# ZEROS IN THE CHARACTER TABLES OF SYMMETRIC GROUPS WITH AN $\ell$-CORE INDEX

ELEANOR MCSPIRIT AND KEN ONO

*In memory of master representation theorist John McKay*

ABSTRACT. Let $\mathcal{C}_n = [\chi_\lambda(\mu)]_{\lambda,\mu}$ be the character table for $S_n$, where the indices $\lambda$ and $\mu$ run over the $p(n)$ many integer partitions of $n$. In this note we study $Z_\ell(n)$, the number of zero entries $\chi_\lambda(\mu)$ in $\mathcal{C}_n$, where $\lambda$ is an $\ell$-core partition of $n$. For every prime $\ell \geq 5$, we prove an asymptotic formula of the form
$$Z_\ell(n) \sim \alpha_\ell \cdot \sigma_\ell(n + \delta_\ell)p(n) \gg_\ell n^{\frac{\ell-5}{2}} e^{\pi\sqrt{2n/3}},$$
where $\sigma_\ell(n)$ is a twisted Legendre symbol divisor function, $\delta_\ell := (\ell^2 - 1)/24$, and $1/\alpha_\ell > 0$ is a normalization of the Dirichlet $L$-value $L\left(\left(\frac{\cdot}{\ell}\right), \frac{\ell-1}{2}\right)$. For primes $\ell$ and $n > \ell^6/24$, we show that $\chi_\lambda(\mu) = 0$ whenever $\lambda$ and $\mu$ are both $\ell$-cores. Furthermore, if $Z_\ell^*(n)$ is the number of zero entries indexed by two $\ell$-cores, then for $\ell \geq 5$ we obtain the asymptotic
$$Z_\ell^*(n) \sim \alpha_\ell^2 \cdot \sigma_\ell(n + \delta_\ell)^2 \gg_\ell n^{\ell-3}.$$

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let $\mathcal{C}_n = [\chi_\lambda(\mu)]_{\lambda,\mu}$ be the usual character table (for example, see [6, 13, 14]) for the symmetric group $S_n$, where the indices $\lambda$ and $\mu$ both vary over the $p(n)$ many integer partitions of $n$. Confirming conjectures of Miller [8], Peluse and Soundararajan [11, 12] recently proved that if $\ell$ is prime, then almost all of the $p(n)^2$ entries in $\mathcal{C}_n$, as $n \to +\infty$, are multiples of $\ell$. We note that Miller conjectured that the same conclusion holds for arbitrary prime powers, a claim which remains open.

In recent papers [8, 9], Miller raised the problem of determining the limiting behavior of $Z(n)$, the number of zero entries in $\mathcal{C}_n$. Despite the remarkable theorem of Peluse and Soundararajan, little is known. Moreover, due to the rapid growth of $p(n)$, it is computationally infeasible to compute many values of $Z(n)$. Consequently, there are no conjectures that are supported with substantial numerics. For example, is there a limiting proportion for the zeros in $\mathcal{C}_n$? Such a proportion would be given by the limit
$$\lim_{n \to +\infty} \frac{Z(n)}{p(n)^2}.$$
Limited numerics suggest that such a limit might exist, and might be $\approx 0.36$ (see Table 3 of [8]). However, this is a dubious guess at best. What's more, the simpler problem of determining whether $\liminf_{n \to +\infty} Z(n)/p(n)^2 > 0$ also seems to be out of reach. In view of these difficulties, McKay [5] posed a less ambitious problem; he asked for lower bounds arising from $\ell$-cores that illustrate the rapid growth of $Z(n)$. Here we answer this question, and for primes $\ell \geq 5$, we obtain asymptotic formulas for
$$Z_\ell(n) := \#\left\{(\lambda, \mu) \ : \ \chi_\lambda(\mu) = 0 \ \text{ with } \lambda \text{ an } \ell\text{-core}\right\}. \tag{1.1}$$

