medicee sculpted by Michelangelo representing Dawn, Dusk, Day, and Night. Gordon said that, when G. N. Watson spoke about the mock theta functions of Ramanujan, he compared the grandeur of Ramanujan's identities to the beauty of these statues, and thus our visit to Florence would be incomplete if we did not see them. I am glad we followed the instructions of the guru!

## References


[1] K. Alladi, G. E. Andrews, and B. Gordon, Generalizations and refinements of a partition theorem of Göllnitz, *J. Reine Angew. Math.* **460** (1995), 165–188.

[2] ———, Refinements and generalizations of Capparelli's conjecture on partitions, *J. Algebra* **174** (1995), 636–658.

[3] K. Alladi, G. E. Andrews, K. Ono, and R. McIntosh, On the work of Basil Gordon, *J. Comb. Th. Ser. A* **113** (2006), 21–38.
[4] K. Alladi and B. Gordon, Partition identities and a continued fraction of Ramanujan, *J. Comb. Th. Ser. A* **63** (1993), 275–300.
[5] ———, Generalizations of Schur's partition theorem, *Manus. Math.* **79** (1993), 113–126.
[6] ———, Schur's partition theorem, companions, refinements and generalizations, *Trans. Amer. Math. Soc.* **347** (1995), 1591–1608.
[7] G. E. Andrews, The theory of partitions, in *Encyclopedia of Math.*, vol. 2, Addison-Wesley, Reading, MA, 1976.
[8] ———, Schur's theorem, Capparelli's conjecture, and the $q$-trinomial coefficients, Contemp. Math., vol. 166, 1994, Amer. Math. Soc., Providence, RI, pp. 141–154.
[9] S. Capparelli, On some representations of twisted affine Lie algebras and combinatorial identities, *J. Algebra* **154** (1993), 335–355.
[10] H. Göllnitz, Partitionen mit Differenzenbedingungen, *J. Reine Angew. Math.* **225** (1967), 154–190.
[11] B. Gordon, A combinatorial generalization of the Rogers-Ramanujan identities, *Amer. J. Math.* **83** (1961), 393–399.
[12] ———, Some continued fractions of the Rogers-Ramanujan type, *Duke Math. J.* **32** (1965), 741–748.
[13] B. Gordon and K. Ono, Divisibility of certain partition functions by powers of primes, *Ramanujan J.* **1** (1997), 25–34.
[14] B. Gordon and R. J. McIntosh, Some eighth order mock theta functions, *J. London Math. Soc. (2)* **62** (2000), 321–335.
[15] ———, A survey of the classical mock theta functions, in *Partitions, q-Series and Modular Forms* (K. Alladi and F. Garvan, eds.), Developments in Math., vol. 23, Springer, New York, 2010, pp. 95–144.

## *George E. Andrews*

Basil Gordon was a reclusive, brilliant mathematician who proved some wonderful theorems on partitions which greatly inspired many, including me.

In the 1960s he published a number of innovative papers on the theory of partitions. Krishna Alladi has devoted much of his article to some of Basil's most prescient and spectacular achievements, including the generalization of the Rogers-Ramanujan identities, the Göllnitz-Gordon identities, and the method of weighted words. Gordon's later work on mock theta functions is one of the topics in Ken Ono's contribution.

Basil was a delightful and kind man but not a great correspondent. He directly answered very few of my letters to him. At first I feared that I might have offended him in some way. However, I was assured by others that his treatment of me was not unique. The late Marco Schützenberger told me that he had offered Basil a visiting position at Paris VII and received no response.

*George E. Andrews is Evan Pugh Professor of Mathematics at The Pennsylvania State University. His email address is* andrews@math.psu.edu.

However, lack of letters was made up for in personal interactions. In the late 1960s I attended an AMS meeting at Cornell University for the sole purpose of meeting Gordon and asking for clarification of some of his papers. He was open, gracious, and immensely helpful.

My most extended contact with Basil came in 1987 in India during the Ramanujan Centenary. We had several long car rides together connecting to the several conferences. During one such venture Basil entertained me and the other passengers with one of his hobbies. The object was to take a word or phrase, use its letters as an anagram, and produce a new word or phrase directly related to the original. The only one I remember was "Mosquitoes" morphing into "O, Moses quit!" This was long before computers dominated such games.

