TURÁN INEQUALITIES FOR THE PLANE PARTITION FUNCTION

KEN ONO, SUDHIR PUJAHARI, AND LARRY ROLEN

Abstract. Heim, Neuhauser and Tröger recently established some inequalities for MacMahon's plane partition function $\mathrm{PL}(n)$ that generalize known results for Euler's partition function $p(n)$. They also conjectured that $\mathrm{PL}(n)$ is log-concave for all $n \geq 12$. We prove this conjecture. Moreover, for every $d \geq 1$, we prove their speculation that $\mathrm{PL}(n)$ satisfies the degree d Turán inequality for sufficiently large n. The case where $d = 2$ is the case of log-concavity.

1. Introduction and Statement of Results

A *partition* of a non-negative integer n is any non-increasing sequence of positive integers that sum to n. Hardy and Ramanujan famously proved that the partition function $p(n)$, which counts the number of integer partitions of n, satisfies the asymptotic formula

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{2n/3}}.$$

In the '70s, Nicolas [13] employed such asymptotics to prove[1] that $p(n)$ is log-concave for $n > 25$, where a sequence of real numbers $\{\alpha(0), \alpha(1), \dots\}$ is said to be *log-concave at* n if

$$\alpha(n)^2 \geq \alpha(n-1)\alpha(n+1). \tag{1.1}$$

The condition of log-concavity for nonvanishing real sequences $\{\alpha(n)\}$ is the first example of the *Turán inequalities*, which can be conveniently formulated in terms of Jensen polynomials. The *Jensen polynomial of degree* d *and shift* n is defined by

$$J_{\alpha}^{d,n}(X) := \sum_{j=0}^{d} \binom{d}{j} \alpha(n+j)\, X^j. \tag{1.2}$$

For degree $d = 2$ and shift $n-1$, the roots are

$$\frac{-\alpha(n) \pm \sqrt{\alpha(n)^2 - \alpha(n-1)\alpha(n+1)}}{\alpha(n+1)}.$$

Therefore, $\alpha(n)$ is log-concave at n if and only if the roots of $J_{\alpha}^{2,n-1}(X)$ are real. Generalizing log-concavity, a real sequence is said to satisfy the *degree* d *Turán inequality at* n if $J_{a}^{d,n-1}(X)$ is hyperbolic, where a polynomial is hyperbolic if all of its roots are real.

There have been several recent works on the higher Turán inequalities for $p(n)$. Chen, Jia and Wang [4] proved that $J_{p}^{3,n}(X)$ is hyperbolic for $n \geq 94$, which inspired them to conjecture, for

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[1] This result was reproved in recent work by DeSalvo and Pak [6].

every integer $d \geq 1$, that there is an integer $N_p(d)$ for which $J_p^{d,n}(X)$ is hyperbolic for $n \geq N_p(d)$. Griffin, Zagier and two of the authors recently proved (see Theorem 5 of [8]) this conjecture for all degrees d. Furthermore, recent work by Larson and Wagner [11] established the optimal values $N_p(4) = 206$ and $N_p(5) = 381$, as well as the effective bound $N_p(d) \leq (3d)^{24d}(50d)^{3d^2}$.

We consider such questions for *plane partitions* (for background, see references by Andrews [2] and Stanley [16]). A *plane partition* of size n is an array of non-negative integers $\pi := (\pi_{i,j})$ for which $|\pi| := \sum_{i,j} \pi_{i,j} = n$, in which the rows and columns are weakly decreasing. The figure below offers a 3d rendering of a plane partition.

5	4	4	3	3	2	1
4	3	2				
2	1					

$\longrightarrow$

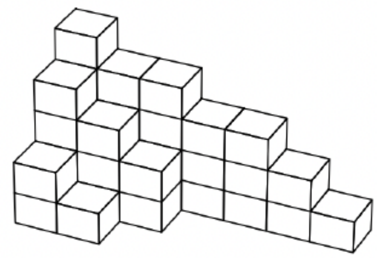

FIGURE 1. Example of a plane partition

If $\mathrm{PL}(n)$ is the number of size n plane partitions, then MacMahon [12] proved that

$$f(x) = \sum_{n=0}^{\infty} \mathrm{PL}(n)x^n := \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^n} = 1 + x + 3x^2 + 6x^3 + 13x^4 + 24x^5 + 48x^6 + \dots . \tag{1.3}$$

This function also appears prominently in physics in connection with the enumeration of small black holes in string theory. Indeed, $f(x)$ is the generating function (for example, see Appendix E of [5]) for the number of BPS bound states between a $D6$ brane and $D0$ branes on $\mathbb{C}^3$.

Heim, Neuhauser, and Tröger [9] have undertaken a study of $\mathrm{PL}(n)$ in analogy with the aforementioned results on $p(n)$. In addition to proving many inequalities satisfied by $\mathrm{PL}(n)$, they pose two further conjectures. They prove (see Theorem 1.2 of [9]) that $\mathrm{PL}(n)$ is log-concave for sufficiently large n, and they pose the following explicit conjecture (see Conjecture 1 of [9]).

Conjecture (Heim, Neuhauser, and Tröger)**.** *The function* $\mathrm{PL}(n)$ *is log-concave for* $n \geq 12$.

Here we resolve this problem.

Theorem 1.1. *The Heim-Neuhauser-Tröger Conjecture on the log-concavity of* $\mathrm{PL}(n)$ *is true.*

Heim et al. also conjectured the direct analog of the Chen-Jie-Wang Conjecture on the higher degree Turán inequalities. We prove this conjecture.

Theorem 1.2. *If d is a positive integer, then $J_{\mathrm{PL}}^{d,n}(X)$ is hyperbolic for all sufficiently large n.*

Remark. In his Ph.D thesis, Pandey will obtain an effective form of Theorem 1.2 that is a counterpart to the bound of $N_p(d) \leq (3d)^{24d}(50d)^{3d^2}$ established [11] by Larson and Wagner for $p(n)$.

The proofs of Theorems 1.1 and 1.2 require a strong asymptotic formula for $\mathrm{PL}(n)$. In the 1930s, Wright [17] adapted the "circle method" of Hardy and Ramanujan to prove asymptotic formulas for $\mathrm{PL}(n)$. He obtained such a formula for every positive integer r, where the implied error terms are smaller with larger choices of r for large n. The bulk of this paper is devoted to the lengthy and delicate task of obtaining the first asymptotics with explicitly bounded error terms.

To state these formulas, we require the two constants

$$A := \zeta(3) \approx 1.202056\ldots, \quad \text{and} \quad c := 2\int_0^\infty \frac{y\log y}{e^{2\pi y}-1}dy = \zeta'(-1) \approx -0.16542\ldots. \tag{1.4}$$

Furthermore, for any pair of non-negative integers s and m, we define coefficients $c_{s,m}(n)$ by

$$\frac{(1+y)^{2s+2m+\frac{13}{12}}}{(3+2y)^{(m+\frac{1}{2})}} =: \sum_{n=0}^{\infty} c_{s,m}(n)y^n.$$

In terms of these coefficients, we define the important numbers

$$b_{s,m} := c_{s,m}(2m). \tag{1.5}$$

The asymptotic formulas we obtain are defined in terms of special numbers $\beta_0, \beta_1, \ldots$. To define them, for every positive integer s we let

$$\alpha_s := \frac{2\Gamma(2s+2)\zeta(2s)\zeta(2s+2)}{s(2\pi)^{4s+2}}. \tag{1.6}$$

The real numbers β_s are the Taylor coefficients of

$$\exp\left(-\sum_{i=1}^{\infty}\alpha_i y^i\right) =: \sum_{n=0}^{\infty}\beta_s y^s. \tag{1.7}$$

For each r, we use the numbers $\beta_0, \ldots, \beta_{r+1}$ to derive the following explicit asymptotic formula.

Theorem 1.3. *If $r \in \mathbb{Z}^+$, then for every integer $n \geq \max(n_r, \ell_r, 87)$ (see (2.8-2.9)) we have*

$$\mathrm{PL}(n) = \frac{e^{c+3AN_n^2}}{2\pi}\sum_{s=0}^{r+1}\sum_{m=0}^{r+1}\frac{(-1)^m\beta_s b_{s,m}\Gamma\left(m+\frac{1}{2}\right)}{A^{m+\frac{1}{2}}N_n^{2s+2m+\frac{25}{12}}} + E_r^{\mathrm{maj}}(n) + E^{\mathrm{min}}(n),$$

where $|E_r^{\mathrm{maj}}(n)| \leq \widehat{E}_r^{\mathrm{maj}}(n)$ (see definition (2.36)), $N_n := \left(\frac{n}{2A}\right)^{\frac{1}{3}}$ and

$$|E^{\mathrm{min}}(n)| \leq \exp\left(\left(3A - \frac{2}{5}\right)n^2/(2A)^{\frac{2}{3}}\right).$$

Three remarks.

(1) The $s = m = 0$ term in Theorem 1.3 gives the well-known asymptotic[2]

$$\mathrm{PL}(n) \sim \frac{(2^{25}A^7)^{\frac{1}{36}}e^c}{\sqrt{12\pi}\cdot n^{\frac{25}{36}}}\exp\left(\sqrt[3]{\frac{27An^2}{4}}\right).$$

Theorem 1.3 has two explicit error terms. The error $E^{\mathrm{min}}(n)$, which is independent of r and arises from "minor arc" integrals, is exponentially smaller as $(3A-\frac{2}{5})/(2A)^{\frac{2}{3}} \approx 1.79 < \sqrt[3]{27A/4} \approx 2.01$. The error term $E_r^{\mathrm{maj}}(n)$, which arises from "major arc" integrals, only offers small power savings in n that improve with larger choices of r. In addition to the expected complications required to make error terms explicit, the proof of Theorem 1.1 is hampered by the small size of these

[2]It is well-known that Wright has a typographical error where his asymptotic is off by a factor $\sqrt{3}$.

power savings. This annoying problem does not arise[3] where working with further terms in the circle method gives exponentially improved error terms. Since the effective bounds must be sufficient to reduce the conjecture to a finite range that can be handled by computer, the task of proving Theorem 1.1 is a delicate balance between theory and practicality. To prove Theorem 1.1, we use the case of $r = 2$, where the major arc power savings is on the order of $n^{-\frac{7}{3}}$, and our theoretical bounds are sufficient to confirm the conjecture for all $n \geq 8820$.

(2) Almkvist [1] and Govindarajan and Prabhakar [7] have refined Wright's asymptotic formula in a different way. At the expense of requiring more summands as a function of n (i.e. $\sim \kappa\sqrt[3]{n}$ many summands), their formulas give precise asymptotics. Given a positive integer n, choosing $r \sim \kappa\sqrt[3]{n}$ in Theorem 1.3 gives similarly strong asymptotics, with the added benefit that the error terms are explicitly bounded.

