

MACMAHON'S SUMS-OF-DIVISORS AND ALLIED q -SERIES

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ABSTRACT. Here we investigate the q -series

$$\mathcal{U}_a(q) = \sum_{n=0}^{\infty} MO(a; n)q^n := \sum_{0 < k_1 < k_2 < \dots < k_a} \frac{q^{k_1+k_2+\dots+k_a}}{(1-q^{k_1})^2(1-q^{k_2})^2 \dots (1-q^{k_a})^2},$$

$$\mathcal{U}_a^*(q) = \sum_{n=0}^{\infty} M(a; n)q^n := \sum_{1 \leq k_1 \leq k_2 \leq \dots \leq k_a} \frac{q^{k_1+k_2+\dots+k_a}}{(1-q^{k_1})^2(1-q^{k_2})^2 \dots (1-q^{k_a})^2}.$$

MacMahon introduced the $\mathcal{U}_a(q)$ in his seminal work on partitions and divisor functions. Recent works show that these series are sums of quasimodular forms with weights $\leq 2a$. We make this explicit by describing them in terms of Eisenstein series. We use these formulas to obtain explicit and general congruences for the coefficients $MO(a; n)$ and $M(a; n)$. Notably, we prove the conjecture of Amdeberhan-Andrews-Tauraso as the $m = 0$ special case of the infinite family of congruences

$$MO(11m + 10; 11n + 7) \equiv 0 \pmod{11},$$

and we prove that

$$MO(17m + 16; 17n + 15) \equiv 0 \pmod{17}.$$

We obtain further formulae using the limiting behavior of these series. For $n \leq a + \binom{a+1}{2}$, we obtain a ‘‘hook length’’ formulae for $MO(a; n)$, and for $n \leq 2a$, we find that $M(a; n) = \binom{a+n-1}{n-a} + \binom{a+n-2}{n-a-1}$.

1. INTRODUCTION AND STATEMENT OF RESULTS

At first glance, one might underestimate the value of the trivial observation that the number of partitions of an integer n into identical parts is also the number of divisors of n . This fact is a glimpse of a rich theory that relates integer partitions and divisor functions. Indeed, MacMahon’s important paper [7] is based on the idea of connecting partitions to divisor sums: partition of n using k_1 repeated s_1 times, and k_2 repeated s_2 times, and so on through k_a repeated s_a times. Using this convention, he considered the sum of products of the *multiplicities* $MO(a; n) := \sum s_1 s_2 \dots s_a$ of size n partitions, which has the generating function

$$(1.1) \quad \mathcal{U}_a(q) := \sum_{n \geq 0} MO(a; n) q^n = \sum_{0 < k_1 < k_2 < \dots < k_a} \frac{q^{k_1+k_2+\dots+k_a}}{(1-q^{k_1})^2(1-q^{k_2})^2 \dots (1-q^{k_a})^2}.$$

His work [7] is populated with beautiful divisor function identities, where $\sigma_\nu(n) := \sum_{d|n} d^\nu$, such as:

$$(1.2) \quad \mathcal{U}_1(q) = \sum_{n \geq 1} \sigma_1(n)q^n \quad \text{and} \quad \mathcal{U}_2(q) = \sum_{n \geq 1} \left(\frac{\sigma_1(n)}{8} - \frac{n\sigma_1(n)}{4} + \frac{\sigma_3(n)}{8} \right) q^n.$$

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To entice the reader, we offer the first few terms of $\mathcal{U}_1(q), \dots, \mathcal{U}_4(q)$:

$$\begin{aligned}\mathcal{U}_1(q) &= q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + 8q^7 + \dots, \\ \mathcal{U}_2(q) &= q^3 + 3q^4 + 9q^5 + 15q^6 + 30q^7 + 45q^8 + 67q^9 + \dots, \\ \mathcal{U}_3(q) &= q^6 + 3q^7 + 9q^8 + 22q^9 + 42q^{10} + 81q^{11} + 140q^{12} + \dots, \\ \mathcal{U}_4(q) &= q^{10} + 3q^{11} + 9q^{12} + 22q^{13} + 51q^{14} + 97q^{15} + 188q^{16} + \dots\end{aligned}$$

The inequalities in definition (1.1) imply that $q^{-\frac{a(a+1)}{2}} \cdot \mathcal{U}_a(q) = 1 + 3q + \dots$, while for $a \geq 2$, we have

$$q^{-\frac{a(a+1)}{2}} \cdot \mathcal{U}_a(q) = 1 + 3q + 9q^2 + \dots$$

Answering the natural question, we show that this sequence converges to a simple infinite product, which, by the theory of Nekrasov-Okounkov [8], gives *hook length* formulae for many of the $MO(a; n)$.

To make this precise, recall that a partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of n , denoted $\lambda \vdash n$, is a non-increasing sequence of positive integers that sum to n . Its *Young diagram* is the left-justified array of boxes where the row lengths are the parts. The *hook* $H(i, j)$ of the box in position (i, j) consists of this box, together with those below it and those to its right. Its *hook length* $h(i, j) := (\lambda_i - i) + (\lambda'_j - j) + 1$ is the number of such boxes, where λ'_j is the number of boxes in column j . Denote the multiset of hook lengths of λ by $\mathcal{H}(\lambda)$. Finally, we recall the ‘‘exponential form’’ of a partition $\lambda = (1^{m_1}, 2^{m_2}, \dots, t^{m_t})$, where m_i is the multiplicity of part i .

Example. The exponential form of $\lambda = (4, 4, 2)$ is $\lambda = (1^0, 2^1, 3^0, 4^2, 5^0, 6^0, 7^0, 8^0, 9^0, 10^0) \vdash 10$. Its Young diagram is given below, and shows that $\mathcal{H}(\lambda) = \{6, 5, 5, 4, 3, 2, 2, 2, 1, 1\}$.

6	5	3	2
5	4	2	1
2	1		

We derive the following result using the work of Andrews-Rose [2] and Nekrasov-Okounkov [8].

Theorem 1.1. *The following are true:*

(i) *If a is a positive integer, then we have that*

$$q^{-\frac{a(a+1)}{2}} \cdot \mathcal{U}_a(q) = \prod_{n \geq 1} \frac{1}{(1 - q^n)^3} + O(q^{a+1}).$$

(ii) *If $n \leq a + \binom{a+1}{2}$, then we have that*

$$MO(a; n) = \sum_{\lambda \vdash n-a} \prod_{h \in \mathcal{H}(\lambda)} \left(\frac{2}{h^2} + 1 \right) = \sum_{\lambda \vdash n-a} \prod_{s=1}^{n-a} \binom{2 + m_s}{2}.$$

Inspired by the $\mathcal{U}_a(q)$, Amdeberhan-Andrews-Tauraso [1] initiated the study of the q -series

$$(1.3) \quad \mathcal{U}_a^*(q) := \sum_{n \geq 0} M(a; n) q^n = \sum_{1 \leq k_1 \leq k_2 \leq \dots \leq k_a} \frac{q^{k_1 + k_2 + \dots + k_a}}{(1 - q^{k_1})^2 (1 - q^{k_2})^2 \dots (1 - q^{k_a})^2},$$

where the strict inequalities in (1.1) are replaced by weak inequalities. One easily sees that

$$\mathcal{U}_a^*(q) = \sum_{n \geq 0} M(a; n) q^n = q^a + (2a + 1)q^{a+1} + \dots$$

To entice the reader, we offer the first few terms of $\mathcal{U}_1^*(q), \dots, \mathcal{U}_4^*(q)$:

$$\begin{aligned}\mathcal{U}_1^*(q) &= q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + \dots, \\ \mathcal{U}_2^*(q) &= q^2 + 5q^3 + 14q^4 + 29q^5 + 55q^6 + 86q^7 + \dots, \\ \mathcal{U}_3^*(q) &= q^3 + 7q^4 + 27q^5 + 77q^6 + 181q^7 + 378q^8 + \dots, \\ \mathcal{U}_4^*(q) &= q^4 + 9q^5 + 44q^6 + 156q^7 + 450q^8 + 1121q^9 + \dots.\end{aligned}$$

In analogy with Theorem 1.1, we consider the limiting behavior of these series. These series converge to specializations of the generating function for the polynomials $p_0(x) := 1, p_1(x) := 2x + 1, p_2(x) := 2x^2 + 3x, p_3(x) := \frac{4}{3}x^3 + 4x^2 + \frac{5}{3}x, \dots$. For $n \geq 1$, these polynomials are defined by

$$(1.4) \quad p_n(x) := \binom{2x+n-1}{n} + \binom{2x+n-2}{n-1}.$$

As a companion to Theorem 1.1, we obtain the following theorem.