To this end, suppose that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ is a partition of $n$. As is typical in the representation theory of symmetric groups, we make use of $\ell$-core partitions, which are defined using *Young diagrams* of partitions, the left-justified arrays of cells where the row lengths are the parts. The *hook* for the cell in position $(k, j)$ is the set of cells below or to the right of that cell, including the cell itself, and so its *hook length* $h_\lambda(k, j) := (\lambda_k - k) + (\lambda'_j - j) + 1$. Here $\lambda'_j$ is the number of boxes in the $j$th column of the diagram. We say that $\lambda$ is an $\ell$-*core partition* if none of its hook lengths are multiples of $\ell$. If $c_\ell(n)$ denotes the number of $\ell$-core partitions of $n$, then we have (for example, see [2, 7]) the generating function

$$\sum_{n=0}^{\infty} c_\ell(n) q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{\ell n})^\ell}{(1 - q^n)}.$$

**Example.** The Young diagram of the partition $\lambda := (5, 4, 1)$, where each cell is labelled with its hook length, is given in Figure 1.

| 7 | 5 | 4 | 3 | 1 |
|---|---|---|---|---|
| 5 | 3 | 2 | 1 |   |
| 1 |   |   |   |   |

FIGURE 1. Hook lengths for $\lambda = (5, 4, 1)$

By inspection, we see that $\lambda$ is an $\ell$-core for every prime $\ell > 7$.

To state the asymptotics formulas, we let $L\left(\left(\frac{\cdot}{\ell}\right), s\right)$ be the Dirichlet $L$-function for the Legendre symbol $\left(\frac{\cdot}{\ell}\right)$, and let

$$\alpha_\ell := \frac{(2\pi)^{\frac{\ell-1}{2}}}{\left(\frac{\ell-3}{2}\right)! \cdot \ell^{\frac{\ell}{2}} \cdot L\left(\left(\frac{\cdot}{\ell}\right), \frac{\ell-1}{2}\right)}. \tag{1.2}$$

By the functional equations of these Dirichlet $L$-functions and the theory of generalized Bernoulli numbers, we have that $1/\alpha_\ell$ is always a positive integer (see p. 339 of [3]). For example, we have $1/\alpha_5 = 1$, $1/\alpha_7 = 8$, $1/\alpha_{11} = 1275$, and $1/\alpha_{13} = 33463$. In addition, we require the integers $\delta_\ell := (\ell^2 - 1)/24$, and the *twisted Legendre symbol divisor functions*

$$\sigma_\ell(n) := \sum_{1 \leq d | n} \left(\frac{n/d}{\ell}\right) d^{\frac{\ell-3}{2}}. \tag{1.3}$$

In terms of these quantities and functions, we obtain the following asymptotics for $Z_\ell(n)$.

**Theorem 1.1.** *If $\ell \geq 5$ is prime, then as $n \to +\infty$ we have*

$$Z_\ell(n) \sim \alpha_\ell \cdot \sigma_\ell(n + \delta_\ell) p(n) \gg_\ell n^{\frac{\ell-5}{2}} e^{\pi \sqrt{2n/3}}.$$

**Remark.** Apart from a density zero subset, we have that $Z_\ell(n) = 0$ when $\ell \in \{2, 3\}$ (see [3]).

As a corollary, we find that $Z(n)/p(n)$ grows faster than any power of $n$.

**Corollary 1.2.** *If $d > 0$, then*

$$\lim_{n \to +\infty} \frac{Z(n)}{p(n) \cdot n^d} = +\infty.$$

**Remark.** We note that Corollary 1.2 is weaker than

$$Z(n)/p(n)^2 \gg 1/\log n,$$

which can be found in the discussion after Lemma 2.3 of Peluse's paper [11].

We turn to the problem of describing the zero entries in $\mathcal{C}_n$ where both indices $\lambda$ and $\mu$ are $\ell$-core partitions. For prime $\ell$ and large $n$, the entries in $\mathcal{C}_n$ indexed by $\ell$-core pairs $(\lambda, \mu)$ always have $\chi_\lambda(\mu) = 0$.