(3) Wright's main asymptotic formula is presented as a single sum, as opposed to the double sum in s and m in Theorem 1.3. This double sum formulation is the main device in Wright's proof of his asymptotic formula. He makes a further simplifcation to obtain a single sum expression. We do not take this extra step as it would introduce further error.

Examples. For all $n \geq 105$, the $r = 1$ case of Theorem 1.3 implies (after some calculation) that

$$\mathrm{PL}(n) = e^{3\cdot 2^{-\frac{2}{3}}A^{\frac{1}{3}}n^{\frac{2}{3}}} n^{-\frac{25}{36}}\Bigg(\frac{2^{\frac{25}{36}}e^{c}A^{\frac{7}{36}}}{\sqrt{12\pi}} - \frac{\sqrt{3}\cdot 2^{\frac{13}{36}}e^{c}(3A+1385)}{25920\sqrt{\pi}A^{\frac{5}{36}}}n^{-\frac{2}{3}} - \frac{\sqrt{3}\cdot 2^{\frac{1}{36}}e^{c}(1377A^2-370650A+12525625)}{1567641600\sqrt{\pi}A^{\frac{17}{36}}}n^{-\frac{4}{3}} + E(n)\Bigg),$$

where $|E(n)| \leq 527n^{-\frac{5}{3}}$. If $\widehat{\mathrm{PL}}_1(n)$ denotes this formula without the error $E(n)$, then we have

$$E(n) := \Big(\mathrm{PL}(n) - \widehat{\mathrm{PL}}_1(n)\Big)\cdot e^{-3\cdot 2^{-\frac{2}{3}}A^{\frac{1}{3}}n^{\frac{2}{3}}} n^{\frac{25}{36}}.$$

Table 1 illustrates the observed strength of the power savings obtained by the $\widehat{\mathrm{PL}}_1(n)$ estimate.

n	$\mathrm{PL}(n)$	$E(n)$	$527n^{-\frac{5}{3}}$
100	$5.92\ldots \times 10^{16}$	$-1.18\ldots \times 10^{-7}$	$0.24\ldots$
200	$4.06\ldots \times 10^{27}$	$-3.00\ldots \times 10^{-8}$	$0.07\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
500	$2.91\ldots \times 10^{52}$	$-4.87\ldots \times 10^{-9}$	$0.01\ldots$

TABLE 1. Numerics for $r = 1$ case of Theorem 1.3

For $n \geq 87$, explicit calculations[4] with the $r = 2$ case of Theorem 1.3 gives (see (3.6))

$$\mathrm{PL}(n) = \widehat{\mathrm{PL}}_2(n) + E_2(n),$$

where $\widehat{\mathrm{PL}}_2(n)$ is defined by (3.3), and $|E_2(n)| \leq \mathcal{E}_2(n) := 227e^{3AN_n^2}n^{-\frac{109}{36}} + e^{\left(3A-\frac{2}{5}\right)N_n^2}$. The bound $\mathcal{E}_2(n)$ is smaller than $\widehat{\mathrm{PL}}_2(n)$ for $n \geq 96$. Table 2 below illustrates the strength of this estimate.

[3]This comment for $p(n)$ applies for the Fourier coefficients of all non-positive weight weakly holomorphic modular forms.

[4]We leave the details of the proof of the simpler $r = 1$ case to the reader.

n	$\widehat{\mathrm{PL}}_2(n) - \mathcal{E}_2(n)$	$\mathrm{PL}(n)$	$\widehat{\mathrm{PL}}_2(n) + \mathcal{E}_2(n)$
100	$5.932\ldots \times 10^{15}$	$5.920\ldots \times 10^{16}$	$1.124\ldots \times 10^{17}$
200	$3.706\ldots \times 10^{27}$	$4.066\ldots \times 10^{27}$	$4.426\ldots \times 10^{27}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
500	$2.913\ldots \times 10^{52}$	$2.915\ldots \times 10^{52}$	$2.917\ldots \times 10^{52}$
1000	$3.542\ldots \times 10^{84}$	$3.542\ldots \times 10^{84}$	$3.542\ldots \times 10^{84}$

Table 2. Numerics for $r = 2$ case of Theorem 1.3

In Section 2 we prove Theorem 1.3 by modifying Wright's implementation of the circle method. As is common for most applications of the circle method, the proof follows by considering integrals over "major" and "minor" arcs. Our analysis of the major arc contributions is essentially a necessarily lengthy and careful refinement of Wright's original work. However, our analysis of the minor arc contributions follows a completely different approach, which relies on work of Zagier and a careful application of Euler-Maclaurin summation. In Section 3 we deduce Theorem 1.1 and Theorem 1.2 from Theorem 1.3. Theorem 1.2 follows from recent work by Griffin, Zagier, and two of the authors in [8] on Jensen polynomials for suitable arithmetic sequences.

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2. Effective form of Wright's asymptotic formulas

For convenience, we begin by outlining Wright's strategy for obtaining asymptotics for $\mathrm{PL}(n)$, which is a modification of the classical "circle method." For positive integers n, we recall that $N_n = (\frac{n}{2A})^{\frac{1}{3}}$, where $A = \zeta(3)$ as in (1.4), and we consider the circle

$$C_{N_n} := \left\{ x : |x| = e^{-\frac{1}{N_n}} \right\}. \tag{2.1}$$

Throughout, we let θ_x denote the principal value of $\arg(x)$, and we divide C_{N_n} into a "major arc" C'_{N_n}, consisting of those x with $|\theta_x| < \frac{1}{N_n}$, and the "minor arc" C''_{N_n} which is its complement.

In terms of the generating function $f(x)$ in (1.3), Cauchy's integral formula immediately gives

$$\mathrm{PL}(n) = J(n) + E^{\min}(n), \tag{2.2}$$

where

$$J(n) := \frac{1}{2\pi i}\int_{C'_{N_n}} \frac{f(x)}{x^{n+1}}dx \qquad \text{and} \qquad E^{\min}(n) := \frac{1}{2\pi i}\int_{C''_{N_n}} \frac{f(x)}{x^{n+1}}dx \tag{2.3}$$

(i.e. with the usual counterclockwise orientation). Wright analyzes $J(n)$ and $E^{\min}(n)$ separately[5]. The asymptotics arise from the major arc piece $J(n)$, and the minor arc $E^{\min}(n)$ piece is a small error term. To prove Theorem 1.3, we improve on Wright's analysis of $E^{\min}(n)$ with a completely different argument, and we meticulously estimate $J(n)$ to obtain explicit estimates.

[5]Wright referred to $J(n)$ as $J_1(n)$ (resp. $E^{\min}(n)$ as $J_2(n)$).

2.1. **Explicit bounds over the minor arcs.** Here we bound $E^{\min}(n)$ using a different method from that of Wright. Instead of working directly with $f(x)$, we use its logarithmic derivative, which turns out to be the generating function for the sums of squares of divisors. Thanks to this interpretation, we make connection with work of Zagier [18], and we can then effectively bound $E^{\min}(n)$ using Euler-Maclaurin summation and calculus.

Proposition 2.1. *For all $n \geq 87$, we have* $|E^{\min}(n)| \leq \exp\left(\left(3A - \frac{2}{5}\right)\left(\frac{n}{2A}\right)^{\frac{2}{3}}\right)$.

2.1.1. *Lemmata for Proposition 2.1.* The next lemma bounds $\log(f(|x|))$ on the minor arcs.

Lemma 2.2. *If $x \in C''_{N_n}$, then we have* $\log\left(f(|x|)\right) \leq A{N_n}^2 + 0.33N_n - 0.5$.

Proof. We begin by noting that

$$\log f(|x|) = \sum_{m\geq 1} \frac{|x|^m}{m(1-|x|^m)^2}.$$

We let $|x| =: q$ and $t := 1/N_n$, and so we have $|x| = q = e^{-t}$. Taking the derivative, we obtain

$$L(q) := q\frac{d}{dq}\sum_{m\geq 1}\frac{q^m}{m(1-q^m)^2} = \sum_{m\geq 1}\frac{q^m(1+q^m)}{(1-q^m)^3}.$$

Thanks to the elementary fact that $X(1+X)/(1-X)^3 = \sum_{k=1}^{\infty} k^2X^k$, we have that

$$L(q) = g_3(q) := \sum_{m\geq 1}\frac{m^2q^m}{1-q^m}.$$

As $q = e^{2\pi i\tau}$ with $\tau = \frac{it}{2\pi}$, we have $q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau} = -\frac{d}{dt}$, we have $\frac{d}{dt}\log f(e^{-t}) = -g_3(e^{-t})$. Integrating this relation and adding the correct constant, we obtain

$$\log f(|x|) = \log f(e^{-t}) = \int_t^1 g_3(e^{-z})dz + \log f(e^{-1}).$$

Since $\log f(e^{-1}) \approx 1.036$, we can bound this by

$$|\log f(|x|)| = \log f(|x|) \leq \left|\int_t^1 g_3(e^{-z})dz\right| + 1.04. \tag{2.4}$$

Estimating the integral in this inequality is more involved. We make use of Zagier's work [18] on generating functions of arbitrary divisor power sums. He considers (see p. 15 of [18]) the function $g_3(e^{-z})$ as $z \searrow 0$. In the $k = 3$ case of Example 3, he applies Proposition 3 of [18] with $F(t) := \frac{t^2}{e^t-1}$ to obtain an asymptotic expansion for $g_3(e^{-t}) = \frac{1}{t^2}\sum_{m\geq 1}F(mt)$.

As we need an estimate with explicitly bounded error, as opposed to an asymptotic expansion, we dig into the proof of Proposition 3 of [18]. This gives, for each $k \geq 1$, an exact formula for $g_3(e^{-t})$, where k controls the number of terms in this asymptotic expansion, and leaves an integral that must be analyzed. The $k = 1$ case[6] gives

$$\sum_{m\geq 1}F(mt) = \frac{1}{t}\int_0^\infty F(z)dz + \frac{(-1)^0B_1F(0)t^0}{1!} + (-t)^0\int_0^\infty \frac{F'(x)\overline{B_1}(x)}{1!}dx,$$

[6] This is $N = 1$ in Zagier's paper.

where $B_1 = -1/2$ is the Bernoulli number, and $\overline{B_1}(x) = x - \lfloor x \rfloor - \frac{1}{2}$ is a periodization of the first Bernoulli polynomial $B_1(x) = x - \frac{1}{2}$. Since $F(t)$ has a removable singularity at $t = 0$ with Taylor expansion $F(t) = t - \frac{1}{2}t^2 + \dots$, we have $F(0) = 0$. Moreover, Zagier computed that

$$\int_0^\infty F(z)dz = (3-1)!\zeta(3) = 2\zeta(3) = 2A.$$

Combining these observations, we obtain

$$g_3(e^{-t}) = \frac{2A}{t^3} - \frac{1}{t^2}\int_0^\infty F'(x)(x - \lfloor x \rfloor - 1/2)dx.$$

As $|x - \lfloor x \rfloor - 1/2| \le \frac{1}{2}$, we have

$$|g_3(e^{-t})| \le \frac{2A}{t^3} + \frac{1}{2t^2}\int_0^\infty |F'(x)|dx = \frac{2A}{t^3} + \frac{1}{2t^2}\int_0^\infty \left|\frac{(xe^x - 2e^x + 2)x}{(e^x-1)^2}\right| dx.$$

Since $\int_0^\infty |F'(x)| \approx 0.6471$, we have $|g_3(e^{-t})| \le 2A/t^3 + 0.33/t^2$. Therefore, (2.4) gives

$$\log f(|x|) \le \int_t^1 \left(\frac{2A}{z^3} + \frac{0.33}{z^2}\right) dz + 1.04 \le \frac{A}{t^2} + \frac{0.33}{t} - 0.5.$$

Letting $t = 1/N_n$ gives the lemma. □

The proof of Proposition 2.1 also requires the following convenient lower bounds.