Theorem 1.2. *The following are true:*

(i) *If a is a positive integer, then we have that*

$$q^{-a} \cdot \mathcal{U}_a^*(q) = \sum_{n=0}^a p_n(a)q^n + O(q^{a+1}).$$

(ii) *If $n \leq 2a$, then we have that $M(a; n) = p_{n-a}(a)$.*

Remark. The $\mathcal{U}_a(q)$ and $\mathcal{U}_a^*(q)$ are multiple q -zeta values. To make this precise, we recall the q -notation $[k]_q := \frac{1-q^k}{1-q}$ and the multiple q -zeta values (for example, see [3])

$$\begin{aligned}\zeta_q(m_1, \dots, m_a) &:= \sum_{0 < k_1 < \dots < k_a} \frac{q^{(m_1-1)k_1 + \dots + (m_a-1)k_a}}{[k_1]_q^{m_1} \dots [k_a]_q^{m_a}}, \\ \zeta_q^*(m_1, \dots, m_a) &:= \sum_{1 \leq k_1 \leq \dots \leq k_a} \frac{q^{(m_1-1)k_1 + \dots + (m_a-1)k_a}}{[k_1]_q^{m_1} \dots [k_a]_q^{m_a}}.\end{aligned}$$

We have that $(1-q)^{2a} \cdot \mathcal{U}_a(q) = \zeta_q(2, \dots, 2)$ and $(1-q)^{2a} \cdot \mathcal{U}_a^*(q) = \zeta_q^*(2, \dots, 2)$.

As divisor functions arise as the coefficients of Eisenstein series, identities such as (1.2) suggest a strong relationship between the $\mathcal{U}_a(q)$ and quasimodular forms. This speculation was confirmed by Andrews-Rose. Indeed, they proved (see [2, Cor. 4]) and [12, Th. 1.12]) that each $\mathcal{U}_a(q)$ is a linear combination of quasimodular forms on $\mathrm{SL}_2(\mathbb{Z})$ with weights $\leq 2a$. Similarly, Amdeberhan-Andrews-Tauraso [1, Th. 6.1] proved that each $\mathcal{U}_a^*(q)$ is a linear combination of quasimodular forms on $\mathrm{SL}_2(\mathbb{Z})$ with weights $\leq 2a$.

Here we make this quasimodularity explicit. In the case of $\mathcal{U}_a(q)$, we employ the standard generators of the graded ring of quasimodular forms: the quasimodular weight 2 Eisenstein series

$$(1.5) \quad E_2(q) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n,$$

and the weight 4 and 6 modular Eisenstein series

$$(1.6) \quad E_4(q) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n \quad \text{and} \quad E_6(q) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n.$$

It is well known [6] that the ring of quasimodular forms is $\mathbb{C}[E_2, E_4, E_6]$, and so our goal is to obtain formulas in terms of the monomials $E_2^\alpha(q)E_4^\beta(q)E_6^\gamma(q)$, where α, β and γ are non-negative integers.

Our formulas for $\mathcal{U}_a(q)$ use of the triple index sequence of rational numbers defined by the recursion

$$(1.7) \quad \begin{aligned} c(\alpha, \beta, \gamma) := & -\frac{1}{3}(2\alpha + 8\beta + 12\gamma + 1) \cdot c(\alpha - 1, \beta, \gamma) + \frac{2}{3}(\alpha + 1) \cdot c(\alpha + 1, \beta - 1, \gamma) \\ & + \frac{8}{3}(\beta + 1) \cdot c(\alpha, \beta + 1, \gamma - 1) + 4(\gamma + 1) \cdot c(\alpha, \beta - 2, \gamma + 1), \end{aligned}$$

where $\alpha, \beta, \gamma \geq 0$. To seed the recursion, we let $c(0, 0, 0) := 1$, and we let $c(\alpha, \beta, \gamma) := 0$ if any of the arguments are negative. Here we list the ‘‘first few’’ values:

$$c(1, 0, 0) = -1, \quad c(0, 1, 0) = -\frac{2}{3}, \quad c(0, 0, 1) = -\frac{16}{9}, \quad c(1, 1, 0) = \frac{14}{3}, \quad c(1, 0, 1) = \frac{64}{3}, \dots$$

We also require constants for the quasimodular summands sorted by weight. For $0 \leq t \leq a$, define

$$(1.8) \quad w_t(a) := \frac{\binom{2a}{a}}{16^a(2a+1)} \sum_{0 \leq \ell_1 < \dots < \ell_t < a} \prod_{j=1}^t \frac{1}{(2\ell_j + 1)^2}.$$

In terms of $w_t(a)$ and the numbers $c(\alpha, \beta, \gamma)$, we have the following explicit formulae for $\mathcal{U}_a(q)$.

Theorem 1.3. *If a is a non-negative integer, then we have that*

$$\mathcal{U}_a(q) = \sum_{t=0}^a w_t(a) \sum_{\substack{\alpha, \beta, \gamma \geq 0 \\ \alpha + 2\beta + 3\gamma = t}} c(\alpha, \beta, \gamma) E_2(q)^\alpha E_4(q)^\beta E_6(q)^\gamma.$$

Example. For $a = 3$, Theorem 1.3 gives

$$\mathcal{U}_3(q) = \frac{5}{7168} - \frac{37E_2(q)}{46080} + \frac{5E_2(q)^2}{27648} - \frac{E_4(q)}{13824} - \frac{E_2(q)^3}{82944} + \frac{E_2(q)E_4(q)}{69120} - \frac{E_6(q)}{181440}.$$

We turn to the $\mathcal{U}_a^*(q)$. Instead of using $E_2(q)$, $E_4(q)$, and $E_6(q)$, we use all of the Eisenstein series

$$(1.9) \quad E_{2k}(q) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n,$$

where B_k is the usual k th Bernoulli number. Namely, we let $\mathbb{E}_0(q) := 1$, and for positive t we define

$$(1.10) \quad \mathbb{E}_{2t}^*(q) := \sum_{(1^{m_1}, \dots, t^{m_t}) \vdash t} \prod_{j=1}^t \frac{1}{m_j!} \left(-\frac{B_{2j} E_{2j}(q)}{(2j) \cdot (2j!)} \right)^{m_j}.$$

We require constants for the summands sorted by weight. We let $w_0^*(0) := 1$, and for $a > 0$, we let

$$(1.11) \quad w_0^*(a) := \sum_{i=1}^a \frac{(-1)^{i-1} \binom{2i}{i}}{16^i (2i+1)} w_0^*(a-i).$$

For $1 \leq t \leq a$, we define

$$(1.12) \quad w_t^*(a) := (-1)^{a+t-1} 4^t (2t+1)! w_{t-1}^*(a-1).$$

With this notation, we obtain the following explicit expressions for $\mathcal{U}_a^*(q)$.