Basil had a wonderful sense of humor. Small absurdities delighted him. I recall one example (at my expense). This is from a rare letter to me, dated October 21, 1981:

> The following filler appeared in the *L.A. Times* of Oct. 18.
>
> **"Ramanujan, Mathematics Giant, Created Formulas**
> State College, Pa. (AP) Srinivasa Ramanujan who died about 60 years ago, when he was 32, is considered to be one of the giants of 20th century mathematics, said George E. Andrews of Penn State University. Ramanujan, a poor Indian, created his own math formulas."

As to Basil's mathematics, I shall restrict my comments to his work on plane partitions, much of it done with his students (Lorne Houten and others). The majority of this work is primarily contained in a sequence of papers titled "Notes on plane partitions" [8], [9], [10], [5], [6], [11]. There are four other papers [3], [4], [7], [12]. Of these, [4] is an early but interesting contribution on two-rowed partitions. The proof of the Bender-Knuth conjecture is given in [7].

The story of plane partitions dates back to P. A. MacMahon [13, p. 673]. There we find his first inkling that there might be appealing generating functions for plane partitions. More than twenty years later, MacMahon [14] effectively proved that

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} = 1 + q + 3q^2 + 6q^3 + \cdots,$$

where the coefficient of $q^n$ is the number of plane partitions of $n$. For example, the six plane partitions of 3 are

$$3 \qquad 2\;\;1 \qquad \begin{matrix}2\\1\end{matrix} \qquad 1\;\;1\;\;1 \qquad \begin{matrix}1&1\\1&\end{matrix} \qquad \begin{matrix}1\\1\\1\end{matrix}$$

Although T. W. Chaundy wrote some papers on plane partitions in the 1930s, it fell to Basil (jointly with his student Lorne Houten) to develop serious methods that really opened up the subject. The first hint that something new was afoot came from their proof that

$$\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{[\frac{n+1}{2}]}} = 1+q+2q^2+4q^3+\cdots$$

is the generating function for plane partitions with strictly decreasing parts along rows. Thus the four plane partitions of 3 subject to this description are

$$3 \qquad 2\;\;1 \qquad \begin{matrix}2\\1\end{matrix} \qquad \begin{matrix}1\\1\\1\end{matrix}$$

These papers illustrate the tremendous insights that Gordon developed in advancing from [4] to [7]. Indeed, it should be emphasized that there is much food for thought and many questions still to be answered arising from these papers. For example, in "Notes on plane partitions, IV" Gordon shows that $C_k(q)$, the generating function for $k$-rowed plane partitions with strictly decreasing parts along columns, can be evaluated in terms of certain classical infinite products and the false theta series

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}.$$

Although Gordon promises [5, p. 98] that a more thorough investigation "will be undertaken elsewhere," unfortunately neither he nor anyone else has followed up on this unique appearance of false theta series in the world of plane partitions.

This wonderful, brilliant man illuminated many aspects of the theory of partitions. It was a joy to collaborate with him on two papers [1], [2] (also joint with Alladi). He gave the subject I love many new ideas and path-breaking insights. I and many others owe him a great debt.

### References


[1] K. Alladi, G. E. Andrews, and B. Gordon, Generalizations and refinements of a partition theorem of Göllnitz, *J. Reine Angew. Math.* **460** (1995), 165–188.

[2] ______, Refinements and generalizations of Capparelli's conjecture on partitions, *J. Algebra* **174** (1995), 636–658.

[3] M. S. Cheema and B. Gordon, Some remarks on two- and three-line partitions, *Duke Math. J.* **31** (1964), 267–273.

[4] B. Gordon, Two new representations of the partition function, *Proc. Amer. Math. Soc.* **13** (1962), 869–873.

[5] ______, Multirowed partitions with strict decrease along columns (Notes on plane partitions, IV), Symposia Amer. Math. Soc., vol. 19, Amer. Math. Soc., Providence, RI, 1971, pp. 91–100.

[6] ______, Notes on plane partitions V, *J. Combin. Theory* **B-11** (1971), 157–168.

[7] ______, A proof of the Bender-Knuth conjecture, *Pacific J. Math.* **108** (1983), 999–1013.

[8] B. Gordon and L. Houten, Notes on plane partitions, I, *J. Combin. Theory* **4** (1968), 72–89.

[9] ______, Notes on plane partitions, II, *J. Combin. Theory* **4** (1968), 118–120.