Lemma 2.3. *If $x \in C''_{N_n}$, then we have*

$$\frac{|x|}{(1-|x|)^2} - \frac{|x|}{|1-x|^2} \ge \frac{N_n^2}{2} - \frac{1}{12}.$$

Proof. We note that

$$\frac{|x|}{(1-|x|)^2} - \frac{|x|}{|1-x|^2} = \frac{|x|}{(1-|x|)^2}\left(1 - \left(\frac{1-|x|}{|1-x|}\right)^2\right). \tag{2.5}$$

We now estimate

$$\frac{1-|x|}{|1-x|} = \frac{1-e^{-1/N_n}}{|1-x|}$$

on C''_{N_n}. Recall that this is the arc of the circle of radius e^{-1/N_n} with angles ranging from $1/N_n$ to $2\pi - 1/N_n$. Geometrically, we have that the closest a point x on this arc can get to the point $(1,0)$ in the plane is when the angle is $1/N_n$ (or $2\pi - 1/N_n$), and so

$$\frac{1-|x|}{|1-x|} \le \frac{1-e^{-1/N_n}}{|1-e^{-1/N_n}e^{i/N_n}|}.$$

We obtain a formula from the denominator using the Law of Cosines. Namely, if we draw a triangle (see Figure 2) with sides consisting of the line from $(0,0)$ to $x = e^{-1/N_n}e^{i/N_n}$ on the circle, the line from $(0,0)$ to $(1,0)$, and the line connecting x to $(1,0)$, then the unknown length $|1-x|$ is the opposite side of the angle $1/N_n$ bounded by side lengths 1 and e^{-1/N_n}.

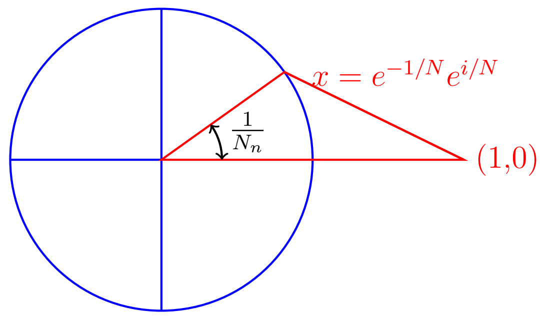

FIGURE 2. Triangle used to evaluate $|1-x|$

After letting $t = 1/N_n$, these facts imply that

$$|1 - e^{-1/N_n}e^{i/N_n}| = \sqrt{1 + e^{-2/N_n} - 2e^{-1/N_n}\cos(1/N_n)} = \sqrt{1 + e^{-2t} - 2e^{-t}\cos(t)}.$$

Thus, we get that

$$h(t) := \frac{1 - e^{-1/N_n}}{|1 - e^{-1/N_n}e^{i/N_n}|} = \frac{1 - e^{-t}}{\sqrt{1 + e^{-2t} - 2e^{-t}\cos(t)}} = \frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{48}t^2 + \dots.$$

Since $t = 1/N_n = (2A/n)^{\frac{1}{3}}$ and $n \geq 1$, we have that $t \leq (2A)^{\frac{1}{3}} \leq 1.34$. On the interval $[0, 1.34]$, the maximum absolute value of $h''(t)$ is $\lim_{t\to 0} h''(t) = \frac{\sqrt{2}}{24}$. Thus, by Taylor's Theorem,

$$|h(t)| = \frac{1}{\sqrt{2}} + O_{\leq}\left(\frac{\sqrt{2}}{48}t^2\right),$$

where $O_{\leq}(\cdot)$ means that the expression is bounded by $\cdot$ in absolute value (i.e. the implied constant can be chosen to be 1 with ordinary O-notation). Hence, for $x \in C''_{N_n}$, we have

$$\frac{1-|x|}{|1-x|} \leq \frac{1}{\sqrt{2}} + O_{\leq}\left(\frac{\sqrt{2}}{48N_n^2}\right). \tag{2.6}$$

Returning to (2.5), we estimate

$$\frac{|x|}{(1-|x|)^2} = \frac{e^{-1/N_n}}{(1-e^{-1/N_n})^2} = \frac{e^{-t}}{(1-e^{-t})^2} = t^{-2} - \frac{1}{12} + \frac{1}{240}t^2 + \dots$$

By a similar use of Taylor's Theorem, we have the strict inequality

$$\frac{e^{-t}}{(1-e^{-t})^2} > t^{-2} - \frac{1}{12} = N_n^2 - \frac{1}{12}.$$

Using this bound, (2.5) and (2.6) completes the proof as

$$\frac{|x|}{(1-|x|)^2} - \frac{|x|}{|1-x|^2} \geq \left(N_n^2 - \frac{1}{12}\right)\left(1 - \left(\frac{1}{\sqrt{2}} + O_{\leq}\left(\frac{\sqrt{2}}{48N_n^2}\right)\right)^2\right) \geq \frac{N_n^2}{2} - \frac{1}{12}.$$

□

2.1.2. *Proof of Proposition 2.1.* To estimate

$$E^{\min}(n)=\frac{1}{2\pi i}\int_{C''_{N_n}}\frac{f(x)}{x^{n+1}}dx,$$

we bound $|f(x)/x^{n+1}|$. Following Wright (see page 184 of [17]), we note that

$$|\log f(x)|\leq \log f(|x|)-\left(\frac{|x|}{(1-|x|)^2}-\frac{|x|}{|1-x|^2}\right). \tag{2.7}$$

Lemma 2.2 proved that $\log f(|x|)\leq A{N_n}^2+0.33N_n-0.5$. Furthermore, Lemma 2.3 establishes that $\frac{|x|}{(1-|x|)^2}-\frac{|x|}{|1-x|^2}\geq\frac{N_n^2}{2}-\frac{1}{12}$. Therefore, (2.7) gives

$$|\log f(x)|\leq AN_n^2+0.33N_n-0.5-\left(\frac{N_n^2}{2}-\frac{1}{12}\right)\leq(A-1/2)N_n^2+0.33N_n,$$

and so $|f(x)|\leq e^{(A-1/2)N_n^2+0.33N_n}$. Thus, the integrand in $E^{\min}(n)$ is bounded by

$$\frac{|f(x)|}{|x|^{n+1}}\leq e^{(A-1/2)N_n^2+0.33N_n+(n+1)/N_n}=e^{(3A-1/2)N_n^2+0.33N_n+1/N_n}.$$

Since the integral defining $E^{\min}(n)$ is along a curve of length bounded by the circumference of the whole circle of radius e^{-1/N_n}, which is $2\pi e^{-\frac{1}{N}n}$, we finally find that

$$|E^{\min}(n)|\leq e^{(3A-1/2)N_n^2+0.33N_n+1/N_n-1/N_n}=e^{(3A-1/2)N_n^2+0.33N_n}.$$

The claimed inequality for $n\geq 87$ follows by analyzing this last expression.

2.2. **Explicit major arc formulas.** The size of $\mathrm{PL}(n)$ is given by the major arc integral $J(n)$. To reduce the complexity of error terms, for each positive integer r we define thresholds

$$n_r:=\min\left\{n\geq 1\ :\ 0.056\cdot\sum_{s=1}^{r+1}\left(\frac{s\cdot A^{\frac{1}{3}}}{2^{\frac{7}{6}}n^{\frac{1}{3}}}\right)^{2s}\left(\frac{\pi^2 n^{\frac{1}{3}}}{(2A)^{\frac{1}{3}}s}+2\right)<1\right\} \tag{2.8}$$

and

$$\ell_r:=\min\left\{n\geq 1\ :\ 2^{r+4}\pi^3\alpha_{r+2}N_n^{-2r-4}+5e^{-4.7N_n}<\frac{1}{2}\right\}. \tag{2.9}$$

Remark. The thresholds ℓ_r and n_r are simple to compute. For instance, we have that $\ell_j=1$ for $j\leq 22$ (resp. $\ell_j=2$ for $23\leq j\leq 30$), and $n_1=1$, $n_2=2,\ldots,n_5=18$.

The following explicit major arc estimate is the main result of this subsection.

Proposition 2.4. *If $r\in\mathbb{Z}^+$, then for every $n\geq\max(\ell_r,n_r,55)$ we have*

$$J(n)=\frac{e^{c+3AN_n^2}}{2\pi}\sum_{s=0}^{r+1}\sum_{m=0}^{r+1}\frac{(-1)^m\beta_s b_{s,m}\Gamma\left(m+\frac{1}{2}\right)}{A^{m+\frac{1}{2}}N_n^{2s+2m+\frac{25}{12}}}+E_r^{\mathrm{maj}}(n),$$

where $|E_r^{\mathrm{maj}}(n)|\leq\widehat{E}_r^{\mathrm{maj}}(n)$ (see (2.36)).

2.2.1. *Lemmata for Proposition 2.4.* For fixed n, Wright sets

$$z = \log\left(\frac{1}{x}\right) = \log\left|\frac{1}{x}\right| - i\vartheta = \log\left(e^{\frac{1}{N_n}}\right) - i\vartheta = \frac{1}{N_n} - i\vartheta =: \rho e^{i\phi}, \tag{2.10}$$

and he defines

$$w := \mathrm{Re}\left(\frac{\pi}{2z}\right) = \frac{\pi\cos\phi}{2\rho}. \tag{2.11}$$

On the major arc C'_{N_n}, we have $|\vartheta| < \frac{1}{N_n}$, and so

$$\rho = \sqrt{N_n^{-2} + \vartheta^2} \in \left[\frac{1}{N_n}, \frac{\sqrt{2}}{N_n}\right]. \tag{2.12}$$

Furthermore, we have

$$|\phi| = |\arctan(\vartheta N_n)| \le \arctan(1) = \frac{\pi}{4}. \tag{2.13}$$

Wright uses the basic integral identity (see (3.17) of [17])

$$\int_0^\infty t\log(1-e^{-tz})dt = -\frac{A}{z^2},$$

which implies

$$-\log f(x) + \frac{A}{z^2} = -\log f(x) - \int_0^\infty t\log(1-e^{-tz})dt.$$

Using the generating function for $f(x)$, this becomes

$$\sum_{m\ge 1} m\log(1-e^{-tz}) - \int_0^\infty t\log(1-e^{-tz})dt.$$

Now since $\frac{1}{e^{2\pi it}-1} - \frac{1}{1-e^{-2\pi it}} = -1$, this can be written as

$$-\log f(x) + \frac{A}{z^2} = \sum_{m\ge 1} m\log(1-e^{-mz}) + \int_\Gamma \left(\frac{t\log(1-e^{-tz})}{e^{2\pi it}-1} - \frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}\right)dt,$$

where Γ is the path from 0 to ∞ which travels along the real axis, apart from sufficiently small semicircles at the positive integers above the real axis to avoid the poles of the integrand.