Theorem 1.4. *If a is a non-negative integer, then we have that*

$$\mathcal{U}_a^*(q) = \sum_{t=0}^a w_t^*(a) \cdot \mathbb{E}_{2t}^*(q).$$

Example. For $a = 5$, Theorem 1.4 gives

$$\mathcal{U}_5^*(q) = \frac{1295803}{12262440960} + \frac{35}{294912} \mathbb{E}_2^*(q) - \frac{3229}{967680} \mathbb{E}_4^*(q) + \frac{47}{1152} \mathbb{E}_6^*(q) - \frac{7}{24} \mathbb{E}_8^*(q) + \mathbb{E}_{10}^*(q).$$

The coefficients of $\mathcal{U}_a(q)$ and $U_a^*(q)$ satisfy surprising congruences. Amdeberhan-Andrews-Tauraso [1] discovered some congruences that are reminiscent of Ramanujan's partition congruences, such as

$$MO(2; 5n + 2) \equiv 0 \pmod{5} \quad \text{and} \quad MO(3; 7n + 3) \equiv MO(3; 7n + 5) \equiv 0 \pmod{7}.$$

Moreover, they conjectured (see Conjecture 9.1 of [1]) that

$$(1.13) \quad MO(10; 11n + 7) \equiv 0 \pmod{11}.$$

Theorem 1.5. *For every non-negative integer n , we have that*

$$MO(10; 11n + 7) \equiv 0 \pmod{11}.$$

We offer two proofs of this result. The first proof uses the explicit description of $\mathcal{U}_{10}(q)$ provided by Theorem 1.3, which allows us to employ the “theory of modular forms mod p ”. This proof illustrates an algorithm that reduces the proof of all conjectured congruences of the form

$$MO(a; pn + r) \equiv 0 \pmod{p} \quad \text{and} \quad M(a; pn + r) \equiv 0 \pmod{p}$$

to finitely many steps. Theorem 1.5 requires computing at most 20 terms of five auxiliary q -series.

The second proof is a special case of one of three new infinite families of congruences.

Theorem 1.6. *The following are true:*

(i) *For every pair of non-negative integers n and m , we have that*

$$MO(3m + 2; 3n + 1) \equiv MO(3m + 2; 3n + 2) \equiv 0 \pmod{3}.$$

(ii) *For every pair of non-negative integers n and m , we have*

$$MO(11m + 10; 11n + 7) \equiv 0 \pmod{11}.$$

(iii) *For every pair of non-negative integers n and m , we have*

$$MO(17m + 16; 17n + 15) \equiv 0 \pmod{17}.$$

Computer searches for congruences suggest that such congruences are rare, thereby underscoring the significance of Theorem 1.5. However, it turns out that congruences are both rare and ubiquitous.

Theorem 1.7. *For positive integers a and m , the following are true:*

(i) *There are infinitely many non-nested arithmetic progressions $tn + r$ (resp. $t^*n + r^*$) for which*

$$\begin{aligned} M(a; tn + r) &\equiv 0 \pmod{m}, \\ MO(a; t^*n + r^*) &\equiv 0 \pmod{m}. \end{aligned}$$

(ii) *There are infinitely many non-nested arithmetic progressions $tn + r$ for which*

$$M(a; tn + r) \equiv MO(a; tn + r) \equiv 0 \pmod{m}.$$

(iii) There exists a positive real number $\alpha(a, m) > 0$ for which

$$\begin{aligned} \#\{n \leq X : M(a; n) \not\equiv 0 \pmod{m}\} &= O\left(X / \log^{\alpha(a, m)} X\right) \\ \#\{n \leq X : MO(a; n) \not\equiv 0 \pmod{m}\} &= O\left(X / \log^{\alpha(a, m)X}\right). \end{aligned}$$

In other words, the values $M(a; n)$ and $MO(a; n)$ are almost always multiples of any integer m .

To conclude, we offer infinite families of congruences when $a \in \{2, 3, 4, 5\}$. For convenience, we let

$$N_a := \begin{cases} 2^3 & \text{if } a = 2, \\ 2^7 3 \cdot 5 & \text{if } a = 3, \\ 2^{10} 3^3 \cdot 5 \cdot 7 & \text{if } a = 4, \\ 2^{15} 3^3 5^2 \cdot 7 & \text{if } a = 5. \end{cases}$$

Corollary 1.8. *If $a \in \{2, 3, 4, 5\}$, then the following are true:*

(i) *If $\ell \in \{2, 3, 5, 7\}$ and $p \equiv -1 \pmod{\ell^{\text{ord}_\ell(N_a)+1}}$ is prime, then for every n coprime to p we have*

$$MO(a; pn) \equiv 0 \pmod{\ell}.$$

(ii) *If $\ell \geq 11$ is prime and $p \equiv -1 \pmod{\ell}$, then for every integer n coprime to p we have*

$$MO(a; pn) \equiv 0 \pmod{\ell}.$$

Example. The following congruence is an example of Corollary 1.8 (i) :

$$MO(2, 19^2n + 19) \equiv MO(2, 19^2n + 38) \equiv MO(2, 19^2n + 57) \equiv MO(2, 19^2n + 76) \equiv 0 \pmod{5},$$

As an example of Corollary 1.8 (ii), for $1 \leq t \leq 18$, we have

$$MO(a, 37^2n + 37t) \equiv 0 \pmod{19}.$$

Remark. Most of the congruences in Theorem 1.7 do not belong to infinite families such as those in Corollary 1.8. For instance, if $p \in \{67, 101, 271, 373\}$, then for every non-negative integer n we have

$$M(6; pn) \equiv MO(6; pn) \equiv 0 \pmod{17}.$$

The coefficients of the expansion of $\mathcal{U}_6(q)$ provided by Theorem 1.3 are units modulo 17, and so these congruences follow from the fact that all of the monomials $E_2(q)^\alpha E_4(q)^\beta E_6(q)^\gamma$, with $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 3\gamma \leq 6$, are annihilated modulo 17 by the Hecke operators T_p for $p \in \{67, 101, 271, 373\}$.

This paper is organized as follows. In Section 2, we recall the Nekrasov-Okounkov hook formulae and relevant results of Andrews-Rose and Amdeberhan-Andrews-Tauraso, which we then employ to prove Theorems 1.1 and 1.2 on the limiting behavior of $U_a(q)$ and $U_a^*(q)$. In Section 3 we recall pertinent facts about symmetric functions, as well as results on the quasimodularity of $U_a(q)$, which we then use to prove Theorems 1.3 and 1.4. Finally, in Section 4 we prove Theorems 1.5 and 1.6, and in Section 5 we prove Theorem 1.7 using modularity.

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2. PROOFS OF THEOREMS 1.1 AND 1.2

Here we prove Theorems 1.1 and 1.2 using earlier work of Nekrasov-Okounkov and Andrews-Rose.

2.1. Proof of Theorem 1.1. We require a beautiful identity of Andrews-Rose for $\mathcal{U}_a(q)$.

Lemma 2.1. [2, Cor. 2] *If a is a positive integer, then as formal power series we have that*

$$\mathcal{U}_a(q) \cdot \prod_{n \geq 1} (1 - q^n)^3 = \frac{(-1)^a}{(2a+1)!} \sum_{n \geq 0} (-1)^n (2n+1) \frac{(n+a)!}{(n-a)!} q^{\frac{n(n+1)}{2}}.$$

We also require the celebrated Nekrasov-Okounkov hook length identity (see (6.12) on page 569 of [8]; see also Th. 1.3 of [5]).

Theorem 2.2. *As a formal power series, we have*

$$\prod_{j \geq 1} \frac{1}{(1 - q^j)^{z+1}} = \sum_{m \geq 0} q^m \sum_{\lambda \vdash m} \prod_{h \in \mathcal{H}(\lambda)} \left(\frac{z}{h^2} + 1 \right).$$

Proof of Theorem 1.1. Thanks to Lemma 2.1 for $\mathcal{U}_a(q)$, we find that

$$\begin{aligned} q^{-\binom{a+1}{2}} \cdot \mathcal{U}_a(q) \cdot \prod_{n \geq 1} (1 - q^n)^3 &= \sum_{j \geq 0} (-1)^j \cdot \frac{2j+2a+1}{2a+1} \cdot \binom{j+2a}{j} q^{aj + \binom{j+1}{2}} \\ &= 1 - (2a+3)q^{a+1} + (a+1)(2a+5)q^{2a+3} + \dots \end{aligned}$$

Claim (i) follows immediately.