**Theorem 1.3.** *Suppose that $\ell$ is prime, and let $N_\ell := (\ell^6 - 2\ell^5 + 2\ell^4 - 3\ell^2 + 2\ell)/24$. If $n > N_\ell$ and $\lambda, \mu \vdash n$ are $\ell-$core partitions, then $\chi_\lambda(\mu) = 0$.*

If $Z_\ell^*(n)$ denotes the number of vanishing entries $\chi_\lambda(\mu) = 0$ indexed by $\ell$-core partitions $\lambda, \mu \vdash n$, then we have the following corollary.

**Corollary 1.4.** *For primes $\ell$, the following are true.*

*(1) Apart from a density zero subset, we have that $Z_\ell^*(n) = 0$ when $\ell \in \{2, 3\}$.*

*(2) If $\ell \geq 5$, then as $n \to +\infty$ we have*

$$Z_\ell^*(n) \sim \alpha_\ell^2 \cdot \sigma_\ell(n + \delta_\ell)^2 \gg_\ell n^{\ell-3}.$$

To obtain these results, we use the well-known vanishing result that follows from the Murnaghan–Nakayama rule and says that $\chi_\lambda(\mu) = 0$ whenever $\mu$ has a part that is not the length of any hook in $\lambda$. Therefore, our goal is reduced to counting pairs of partitions $(\lambda, \mu)$ of large $n$, where $\mu$ has a part that is a multiple of $\ell$, and where $\lambda$ is an $\ell$-core. Theorem 1.1 is obtained by estimating these counts using asymptotics and lower bounds for various partition functions due to Hardy and Ramanujan, Hagis, and Granville and the second author.

Theorem 1.3 concerns the cases where $(\lambda, \mu)$ are both $\ell$-cores, and is a consequence of the fact (see Theorem 4.1) that every large $\ell$-core has a part that is a multiple of $\ell$. This fact is proved using the "abacus theory" of $\ell$-cores, and is a generalization of Section 3 of [10] by Sze and the second author in the case of 4-core partitions. Corollary 1.4 then follows from the asymptotics for $c_\ell(n)$ due to Granville and the second author.

This paper is organized as follows. Section 3 recalls well-known vanishing result and bounds, as well as the asymptotics and estimates for the relevant partition functions. Section 4 gives the abacus theory of $\ell$-cores and the statement and proof of Theorem 4.1. In Section 5 we employ these results to prove Theorems 1.1 and 1.3, and Corollaries 1.2 and 1.4.

## 2. Acknowledgements

## 3. Nuts and Bolts

In this section we recall essential facts that we require for the proofs of our results. We first state a criterion that guarantees the vanishing of character values, and then we give estimates for the relevant partition functions.

3.1. **Criterion for the vanishing of $\chi_\lambda(\mu)$.** Here we recall a standard partition theoretic criterion that guarantees the vanishing of a character value $\chi_\lambda(\mu)$. Suppose that $\lambda = (\lambda_1, \ldots, \lambda_s)$ and $\mu = (\mu_1, \ldots, \mu_t)$ are partitions of size $n$, and let $\{h_\lambda(i, j)\}$ be the multiset of hook lengths for $\lambda$. Thanks to the Murnaghan-Nakayama formula (for example, see Theorem 2.4.7 of [6]), we have that $\chi_\lambda(\mu) = 0$ when $\{\mu_i\}$ is not a subset of $\{h_\lambda(i, j)\}$.

Given a prime $\ell$, this immediately gives natural families of vanishing character table entries indexed by pairs of partitions $(\lambda, \mu)$ of $n$, where $\mu$ has a part that is a multiple of $\ell$, and $\lambda$ is an $\ell$-core partition. To make use of this observation, we recall that a partition $\mu$ is *A-regular* if none of its parts $\mu_i$ are

multiples of $A$. If $p_A(n)$ denotes the number of $A$-regular partitions of $n$, then one easily confirms the generating function

$$\sum_{n=0}^{\infty} p_A(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{An})}{(1-q^n)} = \prod_{n=1}^{\infty} \left(1 + q^n + q^{2n} + \cdots + q^{(A-1)n}\right),$$

which shows that $p_A(n)$ also is the number of partitions of $n$ where parts appear at most $A-1$ times. In terms of $p(n), p_\ell(n)$ and $c_\ell(n)$, we have the following lower bounds for $Z(n)$.