[10] ______, Notes on plane partitions, III, *Duke Math. J.* **36** (1969), 801–824.

[11] B. Gordon, Notes on plane partitions, VI, *Discrete Math.* **26** (1979), 41–45.

[12] B. Gordon and L. Houten, Multi-rowed partitions with totally distinct parts, *J. Number Theory* **12** (1980), 439–444.

[13] P. A. MacMahon, Memoir on the theory of the partition of numbers—Part I, *Phil. Trans.* **187** (1897), 619–673. (Reprinted in *Percy Alexander MacMahon, Collected Papers*, MIT Press, Cambridge, 1978.)

[14] ______, Memoir on the theory of the partition of numbers—Part VI, *Phil. Trans.* **211** (1912), 345–373. (Reprinted in *Percy Alexander MacMahon, Collected Papers*, MIT Press, Cambridge, 1978.)


# *Robert Guralnick*

### Basil Gordon's Work in Algebra

Basil Gordon was a major figure in combinatorial number theory and especially in the theory of partition identities. This work is discussed by Alladi, Andrews, and Ono. I will focus on his work in algebra and group theory.

Gordon grew up in Baltimore and was a student at Johns Hopkins, spending a year as an undergraduate in Germany, where he studied with Ernst Witt and Emil Artin. He was advised to go to Caltech and work with Tom Apostol. Apostol has said that, as a graduate student, Gordon already knew as much as Apostol did. He graduated from Caltech in 1956 and spent one year there as a postdoc. He then accepted a job at UCLA but spent some time in the army before arriving there.

It is also important to note that Gordon was an outstanding teacher at all levels and was one of the most successful Ph.D. advisors at UCLA. He had twenty-six Ph.D. students, starting with David Cantor in 1960 and ending with Ken Ono in 1993. At the moment he has a total of seventy-four descendants, including forty-two grandstudents and six great-grandstudents. I was right in the middle. He was an outstanding advisor, giving precisely the right amount of guidance for each student. In particular, Ono (and Alladi, who was


*Robert Guralnick is professor of mathematics at the University of Southern California. His email address is* guralnic@usc.edu.

Photo courtesy of Keith Kendig.

**Gordon as an undergraduate student at Johns Hopkins University.**

not a Gordon student but quite influenced by him) says how influential he was in their lives and careers. Although most of Gordon's students wrote theses in number theory, he did produce about ten students whose theses were in algebra and group theory.

I had a different sort of relationship with him. I met another of Gordon's students, Michael Miller, while I was an undergraduate at UCLA (he was my TA in an undergraduate abstract algebra course). Mike suggested some problems to me, and he and I and Basil started working on them (with Mike communicating between us). We wrote a joint paper before I ever met Basil. I wrote a second paper on my own, took it to him, and asked him to be my advisor. He encouraged my independence.

He was also a wonderful teacher. He gave crystal-clear, beautiful lectures with no notes. Al Hales tells the story of being absolutely mesmerized watching him during office hours at Caltech. He ran the UCLA Putnam team for decades. The team usually did extremely well (most especially in 1968 with a third place finish). Gordon usually worked the solutions out within an hour or two. Mike Miller recalls that in 1969 Gordon finished all the problems in what seemed to be a matter of minutes.

While Gordon's work in algebra and group theory was not as significant as his other work, there were some gems. Below we discuss a few of these.

One of Gordon's earliest papers was a joint paper with Sol Golomb and Lloyd Welch on comma-free codes [2]. The motivation for this was the genetic code. At one point it was thought that nature gave an optimal solution to a coding theory problem. A set $D$ of $k$-letter words from an $n$-letter alphabet is called a comma-free dictionary if when two such words are adjoined, no set of $k$ consecutive letters, other than those in the chosen words, form a word in $D$. In the paper, various results about the number of words in a comma-free dictionary are obtained. Further results were obtained in the famous paper [7]. (Note that Jiggs is not the author; the authors of that paper were Baumert, Golomb, Gordon, Hales, Jewett, and Selfridge. As far as I know this is the first public acknowledgment of the true authors of that paper.) Gordon had several other interesting papers in coding theory.

In [6] Gordon and Straus consider the lattice of finite Galois extensions in an infinite Galois extension $L/K$. They elegantly describe the set of all possible degrees of finite Galois extensions of $K'/K$ with $K < K' < L$.

