REMARKS ON MACMAHON'S q-SERIES

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Abstract. In his important 1920 paper on partitions, MacMahon defined the partition generating functions

$$A_k(q)=\sum_{n=1}^{\infty}\mathfrak{m}(k;n)q^n:=\sum_{0<s_1<s_2<\cdots<s_k}\frac{q^{s_1+s_2+\cdots+s_k}}{(1-q^{s_1})^2(1-q^{s_2})^2\cdots(1-q^{s_k})^2},$$
$$C_k(q)=\sum_{n=1}^{\infty}\mathfrak{m}_{\mathrm{odd}}(k;n)q^n:=\sum_{0<s_1<s_2<\cdots<s_k}\frac{q^{2s_1+2s_2+\cdots+2s_k-k}}{(1-q^{2s_1-1})^2(1-q^{2s_2-1})^2\cdots(1-q^{2s_k-1})^2}.$$

These series give infinitely many formulas for two prominent generating functions. For each non-negative k, we prove that $A_k(q), A_{k+1}(q), A_{k+2}(q), \dots$ (resp. $C_k(q), C_{k+1}(q), C_{k+2}(q), \dots$) give the generating function for the 3-colored partition function $p_3(n)$ (resp. the overpartition function $\overline{p}(n)$). To be precise, we have

$$\sum_{n=0}^{\infty}p_3(n)q^n=q^{-\frac{k^2+k}{2}}\sum_{m=k}^{\infty}\binom{2m+1}{m+k+1}A_m(q),$$
$$\sum_{n=0}^{\infty}\overline{p}(n)q^n=q^{-k^2}\sum_{m=k}^{\infty}\binom{2m}{m+k}C_m(q).$$

These formulas systematically give infinitely many formulas for the 3-colored partition function and the overpartition function in terms of MacMahon's $\mathfrak{m}(\bullet;n)$ and $\mathfrak{m}_{\mathrm{odd}}(\bullet;n)$ partition functions.

1. Introduction and Statement of Results

In an important paper on integer partitions, MacMahon [12] introduced the family of q-series

$$A_k(q):=\sum_{0<s_1<s_2<\cdots<s_k}\frac{q^{s_1+s_2+\cdots+s_k}}{(1-q^{s_1})^2(1-q^{s_2})^2\cdots(1-q^{s_k})^2}. \tag{1.1}$$

For positive integers k, we have that

$$A_k(q)=\sum_{n=1}^{\infty}\mathfrak{m}(k;n)q^n=\sum_{\substack{0<s_1<s_2<\cdots<s_k\\(m_1,\dots,m_k)\in\mathbb{N}^k}}m_1m_2\dots m_kq^{m_1s_1+m_2s_2+\dots m_ks_k}, \tag{1.2}$$

and so $A_k(q)$ is a natural partition generating function. Indeed, we have that $\mathfrak{m}(k;n)$ is the sum of the products of the *part multiplicities* for partitions of n with k distinct part sizes.

These series connect partitions to disparate areas of mathematics. In elementary number theory, MacMahon realized some of the $A_k(q)$ as generating functions for divisor sums. For

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example, he found that

$$A_1(q) = \sum_{n=1}^{\infty} \sigma_1(n) q^n = 1 + 3q^2 + 4q^3 + 7q^4 + 6q^5 + \dots,$$
$$A_2(q) = \frac{1}{8} \sum_{n=1}^{\infty} ((-2n+1)\sigma_1(n) + \sigma_3(n)) = q^3 + 3q^4 + 9q^5 + \dots,$$

where $\sigma_\nu(n) := \sum_{d|n} d^\nu$. Extending beyond number theory, these series arise in the study of Hilbert schemes, q-multiple zeta-values, representation theory, and topological string theory. Recent research has focused on the quasimodularity of the $A_k(q)$ (for example, see [1, 2, 5, 6, 7, 13]). Andrews and Rose [5, 13] proved that $A_k(q)$ is a linear combination of quasimodular forms on $\mathrm{SL}_2(\mathbb{Z})$ with weights $\leq 2k$.

