JENSEN POLYNOMIALS FOR THE RIEMANN XI-FUNCTION

M. J. GRIFFIN, K. ONO, L. ROLEN, J. THORNER, Z. TRIPP, AND I. WAGNER

Abstract. We investigate $\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$, where $\zeta(s)$ is the Riemann zeta function. The Riemann hypothesis (RH) asserts that if $\xi(s)=0$, then $\mathrm{Re}(s)=\frac{1}{2}$. Pólya proved that RH is equivalent to the hyperbolicity of the Jensen polynomials $J^{d,n}(X)$ constructed from certain Taylor coefficients of $\xi(s)$. For each $d \geq 1$, recent work proves that $J^{d,n}(X)$ is hyperbolic for sufficiently large n. In this paper, we make this result effective. Moreover, we show how the low-lying zeros of the derivatives $\xi^{(n)}(s)$ influence the hyperbolicity of $J^{d,n}(X)$.

1. Introduction and Statement of Results

Let $\zeta(s)$ be the Riemann zeta function. Define $\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ and [1]

$$\xi\left(\frac{1}{2}+z\right) = \sum_{j=0}^{\infty} \frac{\gamma(j)}{j!} z^{2j}. \tag{1.1}$$

It is known that $\gamma(n) > 0$ for all $n \geq 0$ [4, Section 4.4]. For $d, n \geq 0$, the degree d Jensen polynomial $J^{d,n}(X)$ for the n-th derivative $\xi^{(n)}(s)$ is

$$J^{d,n}(X) := \sum_{j=0}^{d} \binom{d}{j} \gamma(n+j) X^j. \tag{1.2}$$

A polynomial with real coefficients is *hyperbolic* if all of its zeros are real. Expanding on notes of Jensen, Pólya [16] proved that the Riemann hypothesis (RH) is equivalent to the hyperbolicity of $J^{d,n}(X)$ for all $d, n \geq 0$. Since RH remains unproved, some research has focused on proving hyperbolicity for all $n \geq 0$ when d is small. Csordas, Norfolk, and Varga [6] and Dimitrov and Lucas [9] proved hyperbolicity for $n \geq 0$ and $d \leq 3$. Building on the work of Borcea and Brändén [3] and Obreschkoff [14], Chasse [5] proved hyperbolicity for $d \leq 2 \times 10^{17}$ and $n \geq 0$.

Recent work [12] provides a complementary treatment. For all $d \geq 1$, there is a threshold $N(d)$ such that $J^{d,n}(X)$ is hyperbolic for $n \geq N(d)$. Specifically, under the transformation (2.2) below, the polynomials $J^{d,n}(X)$ are closely modeled by the Hermite polynomials $H_d(\frac{X}{2})$, where

$$\sum_{d=0}^{\infty} H_d(X)\frac{t^d}{d!} := e^{2Xt-t^2} = 1 + 2Xt + (4X^2-2)\frac{t^2}{2!} + (8X^3-12X)\frac{t^3}{3!} + \cdots \tag{1.3}$$

Thus for large n, $J^{d,n}(X)$ inherits hyperbolicity from $H_d(\frac{X}{2})$. See Bombieri [2] for commentary.

Our main result, which builds on work in [12], provides an effective upper bound for $N(d)$.

Theorem 1.1. *There is a constant $c > 0$ such that $J^{d,n}(X)$ is hyperbolic whenever $n \geq ce^d$.*

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[1] This presentation, which is convenient for us, is not widely used; see the discussion in the proof of Theorem 2.1.

For an integer $m \geq 0$, let RH_m to be the statement that if $\xi^{(m)}(s) = 0$, then $\mathrm{Re}(s) = \frac{1}{2}$. It is well known that $\mathrm{RH} = \mathrm{RH}_0$ implies RH_m for all $m \geq 1$ [16]. The ideas of Pólya lead to the conclusion that $\xi^{(m)}(s)$ satisfies RH_m if and only if $J^{d,n}(X)$ is hyperbolic for $d \geq 1$ and $n \geq m$. For $T \geq 0$, we define $\mathrm{RH}_m(T)$ to be the statement that all zeros $\rho^{(m)}$ of $\xi^{(m)}(s)$ with $|\mathrm{Im}(\rho^{(m)})| \leq T$ satisfy $\mathrm{Re}(\rho^{(m)}) = \frac{1}{2}$. Our second result is a relationship between $\mathrm{RH}_m(T)$ and the hyperbolicity of $J^{d,n}(X)$ for $n \geq m$. In what follows, $\lfloor x \rfloor$ denotes the usual floor function.

Theorem 1.2. *If $\mathrm{RH}_m(T)$ is true and $d \leq \lfloor T \rfloor^2$, then $J^{d,n}(X)$ is hyperbolic for all $n \geq m$.*

This modestly generalizes work of Chasse [5, Theorem 1.8], which Theorem 1.2 recovers when $m = 0$. We include it for the sake of completeness. Since Platt and Trudgian [17] have verified $\mathrm{RH}_0(3000175332800)$, Theorem 1.2 implies the following corollary (cf. [17, Corollary 3]).

Corollary 1.3. *If $d \leq 9 \times 10^{24}$ and $n \geq 0$, then $J^{d,n}(X)$ is hyperbolic.*

Remarks.

(1) One can generalize the notion of a Jensen polynomial by replacing the Taylor coefficients $\gamma(n)$ with other suitable arithmetic functions in (1.2). Questions of hyperbolicity for such polynomials can be of great arithmetic interest [12]. While some of the ideas presented here might apply in other settings, we restrict our consideration and only present the strongest conclusions for $\xi(s)$ that our methods appear to permit.

(2) Our proof quantifies the rate at which a certain transformation of $J^{d,n}(X)$ tends to $H_d(\frac{X}{2})$ as n tends to infinity. See Farmer [11] for an interesting interpretation of this as an instance of a uniform variant of Berry's "cosine is a universal attractor" principle [1].

(3) It would be most desirable to prove a sort of converse to Theorem 1.2 wherein the partial results on hyperbolicity from Theorem 1.1 would directly influence the distribution of zeros of the derivatives of $\xi(s)$, or perhaps even $\xi(s)$ itself. While Theorem 1.2 indicates that a partial understanding of the zeros of $\xi^{(m)}(s)$ influence the hyperbolicity of $J^{d,n}(X)$ for $n \geq m$, a quick inspection of the proofs in [16] indicates that it is highly unlikely that converse influence exists unless one has hyperbolicity for all $n \geq m$ and all $d \geq 1$. While Jensen polynomials can be used to uniformly approximate $\xi^{(n)}(\frac{1}{2}+it)$, they are ultimately quite inefficient at detecting zeros that violate RH_n (should any such zeros exist). One can see this by directly plotting the aforementioned uniform approximation.