In [3] Gordon and Motzkin generalize a result of Herstein about zeroes of polynomials of division algebras $D$. They prove that any degree $n$ polynomial $f(x)$ in $D[x]$ either has at most $n$ roots in $D$ or has infinitely many roots (Herstein assumed the polynomials had central coefficients). They also consider polynomials where the variable is not assumed to commute with coefficients (and so evaluation maps are homomorphisms from the ring of such polynomials to $D$).

In [1] Fein and Gordon study a global Schur index for finite groups. Let $G$ be a finite group and let $K(G)$ denote the field generated by all the entries in the character table of $G$. It is shown that $K(G)$ is an abelian extension of $\mathbb{Q}$ such that all residue fields are splitting fields. A splitting field in characteristic 0 contains a copy of $K(G)$, but $K(G)$ need not be a splitting field for $G$. They define $m(G)$ to be the minimum of $[K : K(G)]$ as $K$ ranges over all possible splitting fields for $G$. They raise the question as to whether any abelian extension of $\mathbb{Q}$ is of the form $K(G)$ for some $G$.

In [4] Gordon and Schacher answer a question from Schacher's thesis. Let $K$ be a number field and $L/K$ be a cubic extension. The authors show that there exists a degree 4 polynomial over $K$ that is irreducible for at least two completions of $K$, has Galois group $A_4$, and whose resolvent cubic polynomial defines $L$. A corollary shows the existence of a division algebra over $K$ of dimension 144 containing a maximal subfield whose Galois group is $A_4$. In [5] the existence of a division algebra over $\mathbb{Q}$ with a maximal subfield Galois with Galois group $A_5$ was constructed. (Schacher in his thesis observed that $A_n, n > 7$, cannot occur in this way.)

## References

[1] Burton Fein and Basil Gordon, Fields generated by characters of finite groups, *J. London Math. Soc.* **4** (1972), 735–740.

[2] S. W. Golomb, Basil Gordon, and L. R. Welch, Comma-free codes, *Canad. J. Math.* **10** (1958), 202–209.

[3] B. Gordon and T. S. Motzkin, On the zeros of polynomials over division rings, *Trans. Amer. Math. Soc.* **116** (1965), 218–226.

[4] Basil Gordon and Murray Schacher, *Quartic coverings of a cubic. Number theory and algebra*, Academic Press, New York, 1977, pp. 97–101.

[5] ______, The admissibility of $A_5$, *J. Number Theory* **11** (1979), 498–504.

[6] Basil Gordon and E. G. Straus, *On the degrees of the finite extensions of a field*, 1965 Proc. Sympos. Pure Math., Vol. VIII, pp. 56–65, Amer. Math. Soc., Providence, RI.

[7] B. H. Jiggs, Recent results in comma-free codes, *Canad. J. Math.* **15** (1963), 178–187.


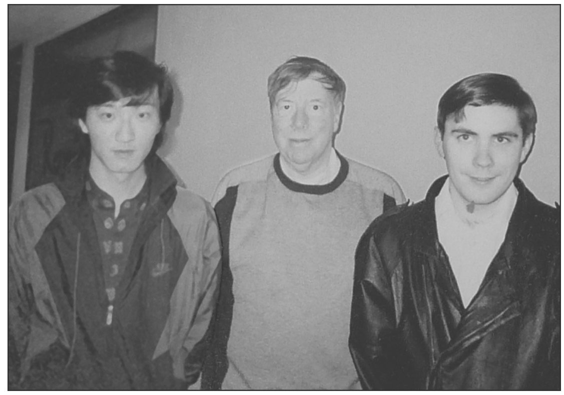
Photo courtesy of Keith Kendig.

**Basil Gordon with two of his Ph.D. students—Ken Ono (on the left) and Doug Bowman (on the right)—at UCLA in spring 1993.**

# *Ken Ono*

### Personal Reflections, and Gordon's Work on Modular Forms and Mock Theta Functions

Basil Gordon changed my life. After completing my undergraduate degree at the University of Chicago in 1989, I moved to Westwood to begin the Ph.D. program in mathematics at UCLA. I started the program with no vision. My passion up to that point had been bicycle racing. I certainly was not committed to the idea of pursuing a career in mathematics. Indeed, I almost dropped out of the program several times during my first year.