The first formula in (ii) follows by letting $z = 2$ in Theorem 2.2, giving

$$\prod_{n \geq 1} \frac{1}{(1 - q^n)^3} = \sum_{m \geq 0} q^m \sum_{\lambda \vdash m} \prod_{h \in \mathcal{H}(\lambda)} \left(\frac{2}{h^2} + 1 \right),$$

while the other claim arises from the interpretation of the q -product in terms of 3-colored partitions. \square

2.2. Proof of Theorem 1.2. Amdeberhan-Andrews-Tauraso express $\mathcal{U}_a^*(q)$ as a single sum.

Lemma 2.3. [1, Prop. 4.1] *We have the identity*

$$\mathcal{U}_a^*(q) = \sum_{k \geq 1} (-1)^{k-1} \frac{(1+q^k) q^{\binom{k}{2} + ak}}{(1-q^k)^{2a}}.$$

Proof of Theorem 1.2. The expansion $(1 - q^k)^{-2a} = \sum_{m \geq 0} \binom{2a+m-1}{m} q^{km}$ and Lemma 2.3 imply that

$$\begin{aligned} q^{-a} \cdot \mathcal{U}_a^*(q) &= \sum_{k \geq 1} (-1)^{k-1} \frac{q^{\binom{k}{2} + a(k-1)}}{(1 - q^k)^{2a}} + \sum_{k \geq 1} (-1)^{k-1} \frac{q^{\binom{k+1}{2} + a(k-1)}}{(1 - q^k)^{2a}} \\ &= \sum_{k \geq 1} \sum_{m \geq 0} (-1)^{k-1} \binom{2a+m-1}{m} q^{km + \binom{k}{2} + a(k-1)} + \sum_{k \geq 1} \sum_{m \geq 0} \binom{2a+m-1}{m} q^{km + \binom{k+1}{2} + a(k-1)}. \end{aligned}$$

We compare coefficients of q^n for $n \leq a$. Namely, in the double sums we require $km + \binom{k}{2} + a(k-1) \leq a$ and $km + \binom{k+1}{2} + a(k-1) \leq a$. The former results in $k = 1, m = n$ and the latter forces $k = 1, m = n - 1$. Consequently, if we let $q^{-a} \cdot \mathcal{U}_a^*(q) =: \sum_{n \geq 0} p_n(a) q^n$, then we find that

$$p_n(a) = \binom{2a+n-1}{n} + \binom{2a+n-2}{n-1}.$$

□

3. PROOF OF THEOREMS 1.3 AND 1.4

Here we prove the explicit descriptions of $\mathcal{U}_a(q)$ and $\mathcal{U}_a^*(q)$ in terms of Eisenstein series.

3.1. Nuts and Bolts. We make use of the differential operator $\Theta := q \frac{d}{dq}$, which acts by

$$(3.1) \quad \Theta \left(\sum a(n) q^n \right) := \sum n a(n) q^n.$$

Ramanujan famously obtained the following formulas [11, p. 181] for the action of Θ :

$$(3.2) \quad \begin{aligned} \Theta(E_2(q)) &= \frac{E_2^2(q) - E_4(q)}{12}, & \Theta(E_4(q)) &= \frac{E_2(q)E_4(q) - E_6(q)}{3}, \\ \Theta(E_6(q)) &= \frac{E_2(q)E_6(q) - E_4^2(q)}{2}. \end{aligned}$$

The q -series $\mathcal{U}_a(q)$ and $\mathcal{U}_a^*(q)$ satisfy the following convenient convolution (see [1, p. 13]).

Lemma 3.1. *If a is a positive integer, then we have that*

$$\sum_{i=0}^a (-1)^i \cdot \mathcal{U}_i(q) \cdot \mathcal{U}_{a-i}^*(q) = 0.$$

Recall the Dedekind eta-function $\eta(q) = q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m)$. The following result of Rose [12, Th. 1.12] describes the structural framework of $\mathcal{U}_a(q)$ in terms of iterated derivatives of $\eta(q)^3$.

Theorem 3.2. *Each $\mathcal{U}_a(q)$ is a finite sum of quasimodular forms with weight $\leq 2a$ on $\mathrm{SL}_2(\mathbb{Z})$. Moreover, the weight $2t$ summand is a (possibly zero) scalar multiple of*

$$2^t \cdot \frac{\Theta^t(\eta(q)^3)}{\eta(q)^3}.$$

Our next result expresses these q -series as a linear combination of monomials $E_2(q)^\alpha E_4(q)^\beta E_6(q)^\gamma$.

Lemma 3.3. *If t is a positive integer, then we have that*

$$(-8)^t \cdot \frac{\Theta^t(\eta(q)^3)}{\eta(q)^3} = \sum_{\substack{\alpha, \beta, \gamma \geq 0 \\ \alpha + 2\beta + 3\gamma = t}} c(\alpha, \beta, \gamma) \cdot E_2(q)^\alpha E_4(q)^\beta E_6(q)^\gamma$$

where the coefficients $c(\alpha, \beta, \gamma)$ are defined by (1.7).

Proof. For convenience, we let $\psi(q) := \eta(q)^3$. We calculate $\frac{\Theta^t(\psi(q))}{\psi(q)}$ by inducting on t . First, it is easy to check $\Theta(\psi(q)) = \frac{1}{8}\psi(q)E_2(q)$. Theorem 3.2 implies the existence of numbers $\tilde{c}(\alpha, \beta, \gamma)$ for which

$$\frac{\Theta^t(\psi(q))}{\psi(q)} = \sum_{\substack{\alpha, \beta, \gamma \geq 0 \\ \alpha + 2\beta + 3\gamma = t}} \tilde{c}(\alpha, \beta, \gamma) \cdot E_2^\alpha(q)E_4^\beta(q)E_6^\gamma(q).$$

This comprises of all weight $2t$ quasimodular summands in $\mathcal{U}_a(q)$. One more derivative $\Theta = q\frac{d}{dq}$ turns the last equation into (for brevity, we write \tilde{c} in place of $\tilde{c}(\alpha, \beta, \gamma)$)

$$\Theta^{t+1}(\psi(q)) = \Theta(\psi(q)) \cdot \left(\sum_{\alpha, \beta, \gamma} \tilde{c} \cdot E_2^\alpha(q)E_4^\beta(q)E_6^\gamma(q) \right) + \psi(q) \cdot \sum_{\alpha, \beta, \gamma} \tilde{c} \cdot \Theta(E_2^\alpha(q)E_4^\beta(q)E_6^\gamma(q)).$$

On the other hand, Ramanujan's identities (3.2) imply that

$$\Theta(E_2^\alpha E_4^\beta E_6^\gamma) = \left(\frac{\alpha}{12} + \frac{\beta}{3} + \frac{\gamma}{2} \right) E_2^{\alpha+1} E_4^\beta E_6^\gamma - \frac{\alpha}{12} E_2^{\alpha-1} E_4^{\beta+1} E_6^\gamma - \frac{\beta}{3} E_2^\alpha E_4^{\beta-1} E_6^{\gamma+1} - \frac{\gamma}{2} E_2^\alpha E_4^{\beta+2} E_6^{\gamma-1}.$$

We find that the homogeneous weight $2t + 2$ form satisfies

$$\begin{aligned} \frac{\Theta^{t+1}(\psi(q))}{\psi(q)} &= \sum_{\substack{\alpha, \beta, \gamma \geq 0 \\ \alpha + 2\beta + 3\gamma = t}} \left(\frac{\alpha}{12} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{1}{8} \right) \tilde{c} \cdot E_2^{\alpha+1} E_4^\beta E_6^\gamma - \sum_{\alpha, \beta, \gamma} \frac{\alpha}{12} \tilde{c} \cdot E_2^{\alpha-1} E_4^{\beta+1} E_6^\gamma \\ &\quad - \sum_{\alpha, \beta, \gamma} \frac{\beta}{3} \tilde{c} \cdot E_2^\alpha E_4^{\beta-1} E_6^{\gamma+1} - \sum_{\alpha, \beta, \gamma} \frac{\gamma}{2} \tilde{c} \cdot E_2^\alpha E_4^{\beta+2} E_6^{\gamma-1}. \end{aligned}$$