**Lemma 3.1.** *If $\ell$ is prime, then the following are true.*

*(1) If $\mu \vdash n$ is not an $\ell$-regular partition and $\lambda \vdash n$ is an $\ell$-core partition, then $\chi_\lambda(\mu) = 0$.*

*(2) If $n$ is a positive integer, then we have*

$$Z_\ell(n) \geq (p(n) - p_\ell(n)) \cdot c_\ell(n).$$

*Proof.* (1) By hypothesis, $\mu$ is not $\ell$-regular, meaning that it has a part that is a multiple of $\ell$. As $\lambda$ is an $\ell$-core, none of its hook lengths are multiples of $\ell$. Therefore, $\chi_\lambda(\mu) = 0$ by Murnaghan-Nakayama. (2) The number of partitions of $n$ that are not $\ell$-regular is $p(n) - p_\ell(n)$. Therefore, (1) gives the conclusion that

$$Z_\ell(n) \geq (p(n) - p_\ell(n))c_\ell(n).$$

$\square$

3.2. **Estimates for some partition functions.** Here we recall asymptotics and lower bounds for the partition functions we require to prove Theorem 1.1 and Corollary 1.4. First we have the celebrated Hardy-Ramanujan asymptotic for $p(n)$.

**Theorem 3.2.** *As $n \to +\infty$, we have*

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot \exp(\pi\sqrt{2n/3}).$$

Hagis obtained asymptotics for $p_A(n)$, the number of $A$-regular partitions of $n$. Letting $t = A - 1$ in Corollary 4.2 of [4], we have the following asymptotic formula.

**Theorem 3.3.** *If $A \geq 2$, then we have*

$$p_A(n) = C_A(24n - 1 + A)^{-\frac{3}{4}} \exp\left(C\sqrt{\frac{A-1}{A}\left(n + \frac{A-1}{24}\right)}\right)\left(1 + O(n^{-\frac{1}{2}})\right),$$

*where $C := \pi\sqrt{2/3}$ and $C_A := \sqrt{12}A^{-\frac{3}{4}}(A-1)^{\frac{1}{4}}$.*

Finally, we recall facts about $c_t(n)$, the number of $t$-core partitions of $n$ that were obtained by Granville and the second author in [3]. In terms of $\alpha_\ell$ defined in (1.2), and the twisted Legendre symbol divisor functions $\sigma_\ell(n)$ defined in (1.3), we have the following theorem.

**Theorem 3.4.** *The following are true.*

*(1) We have that*

$$c_2(n) = \begin{cases} 1 & \text{if } n \text{ is a triangular number,} \\ 0 & \text{otherwise.} \end{cases}$$

*(2) If $n$ is a non-negative integer, then*

$$c_3(n) = \sum_{d|(3n+1)} \left(\frac{d}{3}\right).$$

*In particular, $c_3(n) = 0$ for almost all $n$.*

*(3) If $t \geq 4$ and $n$ is a non-negative integer, then $c_t(n) > 0$.*

*(4) If $n$ is a non-negative integer, then $c_5(n) = \sigma_5(n+1)$.*

*(5) If $\ell \geq 7$ is prime, then as $n \to +\infty$ we have*

$$c_\ell(n) \sim \alpha_\ell \cdot \sigma_\ell(n + \delta_\ell).$$

*(6) If $\ell \geq 11$ is prime and $n$ is sufficiently large, then we have*

$$c_\ell(n) > \frac{2\alpha_\ell}{5} \cdot n^{\frac{\ell-3}{2}}.$$

*Proof.* Claim (1) is a straightforward observation. Claims (2), (4), and (5) are proved on p. 339-340 of [3]. Claim (3) is Theorem 1 of [3], while (6) is Theorem 4 of [3]. $\qquad \square$

## 4. Abaci and large $\ell$-core partitions

Throughout this section, suppose that $\ell$ is prime. The main result here is the following theorem which shows that every sufficiently large $\ell$-core partition has a part that is a multiple of $\ell$.