Basil Gordon's 1990 graduate course in number theory changed my life. His passionate lectures were beautiful and inspiring. Basil, my image of a great nineteenth-century scholar, saw the world through special lenses. I was mesmerized by his ability to make mathematics beautiful by making analogies with classical art, literature, and music. His encyclopedic knowledge of everything, combined with his obvious love of mathematics and his role in the subject, drew me into mathematics. Basil taught me how to find beauty in mathematical research, and he helped me find self-confidence and a genuine passion for mathematical research. He taught me these lessons in the idyllic setting of Santa Monica, in the reading room of his home (two blocks from the beach), in the Bagel Nosh, and in a cute Italian bistro that still serves delicious gnocchi.

Basil steered me in the direction of modular forms, a prescient choice, considering that Andrew Wiles would go on to use the subject in his proof of Fermat's Last Theorem, which he announced a few weeks after I defended my thesis in 1993. Basil encouraged me to think about congruences for the coefficients of modular forms, and he suggested deep works of Deligne, Serre, and Swinnerton-Dyer. Basil understood that these deep works would shed light on classical questions on partitions that date back to seminal works of Euler, Jacobi, and Ramanujan.

Basil was enamored with Ramanujan's work on the partition function $p(n)$, and he liked to say that he wanted "to do for $Q(n)$, the number of partitions of an integer $n$ into distinct parts, everything that Ramanujan had done for $p(n)$." Basil succeeded. I am particularly fond of his work with Kim Hughes [1], which established analogs of Ramanujan's celebrated partition congruences modulo powers of 5. If $k \geq 0$ is an integer, then for every integer $n$ with $24n \equiv -1 \pmod{5^{2k+1}}$ they proved that

$$Q(n) \equiv 0 \pmod{5^k}.$$

The partition functions $p(n)$ and $Q(n)$ are examples of coefficients of modular forms that can be represented as infinite products. Basil understood the importance of developing general theorems about such infinite products, which he referred to as eta-quotients and generalized eta-quotients[1] because of their connection to Dedekind's eta-function (note: $q := e^{2\pi i z}$)

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

In addition to proving general theorems about the modularity properties of such products, Basil was interested in the problem of classifying those


*Ken Ono is the Asa Griggs Candler Professor of Mathematics and Computer Science at Emory University. His email address is* ono@mathcs.emory.edu.


[1] *Sinai Robins studied generalized eta-products in his Ph.D. thesis.*

products which resemble Jacobi's classical identity

$$\prod_{n=1}^{\infty}(1-q^n)^3=\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{(k^2+k)/2}.$$

This series is *lacunary*: it has the property that almost all of its coefficients are zero. Carrying out a generalization of an elegant paper [6] of Serre, Basil and his student Sinai Robins [4] classified all of the lacunary eta-products in certain families of modular forms.

Basil was also very interested in Ramanujan's mock theta functions, an enigmatic collection of $q$-series which Ramanujan described in his "death bed" letter to G. H. Hardy. Until the recent work of Zwegers [8], where these functions are described as holomorphic parts of weight 1/2 harmonic Maass forms, very little was known about the analytic properties of these series, which included such functions as

$$f(q):=1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}.$$

In important work with his student Richard McIntosh, Basil defined a "universal" mock theta function:

$$F(\lambda,r,q):=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{\lambda n(n+1)}}{1-q^{n+r}}.$$

They observed that most of Ramanujan's mock theta functions are related to specializations of this series, and they proceeded to determine the modular transformation laws for certain specializations [2], [3]. These results fit nicely into the comprehensive framework later discovered by Zwegers in his transformational work [8] in the subject (also see [5], [7]).

Basil Gordon was a great man. He taught me how to love mathematics. He taught me how to find the confidence to do mathematics. He is my image of the perfect advisor. I owe him so many debts, and I miss him terribly.

### References


[1] B. Gordon and K. Hughes, Ramanujan congruences for $q(n)$, *Analytic Number Theory* (Philadelphia, Pa., 1980), Springer Lecture Notes in Math., vol. 899, 1981, pp. 333–359.

[2] B. Gordon and R. J. McIntosh, Modular transformations of Ramanujan's fifth and seventh order mock theta functions, *Ramanujan Journal* **7** (2003), 193–222.

[3] ______, *A survey of mock-theta functions. I*, preprint.

[4] B. Gordon and S. Robins, Lacunarity of Dedekind $\eta$-products, *Glasgow Math. J.* **37** (1995), 1–14.