By comparing the coefficients of $E_2^\alpha E_4^\beta E_6^\gamma$ on both sides of the equation above, we obtain the recursion (with $\tilde{c}(\alpha, \beta, \gamma) = \delta_{(0,0,0)}(\alpha, \beta, \gamma)$, a Dirac delta boundary conditions)

$$\begin{aligned} \tilde{c}(\alpha, \beta, \gamma) &= \left(\frac{\alpha}{12} + \frac{\beta}{3} + \frac{\gamma}{2} + \frac{1}{24} \right) \tilde{c}(\alpha - 1, \beta, \gamma) - \frac{\alpha + 1}{12} \cdot \tilde{c}(\alpha + 1, \beta - 1, \gamma) \\ &\quad - \frac{\beta + 1}{3} \cdot \tilde{c}(\alpha, \beta + 1, \gamma - 1) - \frac{\gamma + 1}{2} \cdot \tilde{c}(\alpha, \beta - 2, \gamma + 1). \end{aligned}$$

To determine the exact weight $2t$ term (independent of a), we take into account the factor of $(-8)^{\alpha+2\beta+3\gamma}$ to determine $c(\alpha, \beta, \gamma) := (-8)^{\alpha+2\beta+3\gamma} \cdot \tilde{c}(\alpha, \beta, \gamma)$. As a result, we obtain the desired

$$\begin{aligned} c(\alpha, \beta, \gamma) &= -\frac{1}{3} (2\alpha + 8\beta + 12\gamma + 1) \cdot c(\alpha - 1, \beta, \gamma) + \frac{2}{3} (\alpha + 1) \cdot c(\alpha + 1, \beta - 1, \gamma) \\ &\quad + \frac{8}{3} (\beta + 1) \cdot c(\alpha, \beta + 1, \gamma - 1) + 4(\gamma + 1) \cdot c(\alpha, \beta - 2, \gamma + 1). \end{aligned}$$

□

3.2. Proof of Theorem 1.3. We let $\mathcal{E}_t(q) := (-8)^t \cdot \frac{\Theta^t(\psi)}{\psi}$, and we define

$$\mathbb{E}_{2t}(q) := \sum_{(1^{m_1}, \dots, t^{m_t}) \vdash t} \prod_{j=1}^t \frac{1}{m_j!} \left(\frac{B_{2j} E_{2j}(q)}{(2j) \cdot (2j)!} \right)^{m_j}.$$

By inspection, we see that $\mathbb{E}_{2t}(q)$ has weight $2t$. We claim that

$$(3.3) \quad \mathbb{E}_{2t}(q) = \frac{(-1)^t}{4^t(2t+1)!} \cdot \mathcal{E}_t(q).$$

Let $\mathbf{S}_r(q) := \sum_{m \geq 1} \frac{m^r q^m}{1-q^m} = \sum_{n \geq 1} \sigma_r(n) q^n$. By expanding $\sum_{j,k \geq 1} \frac{q^{kj} \cos(2kx)}{k}$ in two different ways, we find that it equals both of these

$$(3.4) \quad \prod_{j \geq 1} \left[1 + \frac{4(\sin^2 x) q^j}{(1-q^j)^2} \right] = \exp \left(-2 \sum_{r \geq 1} \frac{\mathbf{S}_{2r-1}(q)}{(2r)!} (-4x^2)^r \right).$$

Using the identity [1, p. 13], we obtain

$$(3.5) \quad \prod_{k \geq 1} \left(1 + \frac{4q^k \sin^2 x}{(1-q^k)^2} \right) = \sum_{a \geq 0} 4^a \mathcal{U}_a(q) (\sin x)^{2a},$$

and the Jacobi Triple Product then implies that

$$(3.6) \quad \begin{aligned} \sin x \prod_{k \geq 1} \left(1 + \frac{4q^k \sin^2 x}{(1-q^k)^2} \right) &= \frac{e^{ix} - e^{-ix}}{2i} \prod_{j \geq 1} \frac{(1-q^j e^{2ix})(1-q^j e^{-2ix})}{(1-q^j)^2} \\ &= \frac{1}{2i \cdot \psi(q)} \sum_{j \in \mathbb{Z}} (-1)^j q^{\binom{j+1}{2}} e^{(2n+1)ix} \\ &= \frac{1}{\psi(q)} \sum_{t \geq 0} (-1)^t \frac{x^{2t+1}}{(2t+1)!} \sum_{n \geq 0} (-1)^n (2n+1)^{2t+1} q^{\binom{n+1}{2}} \\ &= \sum_{t \geq 0} \mathcal{E}_t(q) \frac{x^{2t+1}}{(2t+1)!}. \end{aligned}$$

Using (1.9) and the generating function for Pólya's *cycle index formula* [10, (1,5)], we obtain

$$(3.7) \quad \sin x \cdot \exp \left(-2 \sum_{r \geq 1} \frac{\mathbf{S}_{2r-1}(q)}{(2r)!} (-4x^2)^{2r} \right) = \sin x \cdot \frac{x}{\sin x} \cdot \sum_{t \geq 0} \left(\sum_{\lambda \vdash t} \prod_{j=1}^t \frac{1}{m_j!} \left(\frac{B_{2j} \cdot E_{2s}(q)}{(2j) \cdot (2j)!} \right)^{m_j} \right) (-4x^2)^t.$$

Combining (3.4), (3.6), (3.7) and then comparing the coefficients of x^{2t+1} , we confirm (3.3).

To the complete the proof, it suffices to determine the constants $b_t(a)$ for which

$$(3.8) \quad \mathcal{U}_a(q) = \sum_{t=0}^a b_t(a) \cdot \mathbb{E}_{2t}(q).$$

It is convenient to recall the Andrews-Rose recursion [2, Cor. 3]

$$(3.9) \quad \mathcal{U}_a(q) = \frac{1}{2a(2a+1)} [(6\mathcal{U}_1(q) + a(a-1))\mathcal{U}_{a-1}(q) - 2\Theta(\mathcal{U}_{a-1}(q))].$$

The structure of equation (3.8) is preserved by (3.9) because of the identity

$$\Theta(\mathbb{E}_{2t-2}) = t(2t+1)\mathbb{E}_{2t} - 3\mathbb{E}_2\mathbb{E}_{2t-2}.$$

It is straightforward to see that

$$b_t(a) = \frac{1}{8a(2a+1)} \left[(2a-1)^2 \cdot b_t(a-1) - 8t(2t+1) \cdot b_{t-1}(a-1) \right],$$

with initial boundary conditions $b_0(0) = 1$ and $b_t(a) = 0$ when $t < 0$ or $t > a$. Finally, one checks that $(-4)^t(2t+1)!w_t(a)$ satisfies this recurrence, thereby completing the proof of the theorem.

3.3. Proof of Theorem 1.4. By reciprocating (3.5), we have

$$\sum_{n \geq 0} (-4)^n \mathcal{U}_a^*(q) (\sin x)^{2a} = \prod_{k \geq 1} \frac{1}{1 + \frac{4q^k \sin^2 x}{(1-q^k)^2}}.$$

In analogy with the previous formula for $\mathbb{E}_{2t}(q)$ involving the $\mathcal{U}_a(q)$, we use (3.3) to obtain an identity for $\mathcal{U}_a^*(q)$ with $\mathbb{E}_{2t}^*(q)$ (see (1.10)). Arguing as in the proof of Theorem 1.3 with Lemma 3.1, we get

$$\mathcal{U}_a^*(q) = \sum_{t=0}^a w_t^*(a) \cdot \mathbb{E}_{2t}^*(q).$$

4. PROOF OF THEOREMS 1.5 AND 1.6

Here we prove Theorem 1.5 using Serre's theory of modular forms modulo primes p (see [9, Section 2.8], or [17]) and a well-known criterion of Sturm that determines congruences between modular forms. In the sequel, we tacitly assume that $q := e^{2\pi iz}$, the uniformizer for the point at infinity. We also prove Theorem 1.6 by combining work of Andrews-Rose with a classical result of Gordon, together with other allied observations.