In this note, we instead focus on the combinatorial properties of MacMahon's series. We show that they satisfy infinitely many systematic identities, that, in turn, illustrate the ubiquity of the $\mathfrak{m}(k; n)$ partition functions. To set the stage, we offer some terms of $A_0(q), A_1(q), \dots, A_5(q)$:

$$\begin{aligned}
A_0(q) &:= q^0, \\
A_1(q) &= q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + \dots, \\
A_2(q) &= q^3 + 3q^4 + 9q^5 + 15q^6 + 30q^7 + \dots, \\
A_3(q) &= q^6 + 3q^7 + 9q^8 + 22q^9 + 42q^{10} + \dots, \\
A_4(q) &= q^{10} + 3q^{11} + 9q^{12} + 22q^{13} + 51q^{14} + \dots, \\
A_5(q) &= q^{15} + 3q^{16} + 9q^{17} + 22q^{18} + 51q^{19} + \dots
\end{aligned}$$

As these examples suggest, the $A_k(q)$ behave well as $k \to +\infty$. Indeed, in terms of the q-Pochammer symbol

$$(a; q)_\infty := (1-a)(1-aq)(1-aq^2)\dots,$$

and Jacobi's famous identity for $(q;q)_\infty^3$, it is known that (see Theorem 1.1 of [2])

$$\frac{1}{(q;q)_\infty^3} = q^{-\frac{k^2+k}{2}} A_k(q) + O(q^{k+1}). \tag{1.3}$$

In a recent preprint, Bringmann, Craig, van Ittersum and Pandey [7] obtain further such results relating infinite products with MacMahon-type q-series.

It is natural to ask whether (1.3) is a glimpse of explicit identities, one for each non-negative integer k. We show that this is indeed the case, where $q^{-\frac{k^2+k}{2}} A_k(q)$ is simply the first summand of a closed formula involving $A_k(q), A_{k+1}(q), A_{k+2}(q), \dots$.

Theorem 1.1. *If k is a non-negative integer, then we have*

$$\frac{1}{(q;q)_\infty^3} = q^{-\frac{k^2+k}{2}} \sum_{m=k}^{\infty} \binom{2m+1}{m+k+1} A_m(q).$$

To further appreciate these identities, we note, for positive m, that (1.3) implies

$$q^{-\frac{k^2+k}{2}} A_{k+m}(q) = q^{\frac{m(m+1)}{2}+mk} + \dots. \tag{1.4}$$

The first terms of these q-series have exponents that grow quadratically m. Therefore, we can use Theorem 1.1 to compute initial segments of

$$\frac{1}{(q;q)_\infty^3} = \sum_{n\geq 0} p_3(n)q^n, \tag{1.5}$$

the generating function for the 3-colored partition function $p_3(n)$, using a "small number" of summands. In this way, we obtain a doubly infinite family of formulas relating the 3-colored partition function to MacMahon's $\mathfrak{m}(k;n)$ partition functions.

Corollary 1.2. *If k and j are non-negative integers, then*

$$\frac{1}{(q;q)_\infty^3} = q^{-\frac{k^2+k}{2}} \sum_{m=0}^{j} \binom{2m+2k+1}{m+2k+1} A_{m+k}(q) + O\left(q^{\frac{(j+1)(j+2k+2)}{2}}\right).$$

In particular, if $n < (j+1)(j+2k+2)/2$, then we have

$$p_3(n) = \sum_{m=0}^{j} \binom{2m+2k+1}{m+2k+1} \mathfrak{m}\left(m+k; n+\frac{k^2+k}{2}\right).$$

Remark. Letting $j = 1$ in Corollary 1.2 gives Theorem 1.1 (ii) of [1].

Example. If $k = 100$ and $j = 2$, then Corollary 1.2 gives

$$\frac{1}{(q;q)_\infty^3} = q^{-5050} \cdot (A_{100}(q) + 203A_{101}(q) + 20910A_{102}(q)) + O(q^{306}).$$

Therefore, for $n < 306$, we have

$$p_3(n) = \mathfrak{m}(100; n+5050) + 203\mathfrak{m}(101; n+5050) + 20910\mathfrak{m}(102; n+5050).$$

In addition to the $A_k(q)$, MacMahon also introduced [12] the q-series

$$C_k(q) = \sum_{n=1}^{\infty} \mathfrak{m}_{\text{odd}}(k;n)q^n := \sum_{0<s_1<s_2<\cdots<s_k} \frac{q^{2s_1+2s_2+\cdots+2s_k-k}}{(1-q^{2s_1-1})^2(1-q^{2s_2-1})^2\cdots(1-q^{2s_k-1})^2}. \tag{1.6}$$

The numbers $\mathfrak{m}_{\text{odd}}(k;n)$ have the same partition theoretic description as the $\mathfrak{m}(k;n)$, where here the parts are required to be odd. Furthermore, in analogy with the work of Andrews and Rose [5, 13], Bachmann [6] proved that each $C_k(q)$ is a finite linear combination of quasimodular forms on $\Gamma_0(2)$ with weight $\leq 2k$.

Here we show that the $C_k(q)$ also enjoy properties that are analogous to those of $A_k(q)$ described above. Namely, we prove the following theorem, where $C_0(q) := 1$.