[5] K. Ono, Unearthing the visions of a master: Harmonic Maass forms and number theory, *Harvard-MIT Current Developments in Mathematics, 2008*, Int. Press, Somerville, MA, 2009, pp. 347–454.

[6] J-P. Serre, Sur la lacunarité des puissances de $\eta$, *Glasgow Math. J.* **27** (1985), 203–221.

[7] D. Zagier, Ramanujan's mock theta functions and their applications (after Zwegers and Ono-Bringmann), *Astérisque* **326** (2010), 143–164.

[8] S. Zwegers, *Mock theta functions*, Ph.D. thesis (Advisor: D. Zagier), Universiteit Utrecht, 2002.


# *Bruce Rothschild*

## Working with Basil Gordon

When I was just about to move to UCLA in 1969, Ted Motzkin was briefly visiting MIT from UCLA (I was there as a postdoc). Gian-Carlo Rota was involved at that time in reorganizing the *Journal of Combinatorial Theory* (JCT) into two series, JCT-A and JCT-B. He asked Motzkin if he would be interested in becoming the editor-in-chief of Series A. As Motzkin explained it, he knew that Basil Gordon would be available to support the effort, so he agreed to take the job. The decision took him about thirty seconds according to Rota. He also enlisted me, so Basil and I became managing editors for JCT-A.

I had met Basil briefly, but didn't know much about him except for his fearsome reputation among some of my friends in the graduate student body at UCLA. He was known as one who could solve almost any problem, especially the Putnam problems, in real time. As I got to know him, it became immediately obvious to me why Motzkin had been so confident.

Although Motzkin died unexpectedly shortly after I got to UCLA, Basil and I continued to manage JCT-A, first for several years with Marshall Hall at Caltech as editor-in-chief, and then without a "chief". We worked together on the journal for more than thirty years. This was surely, in a unique way, my most satisfying and rewarding collaboration. Although we never actually wrote a joint paper, the amount of mathematics we discussed was enormous. Mostly this meant I would learn about all kinds of things from him. Basil was an incredible scholar (in many things, not just mathematics), and when we had to figure out what to do with a paper submitted to JCT-A, I could always count on learning a great deal.

JCT-A operated in the usual way at the time, receiving papers, finding referees willing and able to review them, corresponding with all concerned about revisions, and ultimately making a decision whether to publish. Atypically for me, I was the one who kept things organized and moving along (with the essential support of our long-time secretary and assistant, Elaine Barth).

---


*Bruce Rothschild is professor of mathematics at the University of California, Los Angeles. His email address is* `blr@math.ucla.edu`.

Both of us had our dilatory episodes, not infrequently—but not always—due to a large number of papers recently submitted. Like many of our reviewers, Basil needed reminders about papers I doled out to him for review or for steering to reviewers, and more often than I'd admit under quantitative scrutiny, I failed to remind him in a timely manner. What was striking—as was so much about Basil—was that when I'd call, he'd immediately know the paper I was referring to, what it was about, and what its editorial problems and strengths were. In the next day or so those previously unrecorded comments would be in my hands. His comments would be written in his amazingly neat and legible handwriting between the lines and in the margins and when necessary in a manuscript on separate pages. His handwriting was so unique that it was essentially a signature. Finding an appropriate referee could also be quite difficult on occasion. At these times an appeal to Basil's familiarity with the area in question and, even more, with related areas in algebra or number theory made it possible to find the right reviewer.

Although Basil had extremely high standards, broad knowledge, and impeccable taste, his comments about papers that he thought were not strong enough would never be anything but constructive. He would never simply dismiss a paper, no matter how weak. His comments were kind and encouraging in such cases. He took all the mathematics seriously and responded accordingly. It was just a pleasure to work with him. Sometimes I would first see a paper that seemed perhaps too elementary or even trivial, but when I showed it to Basil he would see in it an example of a number theory issue and a connection to deep problems. Even though he might agree that the paper was not appropriate for JCT-A, he would himself be quite interested.

Basil and I stepped down from managing JCT-A in 2002, and the management moved, first to Arizona and then to Australia, and the publisher from Academic Press to Elsevier. Basil was retired by then, but I saw him fairly regularly right up to his last days. He was actively engaged in mathematics until the very end, and it continued to be enlightening, entertaining, and rewarding in general to talk to him.