4.1. Modular forms modulo p . We recall some facts from the theory of modular forms mod p . The key tool in the proof of Theorem 1.5 is the following theorem of Sturm (see [16] or p. 40 of [9]).

Theorem 4.1. *Let p be a prime. If $f(z) = \sum_{n \geq 0} a(n)q^n$ and $g(z) = \sum_{n \geq 0} b(n)q^n$ are modular forms of weight k on $\mathrm{SL}_2(\mathbb{Z})$ with integer coefficients, then $f(z) \equiv g(z) \pmod{p}$ if and only if $a(n) \equiv b(n) \pmod{p}$ for all $n \leq k/12$.*

We shall make use of derivatives of modular forms. Although differentiation does not preserve modularity, it does preserve modular forms modulo p (for example, see [13]).

Lemma 4.2. *If $f(z) = \sum_{n \geq 0} a(n)q^n \in M_k \cap \mathbb{Z}[[q]]$, then there is a modular form $g(z) = \sum_{n \geq 0} b(n)q^n \in M_{k+p+1} \cap \mathbb{Z}[[q]]$ for which*

$$g \equiv \Theta(f) := \sum_{n \geq 0} na(n)q^n \pmod{p}.$$

4.2. Proof of Theorem 1.5. We let $\mathcal{U}_{10}(q) = F_0(q) + F_2(q) + F_4(q) + F_6(q) + F_8(q)$, where each $F_{2i}(q)$ is a sum of $E_2^\alpha E_4^\beta E_6^\gamma$ (suppressing the q), where $2\alpha + 4\beta + 6\gamma \equiv 2i \pmod{10}$. Theorem 1.3

then gives

$$\begin{aligned}
F_0(q) &= \frac{46189}{5772436045824} - \frac{2008213E_4E_6}{4271802792542208000} + \cdots + \frac{E_2^{10}}{230078188847156428800}, \\
F_2(q) &= -\frac{25587296781661E_2}{2645567198945303592960} - \frac{604841E_6^2}{48057781416099840000} + \cdots + \frac{7862933E_2^6}{63910608013099008000}, \\
F_4(q) &= -\frac{79923511502753E_4}{67133754108574433280000} + \frac{79923511502753E_2^2}{26853501643429773312000} + \cdots - \frac{16333E_2^7}{4473742560916930560}, \\
F_6(q) &= -\frac{70726885883E_6}{333200617818292224000} + \frac{70726885883E_2E_4}{126933568692682752000} - \cdots + \frac{1819E_2^8}{25564243205239603200}, \\
F_8(q) &= -\frac{316100258731E_4^2}{20732482886471516160000} + \frac{316100258731E_2E_6}{3887340541213409280000} - \cdots - \frac{19E_2^9}{23007818884715642880}.
\end{aligned}$$

Each of these q -series is 11-integral, and so they may be reduced modulo 11 to obtain

$$\begin{aligned}
\widehat{F}_0(q) &:= F_0(q) \pmod{11} \equiv 2q^3 + 6q^4 + 7q^5 + 8q^6 + 5q^7 + 2q^8 + 2q^9 + \dots \pmod{11}, \\
\widehat{F}_2(q) &:= F_2(q) \pmod{11} \equiv 6q^3 + 7q^4 + 10q^5 + 7q^6 + 8q^7 + 7q^8 + 6q^9 + \dots \pmod{11}, \\
\widehat{F}_4(q) &:= F_4(q) \pmod{11} \equiv 7q^3 + 10q^4 + 8q^5 + 2q^6 + 4q^8 + 7q^9 + \dots \pmod{11}, \\
\widehat{F}_6(q) &:= F_6(q) \pmod{11} \equiv 10q^3 + 8q^4 + 9q^5 + 10q^6 + q^7 + 4q^8 + 10q^9 + \dots \pmod{11}, \\
\widehat{F}_8(q) &:= F_8(q) \pmod{11} \equiv 8q^3 + 2q^4 + 6q^5 + 6q^6 + 8q^7 + 5q^8 + 8q^9 + \dots \pmod{11}.
\end{aligned}$$

Using the congruences $E_2(q) \equiv E_{12}(q) \pmod{11}$ and $E_{10}(q) \equiv 1 \pmod{11}$, we observe that $\widehat{F}_0(q)$, $\widehat{F}_2(q)$, $\widehat{F}_4(q)$, $\widehat{F}_6(q)$, and $\widehat{F}_8(q)$ are modular forms modulo 11 of weight 120, 72, 84, 96, and 108, respectively, on $\mathrm{SL}_2(\mathbb{Z})$.

We proceed to isolate the arithmetic progression of coefficients that is relevant for the theorem. We apply the differential operators to $\mathcal{U}_{10}(q)$ to eliminate terms with exponents $n \equiv 0, 1, 3, 4, 5, 9 \pmod{11}$. The non-zero classes are the quadratic residues modulo 11. Using Fermat's Little Theorem and Euler's Criterion, this is achieved by

$$G_1(q) := \sum_{n \equiv 2, 6, 7, 8, 10 \pmod{11}} MO(10; n)q^n \equiv \sum_{i=0}^4 -5[\Theta^{10}(\widehat{F}_{2i}(q)) - \Theta^5(\widehat{F}_{2i}(q))] \pmod{11}.$$

Next, we proceed to remove the terms with exponents that are quadratic non-residues apart from those with $n \equiv 7 \pmod{11}$. For instance, to eliminate $n \equiv 2 \pmod{11}$ from $G_1(q)$ compute

$$G_2(q) := \Theta(G_1(q)) - 2G_1(q) \equiv \sum_{n \equiv 6, 7, 8, 10 \pmod{11}} MO(10; n)q^n \pmod{11}.$$

We repeat this process to remove terms with exponents $n \equiv 6, 8, 10 \pmod{11}$, and we get

$$\begin{aligned}
& \sum_{n \equiv 7 \pmod{11}} MO(10; n)q^n \pmod{11} \\
& \equiv -5(\Theta^4(\widehat{F}_2) - \Theta^9(\widehat{F}_2)) + 9(\Theta^3(\widehat{F}_4) - \Theta^8(\widehat{F}_4)) - 3(\Theta^2(\widehat{F}_6) - \Theta^7(\widehat{F}_6)) + \Theta(\widehat{F}_8) \\
& \quad - \Theta^6(\widehat{F}_8) - 4(\widehat{F}_0 - \Theta^5(\widehat{F}_0)) - 5(\Theta^4(\widehat{F}_4) - \Theta^9(\widehat{F}_4)) + 9(\Theta^3(\widehat{F}_6) - \Theta^8(\widehat{F}_6)) \\
& \quad - 3(\Theta^2(\widehat{F}_8) - \Theta^7(\widehat{F}_8)) + \Theta(\widehat{F}_0) - \Theta^6(\widehat{F}_0) - 4(\widehat{F}_2 - \Theta^5(\widehat{F}_2)) - 5(\Theta^4(\widehat{F}_6) - \Theta^9(\widehat{F}_6)) \\
& \quad + 9(\Theta^3(\widehat{F}_8) - \Theta^8(\widehat{F}_8)) - 3(\Theta^2(\widehat{F}_0) - \Theta^7(\widehat{F}_0)) + \Theta(\widehat{F}_2) - \Theta^6(\widehat{F}_2) - 4(\widehat{F}_4 - \Theta^5(\widehat{F}_4)) \\
& \quad - 5(\Theta^4(\widehat{F}_8) - \Theta^9(\widehat{F}_8)) + 9(\Theta^3(\widehat{F}_0) - \Theta^8(\widehat{F}_0)) - 3(\Theta^2(\widehat{F}_2) - \Theta^7(\widehat{F}_2)) + \Theta(\widehat{F}_4) \\
& \quad - \Theta^6(\widehat{F}_4)4(\widehat{F}_6 - \Theta^5(\widehat{F}_6)) - 5(\Theta^4(\widehat{F}_0) - \Theta^9(\widehat{F}_0)) + 9(\Theta^3(\widehat{F}_2) - \Theta^8(\widehat{F}_2)) \\
& \quad - 3(\Theta^2(\widehat{F}_4) - \Theta^7(\widehat{F}_4)) + \Theta(\widehat{F}_6) - \Theta^6(\widehat{F}_6) - 4(\widehat{F}_8 - \Theta^5(\widehat{F}_8)).
\end{aligned}$$