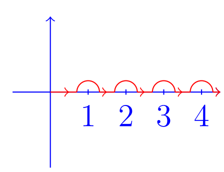

FIGURE 3. The path of integration Γ

Letting Γ' be the reflection of the path Γ across the y-axis, using the Residue Theorem (note that the residue of $\frac{t\log(1-e^{-tz})}{e^{2\pi it}-1}$ at $t = m \in \mathbb{Z}$ is $m\log(1-e^{-mz})/(2\pi i)$, Wright expresses[7] (see

[7]We note that Wright's notation does not clearly indicate that this is an exact identity.

(3.18) of [17]) this as

$$-\log f(x)+\frac{A}{z^2}=\int_{\Gamma'}\frac{t\log(1-e^{-tz})}{e^{2\pi it}-1}dt-\int_{\Gamma}\frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}dt. \tag{2.14}$$

To obtain an effective estimate for $f(x)$, we study these two integrals.

Lemma 2.5. *Assuming the notation and hypotheses above, we have*

$$\int_{\Gamma}\frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}dt=\int_0^{iw}\frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}dt+O_{\leq}\left(35N_n^2e^{-\pi^2N_n}\right),$$
$$\int_{\Gamma'}\frac{t\log\left(1-e^{-tz}\right)}{e^{2\pi it}-1}dt=\int_0^{-iw}\frac{t\log(1-e^{-tz})}{e^{2\pi it}-1}dt+O_{\leq}\left(N_n^2e^{-\pi^2N_n}\right).$$

Proof. By Cauchy's Theorem, Wright showed that the integral over Γ (see p. 182 of [17]) is

$$\int_{\Gamma}\frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}dt=\int_0^{iw}\frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}dt+\int_{iw}^{iw+\infty}\frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}dt, \tag{2.15}$$

where in the second integral on the right hand side the path is horizontal and parallel to the x-axis. To estimate the absolute value of the integrand, note that by the reverse triangle inequality (letting $x\in(0,\infty)$ so that $iw+x$ denotes a typical point on the path of integration)

$$\left|\frac{1}{1-e^{-2\pi i(iw+x)}}\right|\leq\left|\frac{1}{1-|e^{-2\pi i(iw+x)}|}\right|=\left|\frac{1}{1-e^{\pi^2\cos\phi/\rho}}\right|.$$

Using (2.12), this is bounded by $\left|1/(1-e^{\pi^2N_n})\right|\leq 1.007e^{-\pi^2N_n}$, where we used $N_n\geq N_1\approx 0.75$ is monotonically increasing in n and $e^{-\pi^2N_1}|1-e^{\pi^2N_1}|\approx 0.99936$. Therefore, we have

$$\left|\int_{iw}^{iw+\infty}\frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}dt\right|\leq 1.007e^{-\pi^2N_n}\int_{iw}^{iw+\infty}\left|t\log(1-e^{-tz})\right|dt.$$

Following Wright again, and making the change of variables $v=tz$, this is bounded by

$$1.007e^{-\pi^2N_n}\int_L\frac{|t\log(1-e^{-v})|}{|z|^2}dv\leq 1.007N_n^2e^{-\pi^2N_n}\int_L\left|v\log(1-e^{-v})\right|dv=:U(n), \tag{2.16}$$

where L is the ray from $v=\frac{\pi i}{2}e^{i\phi}\cos\phi$ that forms the angle ϕ with the (positive) real axis.

We split the integral in (2.16) into two pieces. Wright also does this, but we make a slightly different choice below to assist us in our goal of obtaining effective estimates. Throughout, let $v_t:=iwz+te^{i\phi}$, so that $L=\{v_t\ :\ t\in[0,\infty)\}$. Noting that

$$wz=\frac{\pi\cos\phi}{2\rho}\cdot\rho e^{i\phi}=\frac{\pi\cos\phi e^{i\phi}}{2}, \tag{2.17}$$

we first estimate the piece $|\log(1-e^{-v_t})|$ in the integrand of (2.16). We use (4.5.6) of DLMF [14], which states that for complex arguments y, to obtain

$$|\log(1+y)|\leq-\log(1-|y|),\qquad \text{when } |y|<1. \tag{2.18}$$

We will break up the line L into a compact piece L_1 and a remaining piece L_2, where we can utilize this bound. To see where it applies, we compute

$$|e^{-v_t}|=e^{-\operatorname{Re}(v_t)}=e^{w\operatorname{Im}(z)-\operatorname{Re}(te^{i\phi})}=e^{-w\vartheta-t\cos\phi}=e^{-\cos\phi\left(\frac{\pi\vartheta}{2\rho}+t\right)}. \tag{2.19}$$

In turn, this is less than or equal to one if and only if $\cos\phi\left(\frac{\pi\vartheta}{2\rho}+t\right)\geq 0$. Using (2.12) and (2.13), and using $\vartheta\geq -1/N$, we have

$$\cos\phi\left(\frac{\pi\vartheta}{2\rho}+t\right)>\frac{\sqrt{2}}{2}\left(-\frac{\pi}{2\sqrt{2}}+t\right)=-\frac{\pi}{4}+\frac{t}{\sqrt{2}},$$

which gives

$$|e^{-v_t}|\leq e^{\frac{\pi}{4}-\frac{t}{\sqrt{2}}}. \tag{2.20}$$

Hence, if we let

$$L_1:=\left\{v_t\colon t\in[0,\pi/\sqrt{8}+1)\right\},\quad L_2:=\left\{v_t\colon t\in[\pi/\sqrt{8}+1,\infty)\right\},$$

then we can use (2.18) to estimate the integrand on L_2. Thanks to (2.20), for $t>\pi/\sqrt{8}$ we have

$$|v_t\log(1-e^{-v_t})|\leq -|v_t|\cdot\log\left(1-|e^{-v_t}|\right)\leq -|v_t|\log\left(1-e^{\frac{\pi}{4}-\frac{t}{\sqrt{2}}}\right).$$

Noting that $|v_t|\leq|\cos\phi|\frac{\pi}{2}+t\leq\frac{\pi}{2}+t$, and using the bound $-\log(1-x)<x/(1-x)$ for $0\neq x<1$ on real-valued logarithms from (4.5.2) of DLMF [14], we find

$$\left|v_t\log(1-e^{-v_t})\right|\leq\frac{\left(\frac{\pi}{2}+t\right)\cdot e^{\frac{\pi}{4}-\frac{t}{\sqrt{2}}}}{1-e^{\frac{\pi}{4}-\frac{t}{\sqrt{2}}}}.$$

Since L_1 is compact, we then find the following estimate for (2.16) (note that when we integrate on L_2, since we are integrating absolute values, the change of variables in the differential goes away as it has absolute value 1):

$$\begin{aligned}U(n)&=1.007N_n^2e^{-\pi^2N_n}\int_{L_1}\left|v\log(1-e^{-v})\right|dv+1.007N_n^2e^{-\pi^2N_n}N_n^2\int_{L_2}\left|v\log(1-e^{-v})\right|dv\\&\leq 1.007N_n^2e^{-\pi^2N_n}\left(\frac{\pi}{\sqrt{8}}+1\right)\cdot\max\left\{|v_t|\cdot|\log(1-e^{-v_t})|\ :t\in[0,\pi/\sqrt{8}+1)\right\}\\&\quad+1.007N_n^2e^{-\pi^2N_n}\int_{\frac{\pi}{\sqrt{8}}+1}^{\infty}\frac{\left(\frac{\pi}{2}+t\right)\cdot e^{\frac{\pi}{4}-\frac{t}{\sqrt{2}}}}{1-e^{\frac{\pi}{4}-\frac{t}{\sqrt{2}}}}dt\\&\leq 1.007N_n^2e^{-\pi^2N_n}\left(\frac{\pi}{\sqrt{8}}+1\right)\cdot\max\left\{|v_t|\cdot|\log(1-e^{-v_t})|\colon t\in[0,\pi/\sqrt{8}+1)\right\}+4.8N_n^2e^{-\pi^2N_n}.\end{aligned}$$

Now we estimate $\log(1-e^{-v_t})$ on for $t\in[0,\pi/\sqrt{8}+1)$. We begin by recalling that for complex y, the principal branch of the logarithm is given by $\log(y)=\log(|y|)+i\arg(y)$. Thus, we have

$$|\log(1-e^{-v_t})|\leq\left|\log|1-e^{-v_t}|\right|+\pi. \tag{2.21}$$

To bound the logarithm, we find the maximum and minimum on the interval $t\in[0,\pi/\sqrt{8}+1)$. The only critical point of $|1-e^{-v_t}|$ is at $t=\cosh(i\phi)$. Thus, the potential extrema of $|1-e^{-v_t}|$ are at 0, $\pi/\sqrt{8}$, and $\cosh(i\phi)$. That is, the maximum of $|\log|1-e^{-v_t}||$ is bounded by

$$\max\left\{|1-e^{-v_0}|,\ |1-e^{-v_{\pi/\sqrt{8}+1}}|,\ |1-e^{-v_{\cosh(i\phi)}}|\right\}.$$

We evaluate these in turn. For instance, we have that $|1-e^{-v_0}|=\left|1-e^{-\frac{\pi i\cos\phi e^{i\phi}}{2}}\right|$. On the interval $\phi\in[-\pi/4,\pi/4]$, we have $0.75\leq|1-e^{-v_0}|\leq 1.85$ and $0.9\leq|1-e^{-v_{\pi/\sqrt{8}+1}}|\leq 1.4$. Similarly, we find $0.7 \leq |1 - e^{-v_{\cosh(i\phi)}}| \leq 1.45$. Combining these observations, we have $|\log |1 - e^{-v_t}|| \leq 0.615\ldots$ on $[0, \pi/\sqrt{8}+1)$. Thus, by (2.21), we have $|\log(1-e^{-v_t})| \leq 3.76$. To bound $|v_t|$ on this interval, using (2.17) and the triangle inequality, we find that $|v_t| \leq \frac{\pi\cos\phi}{2} + t \leq \frac{\pi}{2} + \frac{\pi}{\sqrt{8}} + 1 = 3.68\ldots$. Therefore, we conclude that

$$1.007N_n^2 e^{-\pi^2 N_n}\left(\frac{\pi}{\sqrt{8}}+1\right)\max\left\{|v_t|\cdot|\log(1-e^{-v_t})\colon t \in [0, \pi/\sqrt{8}+1)\right\} \leq 30N_n^2 e^{-\pi^2 N_n}.$$

As a consequence, we obtain

$$1.007N_n^2 e^{-\pi^2 N_n}\int_L \left|v\log(1-e^{-v})\right| dv \leq 35N_n^2 e^{-\pi^2 N_n}.$$

Returning to (2.15), we have shown that

$$\int_\Gamma \frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}dt = \int_0^{iw}\frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}dt + O_\leq\left(35N_n^2 e^{-\pi^2 N_n}\right). \tag{2.22}$$

This bounds the second integral in (2.14), which is the first claim in the lemma.