(4) After this paper was written, O'Sullivan [15] wrote an interesting paper on the Pólya-Jensen criterion for the Riemann Hypothesis. Instead of working directly with the Jensen polynomials $J^{d,n}(X)$, he considers a variant of the original criterion which makes use of $\sum_{j=0}^{d} \binom{d}{j} \gamma(n+j) H_{d-j}(X)$. His paper complements the explicit results obtained here for this modified criterion.

In Section 2, we prove Theorem 1.1 using a small modification of a result of Turán. Our proof assumes two technical results (Theorems 2.2 and 2.4) that we prove in Sections 3 and 4. In Section 5, we prove Theorem 1.2.

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2. Proof of Theorem 1.1

The effective refinement of the work in [12] provided by Theorem 1.1 uses different methods. Our proofs are facilitated by renormalizations of several objects in [12].

2.1. **New conventions and preliminaries.** Recall the setup in [12, Section 5]. It was shown that for each $d \geq 1$, there exist positive numbers $A(n)$, $\delta(n)$, $g_3(n)$, $g_4(n), \ldots, g_d(n)$ such that

$$\log\Big(\frac{\gamma(n+j)}{\gamma(n)}\Big) = A(n)j - \delta(n)^2 j^2 + \sum_{i=3}^{d} g_i(n) j^i + o(\delta(n)^d), \tag{2.1}$$

with $g_i(n) = O(n^{1-i}) = o(\delta(n)^i)$ and $\delta(n) \sim \frac{1}{\sqrt{2n}}$. From these, we define

$$\widehat{J}^{d,n}(X) := \frac{\delta(n)^{-d}}{\gamma(n)} J^{d,n}\Big(\frac{\delta(n)X - 1}{\exp(A(n))}\Big). \tag{2.2}$$

Estimates in [12] are written in terms of the behavior of $\delta(n)$, and there is considerable latitude in the choice of $\delta(n)$. In this sense, $\delta(n)$ serves as a uniformizer for the calculations in [12].

We introduce a more refined uniformizer

$$\Delta(M) := \sqrt{\frac{1}{2}\Big(1 - \frac{\gamma(M-2)\gamma(M)}{\gamma(M-1)^2}\Big)} \tag{2.3}$$

and a corresponding new normalization $\widetilde{J}^{d,n}(X)$ of the polynomials $J^{d,n}(X)$. It will become apparent that $\Delta(M)$ is a more convenient and more accurate uniformizer than $\delta(n)$, which is important for our eventual goal of an effective lower bound for n in terms of d. Before defining $\widetilde{J}^{d,n}(X)$, we reformulate the work of Csordas, Norfolk, and Varga [6, 7] in terms of the notation in (1.1).

Theorem 2.1. *If $n \geq 3$, then $\gamma(n-2)\gamma(n) \leq \gamma(n-1)^2$.*

Proof. This follows from the work of [6, 7], though their normalizations are different from ours. We express the expansion that they consider using our notation:

$$\frac{1}{8}\xi\Big(\frac{1}{4} + \frac{iz}{2}\Big) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!}\Big(\frac{(2j)!}{(j!)} \cdot \frac{\gamma(j)}{2^{2j+3}}\Big) z^{2j}.$$

If we define $\widetilde{\gamma}_j := 2^{-2j-3}\gamma(j)$, then it follows from [6, 7] that if $j \geq 1$, then $\widetilde{\gamma}_j^2 > \widetilde{\gamma}_{j-1}\widetilde{\gamma}_{j+1}$. Our claimed result for $\gamma(n)$ follows immediately. $\square$

By Theorem 2.1, the function $\gamma(n)$ is log-concave; consequently, $\Delta(M) \in \mathbb{R}$ for all $M \geq 3$. The next theorem contains some key results for $\Delta(M)$.

Theorem 2.2. *Let $\Delta(M)$ be as in* (2.3).

(1) We have

$$\lim_{M\to\infty} \Delta(M)\sqrt{2M} = 1.$$

In particular, if $C > 1$, then there exists $M_C > C/(C-1)$ (depending only on C) such that if $M > M_C$, then

$$\frac{1}{\sqrt{2C(M-1)}} \leq \Delta(M) \leq \frac{1}{\sqrt{M}}.$$

(2) For each integer $m \geq 1$, there exists a function $G_m(z)$, holomorphic for $\mathrm{Re}(z) > 1$, such that for all integers $1 \leq j < M$ we have

$$\log\left(\frac{\gamma(M-j)}{\gamma(M)}\right) = -\sum_{m=1}^{\infty} G_m(M)\Delta(M)^{2m-2} j^m. \tag{2.4}$$

With $C > 1$ as in part (1), the bound $|G_m(M)| \ll_C (2C)^m$ holds for all integers $m, M \geq 1$. We also have the limit

$$\lim_{M\to\infty} G_m(M) = \frac{2^{m-1}}{m(m-1)}.$$

(3) We have

$$G_2(M) = 1 + (1 - 3G_3(M))\Delta(M)^2 + O(\Delta(M)^4). \tag{2.5}$$

We will prove Theorem 2.2 in Section 3.

Remark. *The uniform bound on $|G_m(M)|$ is critical for our proofs. While $G_m(M)$ is a bounded function of M for fixed m, we need to bound $|G_m(M)|$ when m and M vary jointly.*

Now that we have listed some key properties of $\Delta(M)$, we define

$$\widetilde{J}^{d,n}(X) := \frac{\gamma(n+d)^{d-1}}{\gamma(n+d-1)^d \cdot \Delta(n+d)^d} J^{d,n}\Big(\frac{\gamma(n+d-1)}{\gamma(n+d)} \cdot (\Delta(n+d)X - 1)\Big). \tag{2.6}$$

For future convenience, we define the coefficients $A_{d,k}(n)$ by the expansion

$$\widetilde{J}^{d,n}(X) = \sum_{k=0}^{d} A_{d,k}(n) X^{d-k}. \tag{2.7}$$

The following lemma explains our reason for working with these new normalizations.

Lemma 2.3. *If $d \geq 1$ and $n \geq 0$, then $A_{d,0}(n) = 1$, $A_{d,1}(n) = 0$, and $A_{d,2}(n) = -d(d-1)$. In particular, $\widetilde{J}^{1,n}(X) = H_1(\frac{X}{2})$, $\widetilde{J}^{2,n}(X) = H_2(\frac{X}{2})$, and $\deg\big(\widetilde{J}^{d,n}(X) - H_d(\frac{X}{2})\big) \leq d-3$ for $d \geq 3$.*

Proof. This is straightforward to verify from (1.2), (1.3), and (2.6). $\square$

We use Theorem 2.2 to prove asymptotics for the coefficients $A_{d,k}(n)$ for $k \geq 3$.