Now we collect these terms so that

$$\sum_{n \equiv 7 \pmod{11}} MO(10; n)q^n \pmod{11} = Y_0(q) + Y_2(q) + Y_4(q) + Y_6(q) + Y_8(q),$$

where $Y_{2i}(q)$ consists of those $\Theta^t(\widehat{F}_j(q))$ with weight congruent to i modulo 10. By Lemma 4.2, we find that $Y_0(q), Y_2(q), Y_4(q), Y_6(q)$, and $Y_8(q)$ are modular forms modulo 11 with weights 228, 180, 192, 204, and 216, respectively on $\mathrm{SL}_2(\mathbb{Z})$. Finally, by Sturm's Theorem 4.1 (i.e. checking at most 20 terms), we find that each of these modular forms vanishes modulo 11, which implies the theorem.

4.3. Proof of Theorem 1.6. The generating function for the 3-colored partition function satisfies

$$P_3(q) = \sum_{n \geq 0} c_3(n)q^n = \prod_{n \geq 1} \frac{1}{(1 - q^n)^3} \equiv \sum_{n \geq 0} p(n)q^{3n} \pmod{3}.$$

Therefore, we have that $3 \mid c_3(n)$ whenever $3 \nmid n$. Furthermore, Gordon [4] proved that

$$c_3(11n + 7) \equiv 0 \pmod{11}.$$

We further claim that $c_3(17n + 15) \equiv 0 \pmod{17}$. To prove this congruence, we employ Ramanujan's weight 12 cusp form $\Delta(z) := \eta(z)^{24}$ through the following observation:

$$q^2 \sum_{n \geq 0} c_3(n)q^n \cdot \prod_{n \geq 1} (1 - q^{17n})^3 \equiv q^2 \prod_{n \geq 1} (1 - q^n)^{48} \pmod{17} = \Delta(z)^2.$$

One easily checks that $\Delta^2 \mid T_{17} \equiv 0 \pmod{17}$, which means that every seventeenth coefficient of $\Delta(z)^2$ vanishes modulo 17. This congruence follows immediately from the fact that

$$\prod_{n \geq 1} (1 - q^{17n})^3 \in (\mathbb{Z}/17\mathbb{Z})[q^{17}].$$

Now suppose that $\ell \in \{3, 11, 17\}$ and $1 \leq a \equiv \ell - 1 \pmod{\ell}$. If $\ell \nmid n(n+1)/2$, then we have

$$\frac{(2n+1)(n+a)!}{(2a+1)!(n-a)!} \equiv 0 \pmod{\ell}.$$

Therefore, the identity in Lemma 2.1 collapses modulo ℓ and gives

$$\mathcal{U}_a(q) = \sum_{n \geq 0} MO(a; n)q^n \equiv P_3(q) \cdot \sum_{n \geq 0} A(a, \ell; \ell n)q^{\ell n} \pmod{\ell}.$$

The power series on the right is in $(\mathbb{Z}/\ell\mathbb{Z})[q^\ell]$, and so the $MO(a; n)$ inherit the $c_3(n)$ congruences.

5. PROOF OF THEOREM 1.7

Here we prove Theorem 1.7 and Corollary 1.8.

5.1. Nuts and Bolts. We first recall the definition of Hecke operators. Let m be a positive integer and $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k$. Then the action of Hecke operator T_m on $f(z)$ is defined by

$$(5.1) \quad f(z) | T_m := \sum_{n=0}^{\infty} \left(\sum_{d|\gcd(n,m)} d^{k-1} a\left(\frac{nm}{d^2}\right) \right) q^n.$$

In particular, if $m = p$ is a prime, we have

$$(5.2) \quad f(z) | T_p := f(z) | U_p + p^{k-1} f(z) | V_p,$$

where $f(z) | U_p := \sum_{n=0}^{\infty} a(pn)q^n$ and $f(z) | V_p := \sum_{n=0}^{\infty} a(n)q^{pn}$.

Let's recall a result of Serre [14] (also see [9, Lemma 2.63 and Theorem 2.65]) on the action of Hecke operator on cusp forms. For a number field K , let \mathcal{O}_K denote its ring of integers.

Lemma 5.1. *For $1 \leq i \leq t$, let $f_i(z) = \sum_{n=1}^{\infty} a_i(n)q^n \in M_k$ be a modular form with coefficients in the ring of integers of a number field \mathcal{O}_K . Then the following are true.*

(i) *If $\mathfrak{m} \subset \mathcal{O}_K$ is an ideal of norm M , then a positive proportion of the primes $p \equiv -1 \pmod{M}$ satisfy*

$$f_1(z) | T_p \equiv f_2(z) | T_p \equiv \cdots \equiv f_t(z) | T_p \equiv 0 \pmod{\mathfrak{m}}.$$

(ii) *There is a constant $a > 0$ such that for every $1 \leq i \leq t$ we have*

$$\#\{n \leq X : a_i(n) \not\equiv 0 \pmod{\mathfrak{m}}\} = O(X/(\log X)^a).$$

We next recall some facts about p -adic modular forms developed by Serre [15]. Let p be a prime. Consider the field of p -adic numbers \mathbb{Q}_p , with its non-archimedean valuation ν_p . We say $x \in \mathbb{Q}_p$ is p -integral if $\nu_p(x) \geq 0$. Let $f = \sum a(n)q^n \in \mathbb{Q}_p[[q]]$ be a formal power series, we define $\nu_p(f) := \inf_n \nu_p(a_n)$. If $\nu_p(f) \geq m$, we write as well $f \equiv 0 \pmod{p^m}$. Assume $\{f_i\}$ to be a sequence of elements in $\mathbb{Q}_p[[q]]$. We say that $f_i \rightarrow f$ if the coefficients of f_i tend uniformly to those of f , i.e., $\nu_p(f - f_i) \rightarrow \infty$. A p -adic modular form f is a formal series with coefficients in \mathbb{Q}_p which is the limit of classical modular forms f_i of weights k_i .

In order to prove Theorem 1.7, we need the following preliminary result.

Lemma 5.2. *The following are true:*

(i) *If m is a positive integer, then we have that*

$$E_2(z) \equiv \frac{1}{(2^m - 1)} \sum_{i=1}^m 2^{i-1} E_{2+3 \cdot 2^{m+1}}(z) | V_{2^{i-1}} \pmod{2^m}.$$

Moreover, $E_2(z) \pmod{2^m}$ is the reduction of a weight $2 + 3 \cdot 2^{m+1}$ modular form on $\mathrm{SL}_2(\mathbb{Z})$.