The second claim in the lemma follows by arguing as above (after suitable sign changes in the integrand) using the path of integration along Γ'. Namely, we get

$$\int_{\Gamma'} \frac{t\log\left(1-e^{-tz}\right)}{e^{2\pi it}-1}dt = \int_0^{-iw}\frac{t\log(1-e^{-tz})}{e^{2\pi it}-1}dt + O_\leq\left(N_n^2 e^{-\pi^2 N_n}\right). \tag{2.23}$$

□

We require bounds for three integrals, two of which make use of the α_s defined by (1.6).

Lemma 2.6. *Assuming the notation and hypotheses above, the following are true.*
(1) If n is a positive integer, then we have

$$\mathcal{I}_1 := 2\int_0^w \frac{y\log(yz)}{e^{2\pi y}-1}dy = c + \frac{\log z}{12} + O_\leq\left(3e^{-4.7N_n}\right).$$

(2) If $n \geq n_r$ is an integer, then we have

$$\mathcal{I}_2 := -2\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_0^w \frac{y^{2s+1}}{e^{2\pi y}-1}dy = -\sum_{s=1}^{r+1}\alpha_s z^{2s} O_\leq\left(e^{-\frac{\pi^2 N_n}{2}}\right).$$

(3) If n is a positive integer, then we have

$$\mathcal{I}_3 := -2\sum_{s\geq r+2}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_0^w \frac{y^{2s+1}}{e^{2\pi y}-1}dy \leq 2^{r+4}\pi^3\alpha_{r+2}N_n^{-2r-4}.$$

Proof. We estimate these three quantities one-by-one. For $\mathcal{I}_1$, we have

$$\mathcal{I}_1 = 2\int_0^w \frac{y\log(yz)}{e^{2\pi y}-1}dy = 2\int_0^\infty \frac{y\log(yz)}{e^{2\pi y}-1}dy - 2\int_w^\infty \frac{y\log(yz)}{e^{2\pi y}-1}dy = c + \frac{\log z}{12} - 2\int_w^\infty \frac{y\log(yz)}{e^{2\pi y}-1}dy.$$

To evaluate the last integral, we first note that $w = \frac{\pi\cos\phi}{2\rho} \geq \frac{\pi\sqrt{2}}{4\rho} \geq \frac{\pi N_n}{4} \geq 0.58$, where we used (2.13) and (2.12) and the fact that $n \geq 1$. By direct manipulation, for $y \geq 0.58$, we have

$$\frac{1}{e^{2\pi y}-1} \leq 1.1\cdot e^{-2\pi y}. \tag{2.24}$$

Then on the interval $w \geq 0.58$, we find that

$$\begin{aligned}\left|-2\int_w^\infty \frac{y\log(yz)}{e^{2\pi y}-1}dy\right| &\leq 2.2\left|\int_w^\infty y\log(yz)e^{-2\pi y}dy\right|\\ &= \frac{2.2}{4\pi^2}\cdot e^{-2\pi w}\left|1-e^{2\pi w}\operatorname{Ei}(-2\pi w)+(1+2\pi w)\log(wz)\right|,\end{aligned}$$

where $\operatorname{Ei}(x) := -\int_{-x}^\infty \frac{e^{-t}dt}{t}$. Straightforward manipulation then gives

$$\left|-2\int_w^\infty \frac{y\log(yz)}{e^{2\pi y}-1}dy\right| \leq e^{-2\pi w}\left(1+0.056\cdot(1+2\pi w)(|\log\rho|+\pi)\right). \tag{2.25}$$

If $n \geq 7$, then $\sqrt{2}/N_n < 1$, and so $w \geq \pi N_n/4$. Therefore, (2.25) is bounded from above by

$$e^{-\frac{\pi^2 N_n}{2}}\left(1+0.056\cdot\left(1+\frac{\pi^2 N_n}{2}\right)(\log N_n+\pi)\right).$$

It is straightforward to show that this less than or equal to $3e^{-4.7N_n}$. By direct computation for $1 \leq n \leq 6$ using (2.25), we find that this bound holds in general, and so we have

$$\mathcal{I}_1 = c + \frac{\log z}{12} + O_\leq\left(3e^{-4.7N_n}\right). \tag{2.26}$$

Using (2.24) and the integral representation of $\zeta(s)$, we can manipulate $\mathcal{I}_2$ to obtain

$$\begin{aligned}\mathcal{I}_2 &= -2\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_0^\infty \frac{y^{2s+1}}{e^{2\pi y}-1}dy + 2\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_w^\infty \frac{y^{2s+1}}{e^{2\pi y}-1}dy\\ &= -2\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_0^\infty \frac{y^{2s+1}}{e^{2\pi y}-1}dy + O_\leq\left(2.2\cdot\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_w^\infty y^{2s+1}e^{-2\pi y}dy\right)\\ &= -2\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\left(\frac{\Gamma(2+2s)\zeta(2+2s)}{(2\pi)^{2+2s}}\right) + O_\leq\left(2.2\cdot\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\left(\frac{\Gamma(2+2s;2\pi w)}{(2\pi)^{2+2s}}\right)\right),\end{aligned}$$

where $\Gamma(a;x) := \int_x^\infty t^{a-1}e^{-t}dt$ is the *incomplete Gamma function*. Using (1.6), this is

$$\mathcal{I}_2 = -\sum_{s=1}^{r+1}\alpha_s z^{2s} + O_\leq\left(2.2\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\left(\frac{\Gamma(2+2s;2\pi w)}{(2\pi)^{2+2s}}\right)\right).$$

To estimate this, we note that the proof of Lemma 2.2 of [3] shows, for $a > 2$, that

$$\Gamma(a;x) \leq \frac{(x+b_a)^a - x^a}{ab_a}e^{-x},$$

where $b_a := \Gamma(a+1)^{\frac{1}{a-1}}$. Combined with the Bernoulli number formula for $\zeta(s)$ at positive even integers, this gives

$$\left|2.2\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\left(\frac{\Gamma(2+2s;2\pi w)}{(2\pi)^{2+2s}}\right)\right| \leq \frac{1.1e^{-2\pi w}}{4\pi^2}\sum_{s=1}^{r+1}\frac{\rho^{2s}B_{2s}}{s(2\pi)^{2s}(2s)!}\left(2\pi w+[(2s+1)!]^{\frac{1}{2s+1}}\right)^{2s+1}.$$

A simple manipulation bounds this by

$$\frac{1.1e^{-2\pi w}}{4\pi^2}\sum_{s=1}^{r+1}\frac{z^{2s}B_{2s}}{s(2\pi)^{2s}(2s)!}(2\pi w+2s)^{2s+1}.$$

Using the Bernoulli number upper bound from (24.9.8) of [14], recalling that $w \geq \pi N_n/4$, and noting that $w = \frac{\pi\cos\phi}{2\rho} < \frac{\pi}{2\rho} < \frac{\pi N_n}{2}$, this is bounded by

$$0.056\cdot e^{-2\pi w}\sum_{s=1}^{r+1}\frac{\rho^{2s}}{s(2\pi)^{4s}}(2\pi w+2s)^{2s+1} = 0.056\cdot e^{-2\pi w}\sum_{s=1}^{r+1}\rho^{2s}\left(\frac{1}{2\pi}+\frac{s}{2\pi^2}\right)^{2s}\left(\frac{2\pi w}{s}+2\right)$$
$$\leq 0.056\cdot e^{-\frac{\pi^2 N_n}{2}}\sum_{s=1}^{r+1}\left(\frac{s\sqrt{2}}{4N_n}\right)^{2s}\cdot\left(\frac{\pi^2 N_n}{s}+2\right) = n_r\cdot e^{-\frac{\pi^2 N_n}{2}}.$$

Therefore, if $n \geq n_r$, then we obtain the claimed inequality for $\mathcal{I}_2$.

Finally, we turn to the bound for $\mathcal{I}_3$, which we recall is

$$\mathcal{I}_3 := -2\sum_{s=r+2}^{\infty}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_0^w \frac{y^{2s+1}}{e^{2\pi y}-1}dy.$$

As $\zeta(2r+4) \geq \zeta(2s)$ for all $s \geq r+2$, we have

$$|\mathcal{I}_3| \leq \frac{\zeta(2r+4)}{r+2}\frac{\rho^{2r+4}}{(2\pi)^{2r+4}}\int_0^w \frac{y^{2r+5}}{e^{2\pi y}-1}\sum_{s=0}^{\infty}\left(\frac{y\rho}{2\pi}\right)^{2s}dy.$$

Therefore, using (1.6) we find that

$$|\mathcal{I}_3| \leq \frac{\zeta(2r+4)}{r+2}\cdot\frac{\rho^{2r+4}}{(2\pi)^{2r+4}}\int_0^w \frac{y^{2r+5}}{(e^{2\pi y}-1)\left(1-(\frac{y\rho}{2\pi})^2\right)}dy$$
$$< \frac{\zeta(2r+4)}{r+2}\cdot\frac{\rho^{2r+4}}{(2\pi)^{2r+4}}\int_0^\infty \frac{y^{2r+5}}{e^{2\pi y}-1}dy$$
$$< \frac{\Gamma(2r+6)\zeta(2r+4)\zeta(2r+6)\rho^{2r+4}}{(r+2)(2\pi)^{2r+7}} = 4\pi^3\alpha_{r+2}\rho^{2r+4}.$$

Since, $\rho \leq \frac{\sqrt{2}}{N_n}$, we obtain the claimed inequality for $\mathcal{I}_3$. □

2.2.2. *Proof of Proposition 2.4.* We begin by recalling (2.14), which asserts that

$$-\log f(x) + \frac{A}{z^2} = \int_{\Gamma'}\frac{t\log(1-e^{-tz})}{e^{2\pi it}-1}dt - \int_{\Gamma}\frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}dt.$$

By combining the two integrals as a single integral, Lemma 2.5 gives

$$\log\left(f(x)\right) = \frac{A}{z^2} + \int_0^w \frac{y\log\left(2\sin\left(\frac{yz}{2}\right)\right)}{e^{2\pi y-1}}dy + O_{\leq}\left(36N_n^2 e^{-\pi^2 N_n}\right). \tag{2.27}$$