Theorem 1.3. *The following are true.*
(1) If k is a non-negative integer, then we have

$$q^{-k^2} C_k(q) = \frac{1}{(q^2;q^2)_\infty (q;q^2)_\infty^2} + O(q^{2k+1}).$$

(2) If k is a non-negative integer, then we have

$$\frac{1}{(q^2;q^2)_\infty (q;q^2)_\infty^2} = q^{-k^2} \sum_{m=k}^{\infty} \binom{2m}{m+k} C_m(q).$$

Theorems 1.1 and 1.3 establish infinitely many formulas, one for each integer k, between MacMahon's two families of q-series and the reciprocals of the theta functions

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n^2+n}{2}} = 1-3q+5q^3-7q^6+9q^{10}-\dots,$$
$$(q^2;q^2)_\infty(q;q^2)_\infty^2 = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} = 1-2q+2q^4-2q^9+2q^{16}-2q^{25}+\dots.$$

Corollary 1.2, which relates the 3-colored partition function to MacMahon's $\mathfrak{m}(k;n)$ partition functions, relies on the fact that $1/(q;q)_\infty^3$ is the generating function of $p_3(n)$. Rather nicely, it turns out that

$$\frac{1}{(q^2;q^2)_\infty(q;q^2)_\infty^2} = \sum_{n=0}^{\infty}\overline{p}(n)q^n = 1+2q+4q^2+8q^3+14q^4+24q^5+\dots, \tag{1.7}$$

where $\overline{p}(n)$ denotes the number of *overpartitions* of size n. Recall that an overpartition of n is an ordered sequence of nonincreasing positive integers, where the first occurrence of each integer may be overlined [9]. Overpartitions have been the focus of intense research in recent years (for example, see [4, 8, 9, 10, 11, 14]). Therefore, in analogy with Corollary 1.2, we obtain the following corollary.

Corollary 1.4. *If k and j are non-negative integers, then*

$$\frac{1}{(q^2;q^2)_\infty(q;q^2)_\infty^2} = q^{-k^2}\sum_{m=0}^{j}\binom{2m+2k}{m+2k}C_{m+k}(q)+O\left(q^{(j+1)(j+2k+1)}\right).$$

In particular, if $n<(j+1)(j+2k+1)$, then we have

$$\overline{p}(n) = \sum_{m=0}^{j}\binom{2m+2k}{m+2k}\mathfrak{m}_{\text{odd}}\left(m+k;n+k^2\right).$$

Example. If $k=100$ and $j=2$, then Corollary 1.4 gives

$$\frac{1}{(q^2;q^2)_\infty(q;q^2)_\infty^2} = q^{-10000}\cdot\left(C_{100}(q)+202C_{101}(q)+20706C_{102}(q)\right)+O(q^{609}).$$

Therefore, for $n<609$, we have

$$\overline{p}(n) = \mathfrak{m}_{\text{odd}}(100;n+10000)+202\mathfrak{m}_{\text{odd}}(101;n+10000)+20706\mathfrak{m}_{\text{odd}}(102;n+10000).$$

The proofs of our results are rather straightforward, and follow from the Jacobi triple product identity. Namely, we recognize the role of MacMahon's q-series as coefficients of power series in $(z+z^{-1})^2$ obtained from this well-known bivariate infinite product.

Remark. The proofs of Theorems 1.1 and 1.3 follow along similar lines. They differ in their choice of specialization (i.e. changes of variable) of the Jacobi triple product identity. It would be interesting to see if other natural partition generating functions emerge from further specializations, to supplement these results on $p_3(n)$ and $\overline{p}(n)$. Finally, we point out that it would be interesting to carry out a similar analysis for the quintuple and septuple infinite product identities.

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2. Proofs

2.1. MacMahon's $A_k(q)$.

Here we prove Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. We recall the Jacobi triple product identity (see Theorem 2.8 of [3])

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = \prod_{n=0}^{\infty}(1-q^{2n+2})(1+z^{-1}q^{2n+1})(1+zq^{2n+1}).$$

By factoring out $(q^2;q^2)_\infty$, and letting $z \to qz^2$, and then letting $q \to \sqrt{q}$, a simple reindex gives

$$\sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} z^{2n} = \left(1+z^{-2}\right)(q;q)_\infty \prod_{n=1}^{\infty}(1+z^{-2}q^n)(1+z^2q^n).$$

After straightforward algebraic manipulation, we find

$$\sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} z^{2n} = \left(1+z^{-2}\right)(q;q)_\infty \prod_{n=1}^{\infty}\left((1-q^n)^2+(z+z^{-1})^2q^n\right).$$