**Theorem 4.1.** *Suppose that $\ell$ is prime, and let $N_\ell := (\ell^6 - 2\ell^5 + 2\ell^4 - 3\ell^2 + 2\ell)/24$. If $n > N_\ell$, then every $\ell$-core partition of size $n$ has a part that is a multiple of $\ell$.*

**Remark.** We note that $N_\ell < \ell^6/24$ is not optimal. Indeed, if we let $N_\ell^{\max}$ be the largest $n$ admitting an $\ell$-regular $\ell$-core partition, then it turns out that $N_3^{\max} = 10$ and $N_3 = 16$.

4.1. **Abaci Theory.** We make use of the theory of *abaci* for partitions (for example, see [1, 6]). In particular, let $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$ be a partition of $n$. For each $1 \leq i \leq s$, define the *$i$th structure number* $B_i := \lambda_i - i + s$, so that $B_i = h_\lambda(i, 1)$, the hook length of cell $(i, 1)$.

Using these structure numbers, we represent the partition $\lambda$ as an $\ell$-abacus $\mathfrak{A}_\lambda$, consisting of beads placed on rods numbered $0, 1, \ldots, \ell - 1$. For each $B_i$, there is a unique pair of integers $(r_i, c_i)$ for which $B_i = \ell(r_i - 1) + c_i$ and $0 \leq c_i \leq \ell - 1$. The abacus $\mathfrak{A}_\lambda$ then consists of $s$ beads, where for each $i$, one places a bead in position $(r_i, c_i)$.

**Lemma 4.2** (Lemma 2.7.13, [6]). *Assuming the notation above, $\lambda$ is an $\ell$-core if and only if all of the beads in $\mathfrak{A}_\lambda$ lie at the top of their respective rods without gaps.*

In view of this lemma, we may represent an abacus of an $\ell$-core partition by $\ell$-tuples of non-negative integers, say $(b_0, \ldots, b_{\ell-1})$, where $b_i$ denotes the number of beads in column $i$. However, such representations are not unique as they generally allow for parts of size zero. We have the following elementary lemma.

**Lemma 4.3** (Lemma 1, [10]). *The following abaci both represent the same $\ell$-core partition:*

$$(b_0, b_1, \ldots, b_{\ell-1}) \quad and \quad (b_{\ell-1} + 1, b_0, b_1, \ldots, b_{\ell-2}).$$

By repeatedly applying this lemma, we may canonically define the unique abacus representation for an $\ell$-core to be the one with zero beads in the first column. Thus, when we talk about *the* abacus representation of an $\ell$-core $\lambda$, we will always mean the abacus of the form $\mathfrak{A}_\lambda = (0, b_1, \cdots, b_{\ell-1})$.[1] Using these abaci, we offer the following lemma that will allow us to rule out the existence of partitions that are simultaneously $\ell$-core and $\ell$-regular for all but finitely many $n$.

**Lemma 4.4.** *Suppose that $\mathfrak{A}_\lambda = (0, b_1, \ldots, b_{\ell-1})$ is the abacus corresponding to an $\ell$-core $\lambda$, and suppose that there is an integer $k \geq 0$ such that for each $1 \leq i \leq \ell - 1$ we have either $b_i \leq k$ or $b_i \geq k + \ell$. If there is at least one $j$ for which $b_j \geq k + \ell$, then $\lambda$ is not an $\ell$-regular partition.*

---

[1] These abaci correspond to those representations of $\lambda$ without parts of size 0.

**Remark.** Let $\mathfrak{A}_\lambda = (0, b_1, \cdots, b_{\ell-1})$ be the abacus of an $\ell$-core $\lambda$. If $\min(b_1, \cdots b_{\ell-1}) \geq \ell$, then the proof of the lemma will show that $\lambda$ has a part of exact size $\ell$. These are the cases where one can choose $k = 0$ in the lemma.