To use this formula, we employ the identity $\sin\tau = \tau\prod_{m\geq 1}\left(1-\frac{\tau^2}{m^2\pi^2}\right)$, which implies that

$$\log(\sin(\tau)) = \log\tau - \sum_{s\geq 1}\sum_{m\geq 1}\frac{\tau^{2s}}{s\cdot m^{2s}\pi^{2s}} = \log\tau - \sum_{s\geq 1}\frac{\zeta(2s)\tau^{2s}}{s\pi^{2s}}.$$

Hence, for every $r \geq 1$, the integral in (2.27) satisfies[8]

$$\begin{aligned}\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3 &= \int_0^w \frac{y\log\left(2\sin\left(\frac{yz}{2}\right)\right)}{e^{2\pi y-1}}dy\\ &= 2\int_0^w \frac{y\log(yz)}{e^{2\pi y}-1}dy - 2\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_0^w \frac{y^{2s+1}}{e^{2\pi y}-1}dy - 2\sum_{s\geq r+2}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_0^w \frac{y^{2s+1}}{e^{2\pi y}-1}dy.\end{aligned}$$

Thanks to Lemma 2.6, for $n \geq n_r$, (2.27) gives[9]

$$\log f(x) = \frac{A}{z^2} + c + \frac{\log z}{12} - \sum_{s=1}^{r+1}\alpha_s z^{2s} + O_{\leq}\left(2^{r+4}\pi^3\alpha_{r+2}N_n^{-2r-4} + 5e^{-4.7N_n}\right). \tag{2.28}$$

We now make use of a complex-analytic version for the remainder terms of the Taylor series of $f(x)$, which is required as our estimates make use of the expressions involving β_s as opposed to α_s. Specifically, the β_s were defined in (1.7) to be the initial $r+2$ Taylor coefficients of

$$g_r(z) := e^{-\sum_{s=1}^{r+1}\alpha_s z^s} = \sum_{s=0}^{r+1}\beta_s z^s + R_r(z),$$

where $R_r(z)$ is the remainder. For convenience, we assume that $n \geq 55$, which guarantees that $|z| = \rho \leq \sqrt{2}N_n^{-1} \leq \frac{1}{2}$. The standard complex Taylor series remainder estimate (for example, see Theorem B.21 of [10] with $R = 1$) gives

$$|R_r(z)| \leq \frac{\max_{|z|=1}(|g_r(z)|)\cdot |z|^{r+2}}{1-|z|} \leq C_r|z|^{r+2},$$

where

$$C_r := 2\max_{|z|=1}\left(\left|e^{-\sum_{s=1}^{r+1}\alpha_s z^s}\right|\right). \tag{2.29}$$

Substituting in $z = z^2$ yields (this is allowed since we demanded that $|z| < 1$, and so $|z^2| < 1$ is still in the range of validity for the remainder estimate) gives

$$g_r(z^2) = e^{-\sum_{s=1}^{r+1}\alpha_s z^{2s}} = \sum_{s=0}^{r+1}\beta_s z^{2s} + O_{\leq}\left(C_r|z|^{2r+4}\right).$$

Therefore, by exponentiating (2.28), for $n \geq 55$ we obtain

$$f(x) = e^c z^{\frac{1}{12}} e^{\frac{A}{z^2}}\left(\sum_{s=0}^{r+1}\beta_s z^{2s} + O_{\leq}\left(C_r|z|^{2r+4}\right)\right)\cdot O_{\leq}\left(\exp(2^{r+4}\pi^3\alpha_{r+2}N_n^{-2r-4} + 5e^{-4.7N_n})\right).$$

To address the error term on the far right above, we assume that $n \geq \ell_r$, which by (2.9) gives

$$2^{r+4}\pi^3\alpha_{r+2}N_n^{-2r-4} + 5e^{-4.7N_n} \leq \frac{1}{2} < 1 \quad\text{and}\quad \frac{1}{1-(2^{r+4}\pi^3\alpha_{r+2}N_n^{-2r-4}+5e^{-4.7N_n})} \leq 2.$$

[8] Wright refers to $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$ as I_4, S_1, and S_2, respectively, on page 183 of [17].

[9] This is an explicit form of the first equation on p. 184 of [17]. We correct a typographical error where the sum on s accidentally starts with $s = 0$ instead of $s = 1$.

Thanks to (4.5.11) of [14], which states that $e^x < 1 + x/(1-x)$ for $x < 1$, this gives

$$\begin{aligned} f(x) &= e^c z^{\frac{1}{12}} e^{\frac{A}{z^2}} \left(\sum_{s=0}^{r+1} \beta_s z^{2s} + O_{\leq}\left(C_r |z|^{2r+4}\right)\right) \cdot O_{\leq}\left(1 + 2^{r+5}\pi^3 \alpha_{r+2} N_n^{-2r-4} + 10e^{-4.7N_n}\right) \\ &= M(x) + \mathcal{X}_r(n) + \mathcal{Y}_r(n), \end{aligned}$$

where we let $M(x) := e^{c+\frac{A}{z^2}} \sum_{s=0}^{r+1} \beta_s z^{2s+\frac{1}{12}}$, and where

$$\mathcal{X}_r(n) := e^{c+AN_n^2} 2^{r+\frac{49}{24}} C_r N_n^{-2r-\frac{49}{12}} \tag{2.30}$$

and

$$\mathcal{Y}_r(n) := \left| e^{c+AN_n^2}\left(2^{r+5}\pi^3\alpha_{r+2}N_n^{-2r-4} + 10e^{-4.7N_n}\right)\left(2^{r+\frac{49}{24}}C_r N_n^{-2r-\frac{49}{12}} + \sum_{s=0}^{r+1} 2^{s+\frac{1}{24}}\beta_s N_n^{-2s-\frac{1}{12}}\right)\right|. \tag{2.31}$$

This encodes the compilation of error on C'_{N_n} using the facts that $\left|e^{\frac{A}{z^2}}\right| \leq e^{\left|\frac{A}{\rho^2}\right|} \leq e^{AN_n^2}$ and $|z| = \rho \leq \sqrt{2}N_n^{-1}$. Now, recalling that $n = 2AN_n^3$, we obtain

$$\begin{aligned} J(n) &= \frac{1}{2\pi i}\int_{\frac{1-i}{N_n}}^{\frac{1+i}{N_n}} f(e^{-z}) e^{2AN_n^3 z} dz = \frac{1}{2\pi i}\int_{\frac{1-i}{N_n}}^{\frac{1+i}{N_n}} (M(x) + \mathcal{X}_r(n) + \mathcal{Y}_r(n)) e^{2AN_n^3 z} dz \\ &= \frac{e^c}{2\pi i}\int_{\frac{1-i}{N_n}}^{\frac{1+i}{N_n}} \left(\sum_{s=0}^{r+1}\beta_s z^{2s+\frac{1}{12}}\right) e^{\frac{A}{z^2}+2AN_n^3 z} dz + O_{\leq}\left(\frac{(\mathcal{X}_r(n)+\mathcal{Y}_r(n))\cdot e^{2AN_n^2}}{N_n \pi}\right), \end{aligned}$$

where we used that on the path of integration, $|e^{2AN_n^3 z}| = e^{2AN_n^2}$, and that the length is $2N_n^{-1}$. We now let $v = N_n z$, and introduce Wright's

$$P_s := \frac{1}{2\pi i}\int_{1-i}^{1+i} v^{2s+\frac{1}{12}} \exp\left(AN_n^2\left(2v + \frac{1}{v^2}\right)\right) dv,$$

to find that

$$J(n) = e^c \cdot \sum_{s=0}^{r+1} \frac{\beta_s P_s}{N_n^{2s+\frac{13}{12}}} + O_{\leq}\left(\frac{(\mathcal{X}_r(n)+\mathcal{Y}_r(n))\cdot e^{2AN_n^2}}{N_n\pi}\right). \tag{2.32}$$

To complete the proof we require an explicit version of Wright's expansion of P_s, that he obtained via the method of steepest descent. Using his notation, we first let $\mathcal{C}$ be the plane curve defined by the equation $(x^2+y^2)^2 = x$, together with the labelled points $E = (2^{-2/3}, 2^{-2/3})$, $D = (2^{-2/3}, -2^{-2/3})$ on $\mathcal{C}$ and the points $O = (0,0)$, $G = (1,1)$, and $F = (1,-1)$ in the plane. This is illustrated in the following figure.

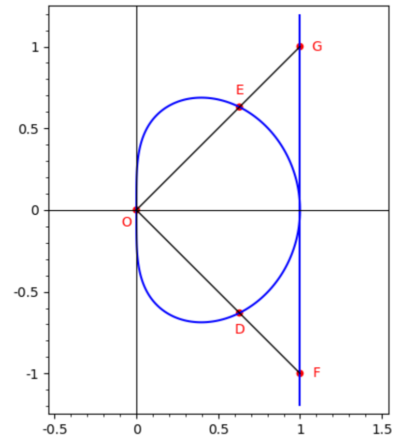

FIGURE 4. The curve $\mathcal{C}$ and the points E, D, O, G, F.

Wright noted that if $\xi_s(v) := (2\pi i)^{-1} v^{2s+1/12} \exp(AN_n^2(2v + 1/v^2))$, then since $|v| \leq \sqrt{2}$, $x \leq 1$ on OG and OF, then we have

$$|\xi_s(v)dv| = \frac{|v|^{2s+\frac{1}{2}}}{2\pi} e^{2AN_n^2 x} \leq \frac{2^{s+\frac{1}{4}} e^{2AN_n^2}}{2\pi}, \quad (v \in OF \cup OG),$$

and he cleverly showed[10] (see p. 186 of [17]), in terms of the arc lengths in the diagram, that
(2.33)

$$\begin{aligned} P_s &= \int_{\mathcal{C}} \xi_s(v)dv + O_{\leq}\left(\left|\int_D^F \xi_s(v)dv\right| + \left|\int_G^E \xi_s(v)dv\right| + \left|\int_E^O \xi_s(v)dv\right| + \left|\int_O^D \xi_s(v)dv\right|\right) \\ &= \int_{\mathcal{C}} \xi_s(v)dv + O_{\leq}\left(\frac{2^{s+\frac{1}{4}} e^{2AN_n^2}}{2\pi}\left(\mathrm{length}(DF) + \mathrm{length}(GE) + \mathrm{length}(EO) + \mathrm{length}(OD)\right)\right) \\ &= \int_{\mathcal{C}} \xi_s(v)dv + O_{\leq}\left(0.64 \cdot 2^s \cdot e^{2AN_n^2}\right). \end{aligned}$$