(ii) *If m is a positive integer, then we have that*

$$E_2(z) \equiv \frac{2}{(3^m - 1)} \sum_{i=1}^m 3^{i-1} E_{2+4 \cdot 3^m}(z) | V_{3^{i-1}} \pmod{3^m}.$$

Moreover, $E_2(z) \pmod{3^m}$ is the reduction of a weight $2 + 4 \cdot 3^m$ modular form on $\mathrm{SL}_2(\mathbb{Z})$.
 (iii) If $p \geq 5$ is prime and m is a positive integer, then we have that

$$E_2(z) \equiv \frac{(p-1)}{(p^m-1)} \sum_{i=1}^m p^{i-1} E_{2+(p-1)p^{m-1}}(z) | V_{p^{m-1}} \pmod{p^m}.$$

In particular, $E_2(z) \pmod{p^m}$ is the reduction of a weight $2 + p^{m-1}(p-1)$ modular form on $\mathrm{SL}_2(\mathbb{Z})$.

Proof. Let $g(z) = \sum_{n=0}^{\infty} b(n)q^n$ be a weight k modular form on $\mathrm{SL}_2(\mathbb{Z})$ with p -integral coefficients. Then $g(z) | T_p = \sum_{n=0}^{\infty} b(pn)q^n + p^{k-1}b(n)q^{pn} \in M_k$. Since $E_{p^{m-1}(p-1)}(z) \equiv 1 \pmod{p^m}$, we have that $\{g(z)E_{p^{m-1}(p-1)}(z)\}$ converges to $g(z)$ p -adically. Hence, $g(z)$ is a p -adic modular form. Also, we have the convergence

$$g(z)E_{p^{m-1}(p-1)}(z) | T_p \longrightarrow g(z) | U_p = \sum_{n=0}^{\infty} b(pn)q^n.$$

Hence, U_p is an operator on M_k and so is V_p , defined as $g(z) | V_p = p^{1-k}(g(z) | T_p - g(z) | U_p)$. Now, our proof of the lemma follows from [15, Example on Page 210]. \square

5.2. Proof of Theorem 1.7. By Theorem 1.3 and 1.4, we have

$$\begin{aligned} \mathcal{U}_a(q) &= \sum_{t=0}^a w_t(a) \sum_{\substack{\alpha, \beta, \gamma \geq 0 \\ \alpha + 2\beta + 3\gamma = t}} c(\alpha, \beta, \gamma) E_2(q)^\alpha E_4(q)^\beta E_6(q)^\gamma = F_0(q) + F_2(q) + \cdots + F_{2a}(q), \\ (5.3) \quad \mathcal{U}_a^*(q) &= \sum_{t=0}^a w_t^*(a) \cdot \mathbb{E}_{2t}^*(q) = F_0^*(q) + F_2^*(q) + \cdots + F_{2a}^*(q), \end{aligned}$$

where $F_{2i}(q)$ and $F_{2i}^*(q)$, for $0 \leq i \leq a$, are quasimodular forms of weight $2i$ on $\mathrm{SL}_2(\mathbb{Z})$. Using Lemma 5.2 and the Chinese Remainder Theorem, we find that $F_{2i}(q)$ and $F_{2i}^*(q)$, for $0 \leq i \leq a$, are modular forms modulo any integer m on $\mathrm{SL}_2(\mathbb{Z})$. Employing Lemma 5.1 (i) on (5.3), we complete the proof of first and second parts of Theorem 1.7 and finally applying Lemma 5.1 (ii) to (5.3), claim (iii) follows.

5.3. Proof of Corollary 1.8. By Theorem 1.3, we find that

$$\begin{aligned} \mathcal{U}_2(q) &= \frac{1}{2^3} \sum_{n \geq 0} [(-2n+1)\sigma_1(n) + \sigma_3(n)]q^n, \\ \mathcal{U}_3(q) &= \frac{1}{2^7 \cdot 3 \cdot 5} \sum_{n \geq 0} [(40n^2 - 100n + 37)\sigma_1(n) + (-30n + 50)\sigma_3(n) + 3\sigma_5(n)]q^n, \\ \mathcal{U}_4(q) &= \frac{1}{2^{10} 3^3 \cdot 5 \cdot 7} \sum_{n \geq 0} [(-840n^3 + 5880n^2 - 9870n + 3229)\sigma_1 \\ &\quad + (756n^2 - 4410n + 4935)\sigma_3 + (-126n + 441)\sigma_5 + 5\sigma_7]q^n, \\ \mathcal{U}_5(q) &= \frac{1}{2^{15} 3^3 5^2 \cdot 7} \sum_{n \geq 0} [(3360n^4 - 50400n^3 + 223440n^2 - 314200n + 96111)\sigma_1(n) \\ &\quad + (-3360n^3 + 45360n^2 - 167580n + 157100)\sigma_3(n) \\ &\quad + (720n^2 - 7560n + 16758)\sigma_5(n) + (-50n + 300)\sigma_7(n) + \sigma_9(n)]q^n. \end{aligned}$$

Let s and t be non-negative integers. If k is a positive odd, then for primes $p \equiv -1 \pmod{s}$ we have

$$(5.4) \quad (pn)^t \sigma_k(pn) = (pn)^t \sigma_k(n) \sigma_k(p) = (pn)^t \sigma_k(n) (1 + p^k) \equiv 0 \pmod{s}$$

for all n coprime to p . Corollary 1.8 follows by applying (5.4) appropriately in each case.

REFERENCES

- [1] T. Amdeberhan, G. E. Andrews, R. Tauraso, *Extensions of MacMahon's sums of divisors*, preprint available at <https://arxiv.org/pdf/2309.03191.pdf> (2023).
- [2] G. E. Andrews, S. C. F. Rose, *MacMahon's sum-of-divisors functions, Chebyshev polynomials, and quasimodular forms*, J. Reine Angew. Math. **676** (2013), 97–103.
- [3] B. Brindle, *A unified approach to qMZVs*, preprint available at <https://arxiv.org/pdf/2111.00051.pdf> (2021).
- [4] B. Gordon, *Ramanujan congruences for $p_{-k} \pmod{11^r}$* , Glasgow J. Math. **24** (1983), 107–123.
- [5] G.-H. Han, *The Nekrasov-Okounkov hook length formula: refinement, elementary proof, extension and applications*, Annales de l'Institut Fourier, Volume 60 (2010) no. 1, 1–29.
- [6] M. Kaneko, D. Zagier, *A generalized Jacobi theta function and quasimodular forms*, The moduli space of curves (Texas Island, 1994), Progr. Math., vol. **129**, Birkhäuser Boston, Boston, MA, 1995, 165–172.
- [7] P. A. MacMahon, *Divisors of Numbers and their Continuations in the Theory of Partitions*, Proc. London Math. Soc. (2) **19** (1920), no.1, 75–113 [also in Percy Alexander MacMahon Collected Papers, Vol.2, pp. 303–341 (ed. G.E. Andrews), MIT Press, Cambridge, 1986].
- [8] N. A. Nekrasov, A. Okounkov, *Seiberg-Witten theory and random partitions*, in *The unity of mathematics*, Progress in Mathematics, Birkhäuser Boston, 2006, vol. 244, 525–596.
- [9] K. Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conference Series in Mathematics, **102**, Amer. Math. Soc., Providence, RI, 2004.
- [10] G. Pólya, *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen*, Acta Mathematica, **68** (0), (1937), 145–254.
- [11] S. Ramanujan, *On certain arithmetical functions*, Trans. Camb. Phil. Soc., **22** (1916), 159–184.
- [12] S. C. F. Rose, *Quasimodularity of generalized sum-of-divisors functions*, Research in Number Theory **1** (2015), Paper No. 18, 11 pp.
- [13] J.-P. Serre, *A course in arithmetic*, Springer-Verlag, New York, 1973.
- [14] J.-P. Serre, *Divisibilité de certaines fonctions arithmétiques*, L'Enseignement Math. **22** (1976), 227–260.
- [15] J.-P. Serre, *Formes modulaires et fonctions zêta p-adiques*, Springer Lect. Notes **350** (1973), 191–268.
- [16] J. Sturm, *On the congruence of modular forms*, Springer Lect. Notes in Math. **1240** (1984), 275–280.
- [17] H. P. F. Swinnerton-Dyer, *On ℓ -adic representations and congruences for coefficients of modular forms*, Springer Lect. Notes. **350** (1973), 1–55.

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