Theorem 2.4. *Let $d \geq 4$, $n \geq 0$, and $3 \leq k \leq d$ be integers, and let $1 < C < 2$. Recall the definition of M_C from Theorem 2.2(1). If $n + d > \max\{10k^3, M_C\}$, then*

$$\frac{(-1)^{\lfloor \frac{k}{2}\rfloor}(d-k)!\lfloor\frac{k}{2}\rfloor!}{d!} A_{d,k}(n) = \begin{cases} 1 + Z_{n+d}(\lfloor\frac{k}{2}\rfloor)\Delta(n+d)^2 + O_C(k^6(4C)^k\Delta(n+d)^4) & \text{if } k \text{ is even,} \\ \lfloor\frac{k}{2}\rfloor(G_3(n+d) - 2)\Delta(n+d) + O_C(k^4(4C)^k\Delta(n+d)^3) & \text{if } k \text{ is odd,} \end{cases}$$

where $Z_{n+d}(t) := t(t-1)(-\frac{2}{3}(3t+2) + 2tG_3(n+d) - \frac{t-2}{2}G_3(n+d)^2 - G_4(n+d))$.

We prove Theorem 2.4 in Section 4.

2.2. **Proof of Theorem 1.1.** We use the following result to prove Theorem 1.1.

Lemma 2.5. *For $0 \le j \le d$, define*

$$c_{d,n,j} := \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(d-j+2i)!}{i!(d-j)!} A_{d,j-2i}(n), \tag{2.8}$$

where $A_{d,k}(n)$ is defined by (2.7)*. If*

$$\sum_{j=3}^{d} 2^{-j} \frac{(d-j)!}{(d-1)!} c_{d,n,j}^2 < 1, \tag{2.9}$$

then $J^{d,n}(X)$ is hyperbolic.

Proof. There exist $A, B, C \in \mathbb{R}$ (depending on n and d) such that $\widetilde{J}^{d,n}(X) = AJ^{d,n}(BX + C)$, hence $J^{d,n}(X)$ is hyperbolic if and only $\widetilde{J}^{d,n}(X)$ is hyperbolic. We apply to (2.7) the inversion formula [10, Equation 18.18.20]

$$x^n = \sum_{\ell=0}^{\lfloor n/2 \rfloor} \frac{(-n)_{2\ell}}{\ell!} H_{n-2\ell}(x),$$

where $(-n)_{2\ell}$ denotes the Pochhammer symbol, thus deducing the existence of constants $c_{d,n,j} \in \mathbb{R}$ such that

$$\widetilde{J}^{d,n}(X) = \sum_{j=0}^{d} c_{d,n,j} H_{d-j}\left(\frac{X}{2}\right).$$

Turán [18, Theorem III] proved that if $c_j \in \mathbb{R}$ for $0 \le j \le N$ and

$$\sum_{j=0}^{N-2} 2^j j! c_j^2 < 2^N (N-1)! c_N^2, \tag{2.10}$$

then all roots of $\sum_{j=0}^{N} c_j H_j(z)$ (hence $\sum_{j=0}^{N} c_j H_j(\frac{z}{2})$) are real and simple. Since $c_{d,n,0} = 1$ and $c_{d,n,1} = c_{d,n,2} = 0$ by Lemma 2.3, the inequality (2.10) applied to our setting reduces to (2.9). $\square$

Proof of Theorem 1.1. We will show that there exists a suitably large absolute constant $c > 0$ such that if $n \ge ce^d$, then (2.9) holds, in which case Lemma 2.5 applies. We now appeal to Theorem 2.4. When $j = 2\ell$, we use the even case of Theorem 2.4, (2.8), and the fact that $A_{d,0} = 1$ and $A_{d,2} = -d(d-1)$ to find that $c_{d,n,2\ell}$ equals

$$\begin{aligned}
&\sum_{i=0}^{\ell} \frac{(d-2i)!}{(\ell-i)!(d-2\ell)!} A_{d,2i}(n) \\
&= \frac{d!}{\ell!(d-2\ell)!} \Big[\sum_{i=0}^{\ell} \binom{\ell}{i} (-1)^i + \Delta(n+d)^2 \sum_{i=2}^{\ell} \binom{\ell}{i} (-1)^i Z_{n+d}(i) + O_C\Big(\Delta(n+d)^4 \sum_{i=2}^{\ell} \binom{\ell}{i} i^6 (4C)^{2i}\Big)\Big] \\
&\qquad = \frac{d!}{\ell!(d-2\ell)!} \Big[\Delta(n+d)^2 \sum_{i=2}^{\ell} \binom{\ell}{i} (-1)^i Z_{n+d}(i) + O_C\Big(\Delta(n+d)^4 \sum_{i=2}^{\ell} \binom{\ell}{i} i^6 (4C)^{2i}\Big)\Big].
\end{aligned}$$

For a function f defined on the nonnegative integers, we define the k-th difference operator

$$\sigma_{k,x}(f(x)) := \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}f(j) \tag{2.11}$$

Note that $f(x)$ is given by polynomial of degree at most d if and only if $\sigma_{k,x}(f(x)) = 0$ for all $k > d$. Since $Z_{n+d}(t)$ is a polynomial in t of degree 3 with $Z_{n+d}(0) = Z_{n+d}(1) = 0$, it follows if $\ell \geq 4$, then

$$\sum_{i=2}^{\ell}\binom{\ell}{i}(-1)^i Z_{n+d}(i) = 0.$$

Thus, if $\ell \geq 4$, then we apply the bound $i^6 \leq \ell^6$ to conclude that

$$c_{d,n,2\ell} \ll_C \frac{d!}{(d-2\ell)!\ell!}\Delta(n+d)^4\ell^6\sum_{i=2}^{\ell}\binom{\ell}{i}(4C)^{2i} \ll_C \frac{d!}{(d-2\ell)!\ell!}\ell^6(16C^2+1)^{\ell}\Delta(n+d)^4. \tag{2.12}$$

The bound (2.12) also holds when $\ell = 2$ and $\ell = 3$ by bounding the main terms directly. A very similar calculation using the odd case of Theorem 2.4 reveals that

$$c_{d,n,2\ell+1} \ll_C \frac{d!}{(d-2\ell-1)!\ell!}\ell^4(16C^2+1)^{\ell}\Delta(n+d)^3. \tag{2.13}$$