*Proof.* By our hypothesis, we may fix $j$ for which $b_j \geq k + \ell$. Let $\delta$ denote the total number of columns with length at least $k + \ell$. Note that if $B_i$ and $B_{i'}$ are structure numbers corresponding to consecutive beads in column $j$ between rows $k + 1$ and $k + \ell$, then $|i - i'| = \delta$. Further, we have $|B_i - B_{i'}| = \ell$. Generalizing the observation that $B_{i-1} - B_i = \lambda_{i-1} - \lambda_i + 1$, we have

$$|\lambda_i - \lambda_{i'}| = |B_i - B_{i'}| - \delta = \ell - \delta.$$

In particular, the difference between parts corresponding to consecutive beads in column $j$ between rows $k + 1$ and $k + \ell$ is fixed and coprime to $\ell$. As a consequence, these parts form a modulus $\ell - \delta$ arithmetic progression consisting of $\ell$ values. Thus, the parts cover all residue classes modulo $\ell$, and so includes a part that is a multiple of $\ell$. □

**Example.** Let $\ell = 3$, and consider the 3-core abacus $(0, 4, 1)$ as shown below.

$$
\begin{array}{c|ccc}
1 & \cdot & \circ & \circ \\
2 & \cdot & \circ & \cdot \\
3 & \cdot & \circ & \cdot \\
4 & \cdot & \circ & \cdot \\
\end{array}
$$

We illustrate Lemma 4.4 with $k = 1$. Since $b_1 = 3 + 1 = 4$ and $b_2 = 1$, the lemma asserts that $\lambda$ has a part that is a multiple of 3. The structure numbers are found to be $B_1 = 10$, $B_2 = 7$, $B_3 = 4$, $B_4 = 2$, and $B_1 = 1$, and we compute that $\lambda_1 = 10 + 1 - 5 = 6$, $\lambda_2 = 4$, $\lambda_3 = 2$, and $\lambda_4 = \lambda_5 = 1$. In particular, $\lambda_1$ is a multiple of 3.

Finally, consider the abacus with the bead in row 4 removed. One easily checks that the corresponding partition is 3-regular, demonstrating that the condition on the size of the gap in column lengths cannot be relaxed.

With two more observations, we will be able to construct an abacus which gives an upper bound for the size of an $\ell$-regular $\ell$-core partition.

First, suppose that $\lambda$ and $\lambda'$ are $\ell$-cores with $\mathfrak{A}_\lambda = (0, b_1, \ldots, b_{\ell-1})$ and $\mathfrak{A}_{\lambda'} = (0, b'_1, \ldots, b'_{\ell-1})$. Then we say $\mathfrak{A}_\lambda \leq \mathfrak{A}_{\lambda'}$ if $b_i \leq b'_i$ for all $1 \leq i \leq \ell - 1$. This relation endows the set of $\ell$-core abaci with the structure of a directed partially ordered set. It is not hard to show that $\mathfrak{A}_\lambda \leq \mathfrak{A}_{\lambda'}$ implies $|\lambda| \leq |\lambda'|$.

For the purpose of obtaining $N_\ell$, the following lemma allows us to restrict our attention to those abaci where the $b_i$ are weakly increasing.

**Lemma 4.5.** *Suppose that $\lambda = (\lambda_1, \ldots, \lambda_s)$ is an $\ell$-core partition of $n$ with abacus $\mathfrak{A}_\lambda = (0, b_1, \ldots, b_{\ell-1})$. If there exist $1 \leq i < j \leq \ell - 1$ for which $b_j < b_i$, then the abacus $\mathfrak{A}'$ obtained by swapping $b_i$ and $b_j$ represents an $\ell$-core partition $\lambda'$ with $\lambda' \vdash n' > n$.*

*Proof.* We may write

$$n = \sum_{k=1}^s \lambda_k = \sum_{k=1}^s (B_k + k - s) = \sum_{k=1}^s B_k + \sum_{k=1}^s (k - s),$$

and likewise $n' = \sum_{k=1}^s B'_k + \sum_{k=1}^s (k - s)$, where $s$ remains the same because we have not changed the total number of beads. Since the second sum is the same in both expressions, it suffices to prove

that $\sum_{i=1}^{s} B_k < \sum_{i=1}^{s} B'_k$. Computing column-wise, we have

$$\sum_{k=1}^{s} B_k = \sum_{m=1}^{b_i} (3(m-1)+i) + \sum_{m=1}^{b_j} (3(m-1)+j) + \sum_{\substack{k=1 \\ k \neq i,j}}^{\ell-1} \sum_{m=1}^{b_k} (3(m-1)+k)$$