Making the change of variables $t^2 = 3 - 2v - v^{-2}$ to estimate the integral $\int_{\mathcal{C}} \xi_s(v)dv$, we have

$$\int_{\mathcal{C}} \xi_s(v)dv = e^{3AN_n^2} \int_{\mathbb{R}} \chi_s(t) e^{-AN_n^2 t^2} dt,$$

where

$$\chi_s(t) = \frac{v^{2s+\frac{37}{12}} \cdot t}{2\pi i(1 - v^3)} = \frac{v^{2s+\frac{25}{12}} \sqrt{2v+1}}{2\pi(v^2 + v + 1)}$$

is a smooth function on $\mathcal{C}$. Expanding $\chi_s(t) = \sum_{m \geq 0} a_{s,m} t^m$, we can estimate the Taylor remainder for all $t \in \mathbb{R}$ by

$$\chi_s(t) = \sum_{m=0}^{2r+3} a_{s,m} t^m + O_{\leq}\left(D_r \cdot |t|^{2r+4}\right),$$

[10]This corrects a typo of Wright's concerning the path of integration of the fourth integral on the right hand side. Wright accidentally wrote $\int_O^F$ instead of $\int_O^D$.

where

$$D_r := \frac{1}{(2r+4)!} \cdot \max\left\{\max\left\{\left|\chi_s^{(2r+4)}(t)\right|\right\}_{t\in\mathbb{R}}\right\}_{s=0}^{r+1}. \tag{2.34}$$

We note that D_r is explicitly computable, as $\chi_s(t)$ is a smooth function on the compact curve $\mathcal{C}$. Thus, we find that[11]

$$\begin{aligned}\int_{\mathcal{C}} \xi_s(v)dv &= e^{3AN_n^2} \sum_{m=0}^{2r+3} \int_{\mathbb{R}} a_{s,m} t^m e^{-AN_n^2 t^2} dt + O_{\leq}\left(D_r \cdot e^{3AN_n^2} \int_{\mathbb{R}} |t|^{2r+4} e^{-AN_n^2 t^2} dt\right)\\ &= e^{3AN_n^2} \sum_{m=0}^{2r+3} \int_{\mathbb{R}} a_{s,m} t^m e^{-AN_n^2 t^2} dt + O_{\leq}\left(D_r \cdot \Gamma\left(r+\frac{5}{2}\right)(AN_n^2)^{-\frac{5}{2}-r} e^{3AN_n^2}\right).\end{aligned}$$

Plugging into (2.33) gives

$$P_s = e^{3AN_n^2} \sum_{m=0}^{r+1} \frac{a_{s,2m}\Gamma\left(m+\frac{1}{2}\right)}{(AN_n^2)^{m+\frac{1}{2}}} + O_{\leq}\left(D_r \cdot \Gamma\left(r+\frac{5}{2}\right)(AN_n^2)^{-\frac{5}{2}-r} e^{3AN_n^2} + 0.64 \cdot 2^s e^{2AN_n^2}\right).$$

Finally, Wright proved that $a_{s,2m} = (-1)^m b_{s,m}/2\pi$, for $s \leq r+1$, and so we obtain

$$P_s = e^{3AN_n^2} \sum_{m=0}^{r+1} \frac{(-1)^m b_{s,m}\Gamma\left(m+\frac{1}{2}\right)}{2\pi \cdot (AN_n^2)^{m+\frac{1}{2}}} + O_{\leq}\left(D_r \cdot \Gamma\left(r+\frac{5}{2}\right)(AN_n^2)^{-\frac{5}{2}-r} e^{3AN_n^2} + 0.64 \cdot 2^{r+1} e^{2AN_n^2}\right).$$

Plugging this into (2.32) gives

$$\begin{aligned}J(n) = \frac{e^{c+3AN_n^2}}{2\pi} \sum_{s=0}^{r+1}\sum_{m=0}^{r+1} \frac{(-1)^m \beta_s b_{s,m}\Gamma\left(m+\frac{1}{2}\right)}{A^{m+\frac{1}{2}} N_n^{2s+2m+\frac{25}{12}}} + O_{\leq}\Bigg(\frac{(\mathcal{X}_r(n)+\mathcal{Y}_r(n))e^{2AN_n^2}}{N_n\pi}\\ + \left|e^c\left(D_r \cdot \Gamma\left(r+\frac{5}{2}\right)(AN_n^2)^{-\frac{5}{2}-r} e^{3AN_n^2} + 0.64 \cdot 2^{r+1} e^{2AN_n^2}\right)\sum_{s=0}^{r+1} \beta_s N_n^{-2s-\frac{13}{12}}\right|\Bigg).\end{aligned} \tag{2.35}$$

Therefore, the proof is complete by letting

$$\widehat{E}_r^{\mathrm{maj}}(n) := \frac{(\mathcal{X}_r(n)+\mathcal{Y}_r(n))e^{2AN_n^2}}{N_n\pi} + |\mathcal{Z}_r(n)|, \tag{2.36}$$

where

$$\mathcal{Z}_r(n) := e^c\left(D_r \cdot \Gamma\left(r+\frac{5}{2}\right)(AN_n^2)^{-\frac{5}{2}-r} e^{3AN_n^2} + 0.64 \cdot 2^{r+1} e^{2AN_n^2}\right)\sum_{s=0}^{r+1} \beta_s N_n^{-2s-\frac{13}{12}}. \tag{2.37}$$

2.3. **Proof of Theorem 1.3.** Cauchy's theorem (2.2) gives $\mathrm{PL}(n) = J(n) + E^{\min}(n)$. Therefore, the theorem follows from Proposition 2.1 and Proposition 2.4.

3. Proof of Theorems 1.1 and 1.2

Here we make use of Theorem 1.3 to prove Theorems 1.1 and 1.2.

[11] This corrects a typo in Wright's work, where he drops the factor $e^{3AN_n^2}$ temporarily in the final displayed equations of page 187.

3.1. **Proof of Theorem 1.1.** To apply Theorem 1.3 with a fixed r, it is natural to express

$$\mathrm{PL}(n) = \widehat{\mathrm{PL}}_r(n) + O_{\leq}\left(\mathcal{E}_r(n)\right), \tag{3.1}$$

where $\widehat{\mathrm{PL}}_r(n)$ is the main term and $\mathcal{E}_r(n)$ is an explicit bound for the error. Then we can write

$$\begin{aligned}\mathrm{PL}(n)^2 - &\mathrm{PL}(n-1)\mathrm{PL}(n+1) \\ &\geq (\widehat{\mathrm{PL}}_r(n) - \mathcal{E}_r(n))^2 - (\widehat{\mathrm{PL}}_r(n-1) + \mathcal{E}_r(n-1)) \cdot (\widehat{\mathrm{PL}}_r(n+1) + \mathcal{E}_r(n+1)).\end{aligned} \tag{3.2}$$

We apply Theorem 1.3 with $r = 2$. We first determine the n to which it applies. By (2.8), n_2 is the point beyond where the following expression is guaranteed to be less than 1:

$$\frac{7\left(243 \cdot 2^{\frac{2}{3}}\pi^2 n^{\frac{1}{3}} A^{\frac{5}{3}} + 32 \cdot 2^{\frac{1}{3}}\pi^2 n^{\frac{5}{3}} A^{\frac{1}{3}} + 64\pi^2 nA + 64 \cdot 2^{\frac{2}{3}} n^{\frac{4}{3}} A^{\frac{2}{3}} + 256 \cdot 2^{\frac{1}{3}} n^{\frac{2}{3}} A^{\frac{4}{3}} + 2916A^2\right)}{32000n^2}.$$

This expression is decreasing in n. At $n = 1$ it is ≈ 2.4, and at $n = 2$ it is ≈ 0.8, and so $n_2 = 2$. To compute ℓ_2, we determine when (2.9) is guaranteed to be less than 1/2. This quantity is bounded from above by $0.00005n^{-\frac{8}{3}} + 5e^{-3.5n^{\frac{1}{3}}}$, which is decreasing in n and is less than 0.16 for $n = 1$, and so we have $\ell_2 = 1$. Therefore, $\max\{\ell_2, n_2, 87\} = 87$, and so Theorem 1.3 holds for $n \geq 87$.

The terms defining $\widehat{\mathrm{PL}}_r(n)$ come from the double sum in Theorem 1.3, and can be organized by the powers of n which are controlled by $s + m$. To this end, we recall

$$\mathrm{PL}(n) \approx \frac{e^{c+3AN_n^2}}{2\pi} \sum_{s=0}^{r+1}\sum_{m=0}^{r+1} \frac{(-1)^m \beta_s b_{s,m} \Gamma\left(m + \frac{1}{2}\right)}{A^{m+\frac{1}{2}} N_n^{2s+2m+\frac{25}{12}}} =: e^{3AN_n^2} n^{-\frac{25}{36}} \sum_{s=0}^{r+1}\sum_{m=0}^{r+1} f_{s,m}(n).$$

The leading asymptotic is given by $f_{0,0}(n) = 2^{\frac{25}{36}} e^c A^{\frac{7}{36}}/\sqrt{12\pi}$. The next largest term, with a power savings of $n^{-\frac{2}{3}}$, is given by the terms with $s + m = 1$:

$$f_{1,0}(n) + f_{0,1}(n) = -\frac{\sqrt{3} \cdot 2^{\frac{13}{36}} e^c (3A + 1385)}{25920\sqrt{\pi} A^{\frac{5}{36}}} \cdot n^{-\frac{2}{3}}.$$

The terms with $s + m = 2$ give

$$f_{1,1}(n) + f_{2,0}(n) + f_{0,2}(n) = -\frac{\sqrt{3} \cdot 2^{\frac{1}{36}} e^c (1377A^2 - 370650A + 12525625)}{1567641600\sqrt{\pi} A^{\frac{17}{36}}} \cdot n^{-\frac{4}{3}}.$$

The final term which contributes to our main is the sum of terms with $s + m = 3$, giving

$$\begin{aligned}&f_{3,0}(n) + f_{0,3}(n) + f_{2,1}(n) + f_{1,2}(n) \\ &\quad = -\frac{\sqrt{3} \cdot 2^{\frac{25}{36}} e^c (609309A^3 - 90985275A^2 + 4957761375A + 37576109375)}{40633270272000\sqrt{\pi} A^{\frac{29}{36}}} \cdot n^{-2}.\end{aligned}$$

These contributions together give our main term

$$\begin{aligned}\widehat{\mathrm{PL}}_2(n) := e^{3AN_n^2} n^{-\frac{25}{36}} (&f_{0,0}(n) + f_{1,0}(n) + f_{0,1}(n) + f_{1,1}(n) + f_{2,0}(n) + f_{0,2}(n) \\ &+ f_{3,0}(n) + f_{0,3}(n) + f_{2,1}(n) + f_{1,2}(n)) \\ \approx &e^{c+3AN_n^2} n^{-\frac{25}{36}} (0.23 - 0.056n^{-\frac{2}{3}} - 0.006n^{-\frac{4}{3}} - 0.001n^{-2}).\end{aligned} \tag{3.3}$$