The bound (2.12) leads to a bound for the even-indexed terms in (2.9), namely

$$\sum_{\substack{3\leq j\leq d \\ j \text{ even}}} 2^{-j}\frac{(d-j)!}{(d-1)!}c_{d,n,j}^2 \ll_C d\Delta(n+d)^8 \sum_{1\leq \ell\leq d/2}\binom{d}{2\ell}\binom{2\ell}{\ell}\ell^{12}\Big(\frac{16C^2+1}{2}\Big)^{2\ell}.$$

Note that $\binom{2\ell}{\ell} \sim \frac{4^\ell}{\sqrt{\pi\ell}}$ by Stirling's formula. Trivially bounding $\ell^{12} \leq d^{12}$, we find that

$$\sum_{\substack{3\leq j\leq d \\ j \text{ even}}} 2^{-j}\frac{(d-j)!}{(d-1)!}c_{d,n,j}^2 \ll_C d^{13}(2+16C^2)^d\Delta(n+d)^8. \tag{2.14}$$

An essentially identical argument bounds the sum over odd terms as well:

$$\sum_{\substack{3\leq j\leq d \\ j \text{ odd}}} 2^{-j}\frac{(d-j)!}{(d-1)!}c_{d,n,j}^2 \ll_C d^{9}(2+16C^2)^d\Delta(n+d)^6. \tag{2.15}$$

We combine (2.14) and (2.15) with the bound for $\Delta(n+d)$ in Theorem 2.2 to conclude that there is a constant $\alpha_C > 0$ such that (2.9) holds if

$$n \geq \alpha_C d^{13/3}(2+16C^2)^{d/3} = \alpha_C \exp\Big(\Big(\frac{1}{3}\log(2+16C^2) + \frac{13\log d}{3d}\Big)d\Big).$$

Since stronger conclusions than Theorem 1.1 follow from Corollary 1.3 when $d \leq 9\times 10^{24}$, we may assume that $d > 9\times 10^{24}$. Choosing $C = 1 + 10^{-5}$, we find that

$$\frac{1}{3}\log(2+16C^2) + \frac{13\log d}{3d} \leq 0.9634\ldots,$$

and the desired result follows. $\square$

3. Proof of Theorem 2.2

The function

$$F(z) := \int_1^\infty (\log t)^z\, t^{-3/4} \left(\sum_{k=1}^\infty e^{-\pi k^2 t} \right) dt$$

is holomorphic for $\mathrm{Re}(z) > 0$. It follows from [12, Equation 13] that

$$\gamma(M) = \frac{M!}{(2M)!} \cdot \frac{32\binom{2M}{2} F(2M-2) - F(2M)}{2^{2M-1}}. \tag{3.1}$$

If we replace the binomial coefficient and factorials in (3.1) with Γ-functions, we see that (3.1) extends to a function of a *complex* variable M which is holomorphic for $\mathrm{Re}(M) > 1$.

For $M > 0$, let L_M be the unique positive solution of the equation $M = L_M(\pi e^{L_M} + \frac{3}{4})$. It is straightforward to show that $L_M \sim \log(\frac{M}{\log M})$. Define $K_M = (L_M^{-1} + L_M^{-2})M - \frac{3}{4}$. The function L_M (and therefore K_M) extends to a function which is holomorphic and non-vanishing for $\mathrm{Re}(M) > 1$. By [12, Equation 16], we have

$$\gamma(M) = \frac{e^{M-2} M^{M+\frac{1}{2}} L_{2M-2}^{2M-2}}{2^{2M-5}(2M-2)^{(2M-2)+\frac{1}{2}}} \sqrt{\frac{2\pi}{K_{2M-2}}} \exp\left(\frac{L_{2M-2}}{4} - \frac{2M-2}{L_{2M-2}} + \frac{3}{4} \right) \left(1 + O_\varepsilon\left(\frac{1}{M^{1-\varepsilon}} \right) \right). \tag{3.2}$$

Ultimately, the analytic continuation of L_M and Stirling's formula imply that even when M is complex, we may keep the existing error term in (3.2) once we replace M with $|M|$.

For fixed $\mathrm{Re}(M) > 1$, there is a function $R_M(j)$ of a *complex* variable j, holomorphic and non-vanishing for $|j| < \mathrm{Re}(M) - 1$, with the property that if $j, M \in \mathbb{Z}$ satisfy $|j| < M$, then

$$R_M(j) = \frac{\gamma(M-j)}{\gamma(M)}. \tag{3.3}$$

Since $R_M(j)$ is holomorphic and nonvanishing when $|j| < \mathrm{Re}(M) - 1$, we have the expansion

$$\log R_M(j) = \sum_{m=1}^\infty a_m(M) j^m, \qquad |j| < \mathrm{Re}(M) - 1. \tag{3.4}$$

By varying M, we find the Taylor coefficents $a_m(M)$ are in fact holomorphic functions in M.

Since $\log R_M(j)$ is holomorphic for M and j in the specified domains, the right hand side of (3.4) converges absolutely and uniformly for j and M in *compact* subsets of their respective domains. We wish to give bounds on the coefficients $a_m(M)$ which are uniform for *all* real M and j in their respective domains. To do so, we must regularize $\log R_M(j)$ to obtain a function $R_M^*(\lambda)$ which extends to a function of M on the extended interval $[3, \infty]$. For convenience, we replace j with $\lambda(M-2)$; it suffices to consider λ in the closed disk $|\lambda| \le 1$ (rather than j in a domain that varies with M). We now define our regularized function

$$R_M^*(\lambda) := \frac{1}{M-2} \log\left(\left(\frac{eL_{2M-2}^2 M}{4(2M-2)^2} \right)^{\lambda(M-2)} R_M(\lambda(M-2)) \right) + (1-\lambda)\log(1-\lambda). \tag{3.5}$$

Our expansion for $R_M^*(\lambda)$ for $|\lambda| \le 1$ naturally incorporates the coefficients $a_m(M)$:

$$R_M^*(\lambda) = \left(a_1(M) + \log\left(\frac{eL_{2M-2}^2 M}{4(2M-2)^2} \right) - 1 \right)\lambda + \sum_{m=2}^\infty \left(a_m(M)(M-2)^{m-1} + \frac{1}{m(m-1)} \right)\lambda^m. \tag{3.6}$$

Lemma 3.1. *The function $R_M^*(\lambda)$ is holomorphic for all $|\lambda| \leq 1$ and all M in the extended interval* $[3, \infty]$. *Moreover, for all $|\lambda| \leq 1$, we have that* $\lim_{M\to\infty} R_M^*(\lambda) = 0$