After factoring out $(q;q)_\infty^2$ from the infinite product, we obtain

$$\sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} z^{2n} = \left(1+z^{-2}\right)(q;q)_\infty^3 \prod_{n=1}^{\infty}\left(1+\frac{q^n}{(1-q^n)^2}\cdot(z+z^{-1})^2\right).$$

Thanks to definition (1.1), we find that the infinite product on the right, as a power series in $(z+z^{-1})^2$, is the generating function for MacMahon's series. Therefore, we find that

$$\sum_{n=-\infty}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_\infty^3}\cdot z^{2n} = \left(1+z^{-2}\right)\sum_{n=0}^{\infty} A_n(q)(z+z^{-1})^{2n}.$$

Thanks to the Binomial Theorem, followed by a simple shift in the index of summation, and culminating with a change in the order of summation, we obtain

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_\infty^3}\cdot z^{2n} &= \left(1+z^{-2}\right)\sum_{n=0}^{\infty} A_n(q)\sum_{j=0}^{2n}\binom{2n}{j}z^{2j-2n} = \left(1+z^{-2}\right)\sum_{n=0}^{\infty} A_n(q)\sum_{j=-n}^{n}\binom{2n}{j+n}z^{2j}\\ &= \left(1+z^{-2}\right)\sum_{j=-\infty}^{\infty}\sum_{n=|j|}^{\infty}\binom{2n}{j+n}A_n(q)z^{2j}.\end{aligned}$$

After multiplying through $(1+z^{-2})$, we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_\infty^3}\cdot z^{2n} = \sum_{j=-\infty}^{\infty}\left(\sum_{n=|j|}^{\infty}\binom{2n}{j+n}A_n(q)+\sum_{n=|j+1|}^{\infty}\binom{2n}{j+n+1}A_n(q)\right)z^{2j}.$$

The theorem follows by comparing the coefficient of z^{2k} on both sides after making use of the binomial coefficient identity $\binom{m}{r}+\binom{m}{r+1}=\binom{m+1}{r+1}$. $\square$

Proof of Corollary 1.2. To prove the corollary, we truncate the infinite sums in Theorem 1.1 after j terms, and we then apply (1.4) and (1.5). $\square$

2.2. **MacMahon's $C_k(q)$.** Here we prove Theorem 1.3 and Corollary 1.4.

Proof of Theorem 1.3. We recall the Jacobi triple product identity (see Theorem 2.8 of [3])

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = \prod_{n=0}^{\infty} (1-q^{2n+2})(1+z^{-1}q^{2n+1})(1+zq^{2n+1}).$$

After factoring out $(q^2;q^2)_\infty$, and then letting $z \to z^2$, a simple reindex gives

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^{2n} = (q^2;q^2)_\infty \prod_{n=1}^{\infty} (1-z^{-2}q^{2n-1})(1+z^2q^{2n-1}).$$

One easily checks that

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^{2n} = (q^2;q^2)_\infty \prod_{n=1}^{\infty} \left((1-q^{2n-1})^2 + (z+z^{-1})^2 q^{2n-1}\right).$$

After factoring out $(q;q^2)_\infty^2$ from the infinite product, we obtain

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^{2n} = (q^2;q^2)_\infty (q;q^2)_\infty^2 \prod_{n=1}^{\infty} \left(1 + \frac{q^{2n-1}}{(1-q^{2n-1})^2} \cdot (z+z^{-1})^2\right).$$

Thanks to definition (1.6), we find that the infinite product on the right, as a power series in $(z+z^{-1})^2$, is the generating function for MacMahon's series. Namely, we have

$$\sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_\infty (q;q^2)_\infty^2} \cdot z^{2n} = \sum_{n=0}^{\infty} C_n(q)(z+z^{-1})^{2n}.$$

Thanks to the Binomial Theorem, followed by a simple shift in the index of summation, and culminating with a change in the order of summation, we get

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_\infty (q;q^2)_\infty^2} \cdot z^{2n} &= \sum_{n=0}^{\infty} C_n(q) \sum_{j=0}^{2n} \binom{2n}{j} z^{2j-2n} = \sum_{n=0}^{\infty} C_n(q) \sum_{j=-n}^{n} \binom{2n}{j+n} z^{2j} \\ &= \sum_{j=-\infty}^{\infty} \sum_{n=|j|}^{\infty} \binom{2n}{j+n} C_n(q) z^{2j}\end{aligned}$$

By comparing the coefficient of z^{2k} on both sides, one easily deduces claim (2), which in turn implies claim (1). $\square$

Proof of Corollary 1.4. By direct computation, for every positive integer m we have

$$q^{-k^2} C_{k+m}(q) = q^{m(m+2k)} + \dots.$$

By truncating the infinite sums in Theorem 1.3 (3) after j terms, the corollary now follows from this fact and (1.7).

$\square$

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