The remaining terms with $s+m\geq 4$ are of equal or smaller magnitude than the error which will come from $E_2^{\mathrm{maj}}(n)$ and $E^{\mathrm{min}}(n)$. Thus, we will put these terms into our error estimate $\mathcal{E}_2(n)$. A simple manipulation shows that for $n\geq 87$, we have

$$f_{2,2}(n)+f_{3,1}(n)+f_{1,3}(n)+f_{3,2}(n)+f_{2,3}(n)+f_{3,3}(n)=O_{\leq}(10^{-5}n^{-\frac{7}{3}}).$$

Thus, Theorem 1.3 implies that for $n\geq 87$, we have

$$\mathrm{PL}(n)=\widehat{\mathrm{PL}}_2(n)+O_{\leq}\left(10^{-5}e^{3AN_n^2}n^{-\frac{109}{36}}+\widehat{E}_2^{\mathrm{maj}}(n)+E_2^{\mathrm{min}}(n)\right). \tag{3.4}$$

We now aim to bound $\widehat{E}_2^{\mathrm{maj}}(n)$. This error is $O(e^{3AN_n^2}n^{-\frac{109}{36}})$.[12] Thus, we will compare our error terms to this expression. We now turn to computing the relevant constants in turn. After computing $\alpha(1)$, $\alpha(2)$, and $\alpha(3)$, we see that computing the constant C_2 is equivalent to maximizing

$$\left|e^{-\frac{e^{it}}{2880}-\frac{e^{2it}}{725760}-\frac{e^{3it}}{43545600}}\right|^2.$$

The derivative of this function is

$$\frac{1}{7257600}\left(4\cos(t)^2+80\cos(t)+5039\right)e^{\left(-\frac{1}{5443200}\cos(t)^3-\frac{1}{181440}\cos(t)^2-\frac{5039}{7257600}\cos(t)+\frac{1}{362880}\right)}\sin(t),$$

which shows that there is a local maximum at $t=\pi$, which is the global maximum. This directly gives $C_2=2\cdot e^{\frac{15061}{43545600}}\leq 2.0007$.

The computation of the constants D_r is more involved. To compute D_r, one recursively computes the derivatives $v^{(m)}(t)=v^{(m)}$ for $m=1,\dots,r+3$ by repeatedly differentiating the equation $t^2=3-2v-v^{-2}$. Using the definition of $\chi_s(t)$ and the relation $t=-i(v-1)\sqrt{2v+1}/v$ gives an expression for $\chi_s(t)$ as a function of v. Here we need D_2, which requires the derivatives

$$\begin{aligned}
v'&=tv^3/(v^3-1),\\
v''&=v^3(1+3t^2v^2-2v^3+v^6)/(v^3-1)^3,\\
v^{(3)}&=3tv^5(3+5t^2v^2-6v^3+4t^2v^5+3v^6)/(1-v^3)^5,\\
v^{(4)}&=3v^5(3+30t^2v^2-12v^3+\cdots+20t^4v^{10}+24t^2v^{11}+3v^{12})/(1-v^3)^7,\\
v^{(5)}&=15tv^7(15+70t^2v^2-48v^3+\cdots+24t^4v^{13}+40t^2v^{14}+12v^{15})/(1-v^3)^9,\\
v^{(6)}&=47v^7(5+105t^2v^2-26v^3+\cdots+120t^4v^{19}+60t^2v^{20}+4v^{21})/(1-v^3)^{11},\\
v^{(7)}&=315tv^9(35+315t^2v^2+\ldots+120t^2v^{23}+20v^{24})/(1-v^3)^{13},\\
v^{(8)}&=315v^9(35+1260t^2v^2+\ldots+480t^2v^{29}+20v^{30})/(1-v^3)^{15}.
\end{aligned}$$

Differentiating and plugging into the definition of $\chi_s(t)$, we obtain expressions such as

$$\chi_1^{(6)}(t)=\frac{iv^{\frac{121}{12}}\sqrt{2v+1}}{859963392\pi(v^2+v+1)^{17}}\cdots(181387629768625+\ldots+2444688400v^{20}). \tag{3.5}$$

Now that these are expressions of v, we can substitute $v=x\pm i\sqrt{\sqrt{x}-x^2}$, which traces out the curve $\mathcal{C}$ as x ranges from 0 to 1. That is, expressions such as the absolute value of (3.5) can be evaluated for $x\in[0,1]$ numerically to obtain an estimate for D_2. By symmetry, we only

[12] At the top of page 189, Wright makes a claim about the magnitude of this error term which is incorrect in the power of N_n. This can be seen by comparing with the error estimate in the final equation of page 188, which is correct and matches our power $-109/12$ here for $r=2$.

need to bound for $v = x + i\sqrt{\sqrt{x} - x^2}$ with $x \in [0,1]$. Performing this analysis, we find that $D_2 \leq 5.3$.

These constants allow us to bound $\widehat{E}_2^{\mathrm{maj}}(n)$ by (2.36) for $n \geq 87$. A simple calculation gives

$$\frac{\mathcal{X}_2(n) + \mathcal{Y}_2(n)}{N_n \pi} \cdot e^{2AN_n^2} = O_{\leq}\left((127 - 10^{-5}) e^{3AN_n^2} n^{-\frac{109}{36}}\right).$$

Further simplification gives (see (2.37)) $\mathcal{Z}_2(n) = O_{\leq}\left(100 e^{3AN_n^2} n^{-\frac{109}{36}}\right)$. Therefore, we find that

$$\widehat{E}_2^{\mathrm{maj}}(n) = O_{\leq}\left((227 - 10^{-5}) e^{3AN_n^2} n^{-\frac{109}{36}}\right).$$

Since we have $E^{\min}(n) = O_{\leq}\left(e^{\left(3A - \frac{2}{5}\right)N_n^2}\right)$, expression (3.4) yields

$$\mathrm{PL}(n) = \widehat{\mathrm{PL}}_2(n) + O_{\leq}\left(227 e^{3AN_n^2} n^{-\frac{109}{36}} + e^{\left(3A - \frac{2}{5}\right)N_n^2}\right). \tag{3.6}$$

In other words, we can let $\mathcal{E}_2(n) := 227 e^{3AN_n^2} n^{-\frac{109}{36}} + e^{\left(3A - \frac{2}{5}\right)N_n^2}$. A simple calculation with (3.2) and (3.3) establishes that $\mathrm{PL}(n)$ is log-concave for all $n \geq 8820$. This completes the proof as log-concavity has been confirmed on a computer for all $12 \leq n \leq 10^5$ by Heim et al. [9].

3.2. **Proof of Theorem 1.2.** The proof of Theorem 1.2 makes use of Theorem 1.3 and recent work by Griffin, Zagier, and two of the authors of [8] on Jensen polynomials of suitable sequences of real numbers. The main observation is that suitable real sequences have Jensen polynomials that can be modeled by the *Hermite polynomials* $H_d(X)$, which are orthogonal polynomials for the measure $\mu(X) = e^{-X^2/4}$, and are given by the generating function

$$\sum_{d=0}^{\infty} H_d(X)\, \frac{t^d}{d!} \;=\; e^{-t^2 + Xt} \;=\; 1 \;+ X\,t \;+\; (X^2 - 2)\,\frac{t^2}{2!} \;+\; (X^3 - 6X)\,\frac{t^3}{3!} \;+\; \cdots$$

Theorem 3.1 (Theorems 3 and 6 of [8])**.** *Let $\{\alpha(n)\}$, $\{A(n)\}$, and $\{\delta(n)\}$ be sequences of positive real numbers, with $\delta(n)$ tending to 0. For an integer $d \geq 3$, suppose that there are real numbers $g_3(n), g_4(n), \ldots g_d(n)$, for which*

$$\log\left(\frac{\alpha(n+j)}{\alpha(n)}\right) \;=\; A(n)j - \delta(n)^2 j^2 + \sum_{i=3}^{d} g_i(n) j^i + \mathrm{o}\big(\delta(n)^d\big) \qquad \text{as } n \to \infty, \tag{3.7}$$

with $g_i(n) = o(\delta(n)^i)$ for each $3 \leq i \leq d$. Then we have

$$\lim_{n \to \infty} \left(\frac{\delta(n)^{-d}}{\alpha(n)}\, J_{\alpha}^{d,n}\left(\frac{\delta(n)\, X \;-\; 1}{\exp(A(n))}\right)\right) \;=\; H_d(X). \tag{3.8}$$

Remark. Theorem 3.1 holds for $d \in \{1, 2\}$. In these cases there simply are no numbers $g_i(n)$.

Since the Hermite polynomials are hyperbolic, and since this property of a polynomial with real coefficients is invariant under small deformation, we have the following key consequence.

Corollary. *Assuming the hypotheses in Theorem 3.1, we have that the Jensen polynomials $J_{\alpha}^{d,n}(X)$ are hyperbolic for all but finitely many values n.*

Proof of Theorem 1.2. Theorem 1.1 implies that $J_{\mathrm{PL}}^{2,n}(X)$ is hyperbolic for all $n \geq 12$. Therefore, without loss of generality we may assume that $d \geq 3$. To complete the proof, we apply Theorem 3.1.

Theorem 1.3 gives constants $\kappa > 0$ and $\nu_0, \nu_1, \ldots$ for which $\mathrm{PL}(n)$ has an asymptotic expansion to all orders of $1/n$ of the form

$$\mathrm{PL}(n) \sim \exp\left(\sqrt[3]{\kappa n^2}\right) n^{-\frac{25}{36}} \cdot \left(\nu_0 - \sum_{m=1}^{\infty} \frac{\nu_m}{n^{\frac{2m}{3}}}\right).$$

We choose ν_0 so that the leading factor is a product of an exponential with a power of n. Using $\log(1-X) = -X - X^2/2 - X^3/3 - X^4/4 - \ldots$, we can rewrite this expression as

$$\mathrm{PL}(n) \sim \exp(\sqrt[3]{\kappa n^2}) n^{-\frac{25}{36}} \cdot \exp\left(c_0 + \frac{c_1}{n^{2/3}} + \frac{c_2}{n^{4/3}} + \frac{c_3}{n^2} + \ldots\right),$$

which in turn gives

$$\log\left(\frac{\mathrm{PL}(n+j)}{\mathrm{PL}(n)}\right) \sim \sqrt[3]{\kappa} \sum_{i=1}^{\infty} \binom{2/3}{i} \frac{j^i}{n^{i-\frac{2}{3}}} - \frac{25}{36} \sum_{i=1}^{\infty} \frac{(-1)^{i-1} j^i}{i n^i} + \sum_{s,t\geq 1} c_t \binom{-t}{s} \frac{j^s}{n^{2(s+t)/3}}.$$

As $\binom{2/3}{1} = 2/3 > 0$ and $\binom{2/3}{2} = -1/9 < 0$, Theorem 3.1 applies, and so its corollary proves the theorem. □

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