Proof. For all finite M and $|\lambda| \leq 1$, the function $R_M^*(\lambda)$ is holomorphic since each such point corresponds to a value of $R_M(j)$ with $|j| \leq M - 2$. In order to understand the behavior of $R_M^*(\lambda)$ as $M \to \infty$, we consider the regularized limit

$$\lim_{M\to\infty} \frac{1}{M-2} \log\Big(\Big(\frac{eL_{2M-2}^2 M}{4(2M-2)^2}\Big)^{\lambda(M-2)} R(\lambda(M-2); M)\Big). \tag{3.7}$$

Let $j = \lambda(M-2)$, as above. The asymptotic (3.2) implies that as $M \to \infty$, we have

$$\frac{1}{M-2} \log\Big(\Big(\frac{eL_{2M-2}^2 M}{4(2M-2)^2}\Big)^{j} R_M(j)\Big) = A + B + C + O_\varepsilon\Big(\frac{\log M}{M}\Big),$$

where

$$A_M(\lambda) = \frac{1}{M-2} \log\Big(\frac{M^j (M-j)^{M-j-2}(2M-2)^{2M-2+\frac{1}{2}}}{(2M-2)^{2j}(2M-2-2j)^{2M-2-2j+\frac{1}{2}} M^{M-2}}\Big),$$

$$B_M(\lambda) = \frac{1}{M-2}\Big((2M-2-2j)\log\Big(\frac{L_{2M-2-2j}}{L_{2M-2}}\Big) - \frac{1}{2}\log\Big(\frac{K_{2M-2-2j}}{K_{2M-2}}\Big)\Big), \quad \text{and}$$

$$C_M(\lambda) = \frac{1}{M-2}\Big(\frac{L_{2M-2-2j}}{4} - \frac{2M-2-2j}{L_{2M-2-2j}} - \frac{L_{2M-2}}{4} + \frac{2M-2}{L_{2M-2}}\Big).$$

Since $L_M \sim \log(\frac{M}{\log M})$, a calculus exercise shows that $\lim_{M\to\infty} B_M(\lambda) = \lim_{M\to\infty} C_M(\lambda) = 0$.

Simplifying $A_M(\lambda)$, we find that

$$A_M(\lambda) = \frac{M-j-2}{M-2}\log\Big(1 - \frac{j}{M}\Big) - \frac{2M-2-2j+\frac{1}{2}}{M-2}\log\Big(1 - \frac{2j}{2M-2}\Big).$$

Since $j = \lambda(M-2)$, it follows that

$$\lim_{M\to\infty} A_M(\lambda) = -(1-\lambda)\log(1-\lambda) = \lambda - \sum_{m=2}^{\infty} \frac{1}{m(m-1)} \lambda^m. \tag{3.8}$$

The rightmost sum converges absolutely for $|\lambda| < 1$, but is not holomorphic at 1, hence we remove the term in (3.5) so that (3.6) converges on the boundary of the disk. □

Since $R_M^*(\lambda)$ is holomorphic for $|\lambda| \leq 1$ and all $M \in [3, \infty]$, the Taylor series given in (3.6) converges absolutely and uniformly for all such λ and M. Taking $\lambda = 1$ and $M \geq 3$, we find that for all $\varepsilon > 0$, there exists an integer $W_\varepsilon \geq 1$, *depending only on* ε, such that

$$|a_m(M)(M-2)^{m-1} + (m(m-1))^{-1}| < \varepsilon \qquad \text{whenever } m \geq W_\varepsilon. \tag{3.9}$$

Proof of Theorem 2.2. To prove our claimed asymptotic for $\Delta(M)$, we write $\Delta(M)$ in terms of $a_m(M)$. We extend $\Delta(M)$, originally defined in (2.3), to a holomorphic function by the identity

$$\Delta(M) = \sqrt{\frac{1}{2}\Big(1 - \frac{R_M(2)}{R_M(1)^2}\Big)}. \tag{3.10}$$

We use (3.4) to expand (3.10) and then apply (3.9) to bound $a_m(M)$ for $m \geq 4$, thus obtaining

$$\Delta(M) = \sqrt{-a_2(M) - 3a_3(M) - a_2(M)^2 + O(M^{-3})}. \tag{3.11}$$

The asymptotic $\Delta(M) \sim \frac{1}{\sqrt{2M}}$ now follows from (3.9) as we let $\varepsilon \to 0$.

We define $G_m(M)$ by the identity $-a_m(M) = G_m(M)\Delta(M)^{2m-2}$. The expansion (2.4) now has the desired properties, and the claimed bounds and asymptotics for $G_m(M)$ follow from (3.9) and the fact that $\Delta(M) \sim \frac{1}{\sqrt{2M}}$. To recover (2.5), we square both sides of (3.11), and notice that $G_2(M)$ satisfies the quadratic equation

$$\Delta(M)^2 G_2(M)^2 - G_2(M) + 1 - 3G_3(M)\Delta(M)^2 = O(\Delta(M)^4).$$

The desired result follows. $\square$

Remark. *These methods provide an effective alternative to the approach to asymptotics for $g_m(n)$ and $\delta(n)$ in [12]. Greater care is required here than in [12] because of the uniformity required in Theorem 1.1. Comparing (2.1) and (2.4), and noticing the sign change of j on the left hand side both equations, we see that $G_m(M)\Delta(M)^{2m-2} \sim (-1)^{m+1} g_m(M)$. In particular, we see that $g_2(M) \sim -1/(2M)$, which implies that $\delta(M) \sim \Delta(M)$.*

4. Proof of Theorem 2.4

Using the functions $G_m(M)$ given by Theorem 2.2, we define $S(j;M)$ and $Q_m(M)$ as follows:

$$S(j;M) = \frac{R_M(j)}{R_M(1)^j} = \exp\Big(\sum_{m=2}^{\infty} G_m(M)\Delta(M)^{2m-2}(j - j^m)\Big) = \sum_{m=0}^{\infty} Q_m(M) j^m. \tag{4.1}$$

This definition of $S(j;M)$ is critical because, by (3.3), we have for integers $j \in [0, M-1]$ that

$$S(j;M) = \frac{\gamma(M-j)\gamma(M)^{j-1}}{\gamma(M-1)^j}.$$

Using (2.6), we may rewrite the coefficients $A_{d,k}(n)$ as

$$A_{d,k}(n) = \binom{d}{k}\Delta(n+d)^{-k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}S(j;n+d). \tag{4.2}$$

Recall (2.11), and define $y_{m,k} = \sigma_{k,x}(x^m)$. This leads to the identity

$$A_{d,k}(n) = \binom{d}{k}\Delta(n+d)^{-k}\sum_{m=0}^{\infty} y_{m,k} Q_m(n+d). \tag{4.3}$$

We have the following lemma about the size of the $y_{m,k}$.

Lemma 4.1. *Let $y_{m,k}$ be defined as above. Then $y_{m,k} = 0$ if $m < k$, and*

$$y_{k,k} = k!, \quad y_{k+1,k} = k!\binom{k+1}{2}, \quad y_{k+2,k} = k!\binom{k+2}{3}\frac{3k+1}{4}, \quad y_{k+3,k} = k!\binom{k+3}{4}\frac{k^2+k}{2}.$$

More generally, for all $i \geq 1$, there exists a polynomial $P_i(k)$ of degree $i-1$, satisfying $P_i(1) = 1$ and $P_i(k) \leq k^{i-1}$ for all positive integers k, such that $y_{k+i,k} = k!\binom{k+i}{1+i}P_i(k)$.

Proof. If $m < k$, the identity $y_{m,k} = 0$ follows the discussion following (2.11). For $m \geq k$, we have the identity $(e^X - 1)^k = (X + \frac{X^2}{2} + \frac{X^3}{3!} + \dots)^k = \sum_{m=0}^{\infty} \frac{y_{m,k}}{m!} X^m$. For integers $t > i$, we now consider $\sigma_{t,k}(\frac{y_{k+i,k}}{(k+i)!})$ as a function of k. For fixed t, we have the generating function

$$\begin{aligned}
\sum_{i=0}^{\infty} \sigma_{t,k}\Big(\frac{y_{k+i,k}}{(k+i)!}\Big) X^{i+t} &= \sum_{i=0}^{\infty}\sum_{j=0}^{t} (-1)^{t-j}\binom{t}{j}\Big(\frac{y_{j+i,j}}{(j+i)!}\Big) X^{i+t} \\
&= \sum_{j=0}^{t} (-1)^{t-j}\binom{t}{j} X^{t-j} \sum_{i=0}^{\infty} \Big(\frac{y_{j+i,j}}{(j+i)!}\Big) X^{j+i} \\
&= \sum_{j=0}^{t} (-1)^{t-j}\binom{t}{j} X^{t-j}(e^X - 1)^j = (e^X - X - 1)^t = \frac{1}{2^t} X^{2t} + \cdots .
\end{aligned}$$

Hence $\sigma_{t,k}(\frac{y_{k+i,k}}{(k+i)!}) = 0$ for $t > i$. This implies that $\frac{y_{k+i,k}}{(k+i)!}$ is a polynomial in k of degree at most i. For $i \geq 1$, note that $y_{i,0} = 0$, and $y_{i,1} = 1$. Thus, we can factor $y_{k+i,k}$ as

$$y_{k+i,k} = kP_i(k) \prod_{j=0}^{i-1} \frac{k+i-j}{1+i-j} = k!\binom{k+i}{i+1} P_i(k),$$

where $P_i(1) = 1$. A short calculation gives the claimed expressions for $y_{k+1,k}$, $y_{k+2,k}$, and $y_{k+3,k}$.

We prove that $P_i(k) \leq k^{i-1}$ for all positive integers k by comparing the Taylor coefficients of

$$\Big(\frac{e^X - 1}{X}\Big)^k = \sum_{i=0}^{\infty} \frac{k \cdot P_i(k)}{(i+1)!} X^i \qquad \text{and} \qquad \frac{e^{kX} - 1}{kX} = \sum_{i=0}^{\infty} \frac{k^i}{(i+1)!} X^i. \tag{4.4}$$

Given functions $f = f(x)$ and $g = g(x)$ which are analytic at 0, let $f \prec g$ denote the condition that $f^{(i)}(0) \leq g^{(i)}(0)$ for all integers $i \geq 0$. In other words, the i-th Taylor coefficient of g is at least the i-th Taylor coefficient of f in the expansions at zero. This statement has transitivity—if $f \prec g$ and $g \prec h$, then $f \prec h$. If $h^{(i)}(0) \geq 0$ for all $i \geq 0$, then $f \prec g$ implies $fh \prec gh$.

By comparing the expansions in (4.4), the bound $P_i(k) \leq k^{i-1}$ is equivalent to

$$\Big(\frac{e^x - 1}{x}\Big)^k \prec \frac{e^{kx} - 1}{kx}. \tag{4.5}$$

Define $F_k = F_k(x) := (e^{kx/2} - e^{-kx/2})/(kx)$. We rewrite (4.5) as $e^{kx/2} F_1^k \prec e^{kx/2} F_k$. Since $(e^{kx/2})^{(n)}(0) > 0$ for all $n \geq 0$, $e^{kx/2} F_1^k \prec e^{kx/2} F_k$ follows from $F_1^k \prec F_k$.

We will prove $F_1^k \prec F_k$ by induction on k. The result when $k = 1$ is trivial. Suppose now that $F_1^k \prec F_k$ is true for some integer $k \geq 1$. By transitivity, the truth of $F_1^{k+1} \prec F_{k+1}$ follows from that of $F_1^{k+1} \prec F_1 F_k$ and $F_1 F_k \prec F_{k+1}$. Since $F_k^{(i)}(0) \geq 0$ for all $i \geq 0$, our inductive hypothesis $F_1^k \prec F_k$ implies $F_1^{k+1} \prec F_1 F_k$.

It remains to prove $F_1 F_k \prec F_{k+1}$ for all $k \geq 1$. We have the expansions

$$F_1(x)F_k(x) = \sum_{i=1}^{\infty} \frac{2}{k(2i)!}\Big(\Big(\frac{k+1}{2}\Big)^{2i} - \Big(\frac{k-1}{2}\Big)^{2i}\Big) x^{2i-2}, \quad F_{k+1}(x) = \sum_{i=1}^{\infty} \frac{1}{(2i-1)!}\Big(\frac{k+1}{2}\Big)^{2i-2} x^{2i-2}.$$

Thus $F_1 F_k \prec F_{k+1}$, and hence (4.5), follows from the bound

$$0 \leq \frac{4ik}{(k+1)^2} + \Big(\frac{k-1}{k+1}\Big)^{2i} - 1, \qquad i, k \geq 1. \tag{4.6}$$

Denote the right side of (4.6) as $\omega_k(i)$. Observe that $\omega_k(i+1)-\omega_k(i)=\frac{4k}{(k+1)^2}(1-(\frac{k-1}{k+1})^{2i})>0$ for all $i,k\geq 1$. It follows that $\omega_k(i)\geq\omega_k(1)=0$ for all $i,k\geq 1$, which proves (4.6). $\square$

The desired asymptotic for $A_{d,k}(n)$ will follow from a suitable bound for $Q_m(M)$, which we prove using Theorem 2.2 and Lemma 4.1.

Lemma 4.2. *If $m,\ell\geq 1$ are integers, $C>1$, and $M\geq\max(\ell^3,M_C)$, then $|Q_m(M)|\ell!\ll_C (4C)^m\ell^{\ell-\frac{1}{2}m}\Delta(M)^m$.*

Proof. Let λ be a partition of m, denoted $\lambda\vdash m$. Let λ_i be the number of parts equal to i so that $\sum_{i=1}^m i\lambda_i=m$. Define $\mathcal{L}(\lambda)=\sum_{i=1}^m\lambda_i$. From (4.1) and the multinomial theorem, we obtain

$$\begin{aligned}\frac{Q_m(M)\ell!}{\Delta(M)^m}&=\frac{\ell!}{\Delta(M)^m}\sum_{\lambda\vdash m}\frac{(\widetilde{G}_1(M)\Delta(M)^2)^{\lambda_1}}{\lambda_1!}\frac{(-G_2\Delta(M)^2)^{\lambda_2}}{\lambda_2!}\cdots\frac{(-G_m\Delta(M)^{2m-2})^{\lambda_m}}{\lambda_m!}\\ &=\sum_{\lambda\vdash m}(-1)^{\mathcal{L}(\lambda)-\lambda_1}\frac{\ell!}{\lambda_1!\lambda_2!\cdots\lambda_m!}\widetilde{G}_1(M)^{\lambda_1}G_2(M)^{\lambda_2}\cdots G_m(M)^{\lambda_m}\Delta(M)^{m-2\mathcal{L}(\lambda)+2\lambda_1},\end{aligned}\tag{4.7}$$

where $\widetilde{G}_1(M):=\sum_{m=2}^\infty G_m(M)\Delta(M)^{2m-4}$. Since $|G_i(M)|\ll_C(2C)^i$ by Theorem 2.2, it follows that $\widetilde{G}_1(M)=1+O_C(\Delta(M)^2)$ and $|\widetilde{G}_1(M)^{\lambda_1}G_2(M)^{\lambda_2}\cdots G_m(M)^{\lambda_m}|\ll_C(2C)^{\sum_{i=1}^m i\lambda_i}=(2C)^m$. Since $\Delta(M)\leq M^{-\frac{1}{2}}\leq\ell^{-\frac{3}{2}}$ by Theorem 2.2 and our hypotheses, the definition of $\mathcal{L}(\lambda)$ yields

$$\frac{\ell!}{\lambda_1!\lambda_2!\cdots\lambda_m!}\Delta(M)^{m-2\mathcal{L}(\lambda)+2\lambda_1}\leq\frac{\ell!}{\lambda_2!}\ell^{-\frac{3}{2}(m-2\mathcal{L}(\lambda)+2\lambda_1)}\leq\ell^{\ell-\lambda_2-\frac{3}{2}(m-2\mathcal{L}(\lambda)+2\lambda_1)}\leq\ell^{\ell-\frac{1}{2}m}.$$

The desired result now follows since there are at most 2^m partitions of m. $\square$

Proof of Theorem 2.4. Recall (4.3), which expresses $A_{d,k}(n)$ as a sum of $y_{m,k}Q_m(n+d)$ over $m\geq 0$. We use Lemma 4.1 to rewrite the contribution from $y_{m,k}$ in (4.3) and arrive at

$$\begin{aligned}A_{d,k}(n)=\binom{d}{k}k!\Big[\frac{Q_k(n+d)}{\Delta(n+d)^k}+\binom{k+1}{2}\frac{Q_{k+1}(n+d)}{\Delta(n+d)^k}+\binom{k+2}{3}\frac{3k+1}{4}\frac{Q_{k+2}(n+d)}{\Delta(n+d)^k}\\ +\binom{k+3}{4}\frac{k^2+k}{2}\frac{Q_{k+3}(n+d)}{\Delta(n+d)^k}+\sum_{i=4}^\infty\binom{k+i}{1+i}\frac{P_i(k)Q_{k+i}(n+d)}{\Delta(n+d)^k}\Big].\end{aligned}$$

Let $j=\lfloor k/2\rfloor$. Since $\binom{d}{k}k!=\frac{d!}{(d-k)!}$, it follows that $A_{d,k}(n)$ equals

$$\begin{aligned}\frac{(-1)^jd!}{j!(d-k)!}\Big[\frac{(-1)^jj!Q_k(n+d)}{\Delta(n+d)^k}+\binom{k+1}{2}\frac{(-1)^jj!Q_{k+1}(n+d)}{\Delta(n+d)^k}\\ +\binom{k+2}{3}\frac{3k+1}{4}\frac{(-1)^jj!Q_{k+2}(n+d)}{\Delta(n+d)^k}+\binom{k+3}{4}\frac{k^2+k}{2}\frac{(-1)^jj!Q_{k+3}(n+d)}{\Delta(n+d)^k}\\ +\sum_{i=4}^\infty\binom{k+i}{1+i}P_i(k)\frac{(-1)^jj!Q_{k+i}(n+d)}{\Delta(n+d)^k}\Big].\end{aligned}$$

Suppose that $n+d > \max\{j^3, M_C, 64C^2 j\}$. The asymptotic bounds for $\Delta(n+d)$ from Theorem 2.2 and the bound for $Q_{k+i}(n+d)$ in Lemma 4.2 imply that $A_{d,k}(n)$ equals

$$\begin{aligned}
&\frac{(-1)^j d!}{j!(d-k)!}\Big[\frac{(-1)^j j! Q_k(n+d)}{\Delta(n+d)^k} + \binom{k+1}{2}\frac{(-1)^j j! Q_{k+1}(n+d)}{\Delta(n+d)^k} \\
&\qquad + \binom{k+2}{3}\frac{3k+1}{4}\frac{(-1)^j j! Q_{k+2}(n+d)}{\Delta(n+d)^k} + \binom{k+3}{4}\frac{k^2+k}{2}\frac{(-1)^j j! Q_{k+3}(n+d)}{\Delta(n+d)^k} \\
&\qquad + O((4C)^k k^{9/2}\Delta(n+d)^4)\Big].
\end{aligned} \tag{4.8}$$

Let $m \in \{k, k+1, k+2, k+3\}$. As in Lemma 4.2, we use (4.7) to expand $Q_m(n+d)$, bounding the contribution from the partitions λ such that $m - 2\mathcal{L}(\lambda) + 2\lambda_1 \geq 3$ using the bound for $|G_m(n+d)|$ in Theorem 2.2. Since $m - 2\mathcal{L}(\lambda) + 2\lambda_1 = \lambda_1 + \sum_{i=3}^m (i-2)\lambda_i$, we must separately consider the cases where m is even (where the powers of $\Delta(n+d)$ are even) and m is odd (where the powers of $\Delta(n+d)$ are odd). When $M = n+d$ and m is even, it then follows from (4.7) that $Q_m(M)$ equals $(-1)^{m/2}\Delta(M)^m/(\frac{m}{2})!$ times

$$\begin{aligned}
&G_2(M)^{\frac{m}{2}} - \frac{m}{4}\Big(G_2(M)^{\frac{m}{2}-1}\widetilde{G}_1(M)^2 + (m-2)G_4(M)G_2(M)^{\frac{m}{2}-2} \\
&+ (m-2)G_3(M)G_2(M)^{\frac{m}{2}-2}\widetilde{G}_1(M) + \frac{(m-2)(m-4)}{4}G_3(M)^2 G_2(M)^{\frac{m}{2}-3}\Big)\Delta(M)^2 \\
&+ O_C(m^6(4C)^m\Delta(M)^4).
\end{aligned} \tag{4.9}$$

Similarly, when m is odd, $Q_m(M)$ equals $(-1)^{\lfloor \frac{m}{2}\rfloor}\Delta(M)^m/(\lfloor\frac{m}{2}\rfloor)!$ times

$$(G_2(M)^{\lfloor\frac{m}{2}\rfloor}\tilde{G}_1(M) + \lfloor\tfrac{m}{2}\rfloor G_3(M)G_2(M)^{\lfloor\frac{m}{2}\rfloor-1})\Delta(M) + O_C(m^4(4C)^m\Delta(M)^3). \tag{4.10}$$

The theorem follows by substituting (4.9) and (4.10) into (4.8). $\square$

5. Proof of Theorem 1.2

We introduce some notation. For $0 < \delta < \pi/2$, define $S(\theta,\delta) := \{z \in \mathbb{C}^\times : |\arg(z) - \theta| \leq \delta\}$. Let $C(\theta,\delta)$ to be the set of entire functions F such that there exist a sequence of complex numbers $(\beta_k)_{k\geq 1}$, an integer $q \geq 0$, and constants $c, \sigma \in \mathbb{C}$ such that $\sum_{k=1}^\infty \frac{1}{|\beta_k|} < \infty$, $\beta_k, \sigma \in S(\theta,\delta)$, and

$$F(z) = cz^q e^{-\sigma z}\prod_{k=1}^\infty\Big(1 - \frac{z}{\beta_k}\Big).$$

Lemma 5.1. *Let $0 < \delta < \pi/2$. If $F \in C(\theta,\delta)$, then F is locally uniformly approximated by polynomials, each of whose zeros lie in $S(\theta,\delta)$, and conversely. Moreover, if $m \geq 1$ is an integer and the m-th derivative $F^{(m)}$ is not identically zero, then $F^{(m)} \in C(\theta,\delta)$.*

Proof. The first claim is proved in [13, Chapter VIII]. For the second claim, suppose that $F \in C(\theta,\delta)$ is non-constant. By the first claim, there exists a sequence of nonzero polynomials (g_n) which locally uniformly approximate F, and each zero of g_n lies in $S(\theta,\delta)$. By the Gauss-Lucas theorem, the zeros of g_n' belong to the convex hull of the set of zeros of g_n; thus each zero of g_n' lies in $S(\theta,\delta)$. Since the sequence (g_n') locally uniformly approximates F', it follows by the first claim that $F' \in C(\theta,\delta)$. For higher derivatives, we proceed by induction. $\square$

Lemma 5.2. *If $\frac{d^n}{dz^n}\xi(\sqrt{z}+\frac{1}{2}) \in C(\pi,\delta)$, then $J^{d,n}(X)$ is hyperbolic for $d \leq |\sin(\delta)|^{-2}$.*

Proof. In (1.1), all powers of z are even, so $\xi(\sqrt{z}+\frac{1}{2})$ is entire. Since $\gamma(j)>0$ for all $j\geq 0$ and

$$\frac{d^n}{dz^n}\xi(\sqrt{z}+\frac{1}{2})=\sum_{j=0}^{\infty}\frac{\gamma(j+n)}{j!}z^j=:\psi_n(z),$$

the Taylor coefficients of $\psi_n(z)$ are positive. Hence the lemma follows immediately from [5, Theorem 3.6] with $\varphi=\psi_n(z)$. □

Proof of Theorem 1.2. We follow [5]. Let $m\geq 0$ be an integer. Suppose that $\mathrm{RH}_m(T)$ holds for some $T>\frac{1}{2}$. Then the zeros of $\frac{d^m}{dz^m}\xi(z+\frac{1}{2})$ in the rectangle $\{z\in\mathbb{C}\colon |\mathrm{Re}(z)|<1/2,\ |\mathrm{Im}(z)|\leq T\}$ are imaginary. Therefore, the zeros of $\psi_m(z)$ must lie in $S(0,2\arctan(\frac{1}{2T}))\cup S(\pi,2\arctan(\frac{1}{2T}))$. Since $\gamma(j)>0$ for all $j\geq 0$, the zeros of $\psi_m(z)$ lie in the half-plane $\mathrm{Re}(z)<0$, and hence must lie in $S(\pi,2\arctan(\frac{1}{2T}))$. Hence $\psi_m(z)\in C(\pi,2\arctan(\frac{1}{2T}))$. We see from Lemma 5.2 that $J^{d,m}(X)$ is hyperbolic for $d\leq\lfloor|\sin(2\arctan(\frac{1}{2T}))|^{-2}\rfloor=\lfloor T^2+\frac{1}{2}+\frac{1}{16T^2}\rfloor$. Thus if $d\leq\lfloor T\rfloor^2$, then $J^{d,m}(X)$ is hyperbolic. Since $C(\theta,\delta)$ is closed under differentiation per Lemma 5.1, we have $\psi_n(z)\in C(\pi,2\arctan(\frac{1}{2T}))$ for all $n\geq m$, finishing the proof. □

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Department of Mathematics, 275 TMCB, Brigham Young University, Provo, UT 84602
Email address: `mjgriffin@math.byu.edu`

Department of Mathematics, University of Virginia, Charlottesville, VA 22904
Email address: `ko5wk@virginia.edu`

Department of Mathematics, Vanderbilt University, Nashville, TN 37240
Email address: `larry.rolen@vanderbilt.edu`

Department of Mathematics, University of Illinois, Urbana, IL 61801
Email address: `jesse.thorner@gmail.com`

Department of Mathematics, Vanderbilt University, Nashville, TN 37240
Email address: `zachary.d.tripp@vanderbilt.edu`

Department of Mathematics, Vanderbilt University, Nashville, TN 37240
Email address: `ian.c.wagner@vanderbilt.edu`