$$< \sum_{m=1}^{b_i} (3(m-1)+j) + \sum_{m=1}^{b_j} (3(m-1)+i) + \sum_{\substack{k=1 \\ k \neq i,j}}^{\ell-1} \sum_{m=1}^{b_k} (3(m-1)+k)$$

$$= \sum_{k=1}^{s} B'_k$$

as desired, where the inequality holds since $i < j$ and $b_i > b_j$. $\qquad\square$

4.2. **Proof of Theorem 4.1.** We aim to find an upper bound on $n$ such that $\lambda \vdash n$ can be an $\ell$-regular $\ell$-core partition. To do this, we will construct a partition $\Lambda$ such that $\lambda$ being an $\ell$-regular $\ell$-core implies $|\lambda| \leq |\Lambda|$. By Lemma 4.5, it suffices to restrict our attention to those $\ell$-cores whose abaci have weakly increasing column lengths. Suppose $\lambda$ is a weakly increasing $\ell$-regular $\ell$-core with abacus $\mathfrak{A}_\lambda = (0, b_1, \ldots, b_{\ell-1})$. By Lemma 4.4, we must have $\min(b_1, \ldots, b_{\ell-1}) = b_1 \leq \ell - 1$. By the same logic, we must have $b_i \leq i(\ell - 1)$ for all $1 \leq i \leq \ell - 1$.

Then if $\Lambda$ is the partition with abacus $\mathfrak{A}_\Lambda = (0, \ell - 1, 2(\ell - 1), \ldots, (\ell - 1)^2)$, we immediately have $\mathfrak{A}_\lambda \leq \mathfrak{A}_\Lambda$, which implies $|\lambda| \leq |\Lambda|$. Then $N_\ell := |\Lambda|$ gives an upper bound on $n$. By direct calculation, we find that

$$|\Lambda| = \sum_{i=1}^{s}(B_i + i - s) = \sum_{i=1}^{\ell-1}\sum_{j=1}^{i(\ell-1)} (\ell(j-1)+i) + \sum_{i=1}^{s}(i-s) = \frac{\ell^6 - 2\ell^5 + 2\ell^4 - 3\ell^2 + 2\ell}{24},$$

where $s = \ell(\ell-1)^2/2$, giving the desired conclusion.

## 5. PROOFS OF OUR RESULTS

We are now in a position to prove Theorems 1.1 and 1.3, and Corollaries 1.2 and 1.4.

*Proof of Theorem 1.1.* We note that Theorems 3.2 and 3.3 imply that

$$\lim_{n \to +\infty} \frac{p(n) - p_\ell(n)}{p(n)} = 1.$$

The claim now follows by combining Lemma 3.1 (2), Theorem 3.2 , and Theorem 3.4 (4-6). $\qquad\square$

*Proof of Corollary 1.2.* This claim follows from Theorem 1.1 by choosing primes $\ell \to +\infty$. $\qquad\square$

*Proof of Theorem 1.3.* By Theorem 4.1, every $\ell$-core partition of size $n > N_\ell$ has a part that is a multiple of $\ell$. Since every hook length of an $\ell$-core is not a multiple of $\ell$, it follows from Murnaghan-Nakayama that whenever $\lambda, \mu \vdash n$ are $\ell$-cores with $n > N_\ell$, we have $\chi_\lambda(\mu) = 0$. $\qquad\square$

*Proof of Corollary 1.4.* Thanks to Theorem 1.3, we have that $Z_\ell^*(n) = c_\ell(n)^2$ for sufficiently large $n$. The claimed asymptotics and inequalities follow from Theorem 3.4 (4-6). $\qquad\square$

## References

[1] K. Erdmann and G. Michler, *Blocks for symmetric groups and their covering groups and quadratic forms*, Contributions to algebra and geometry **37** (1996), 103-118.

[2] F. Garvan, D. Kim, and D. Stanton, *Cranks and t-cores*, Invent. Math. **101** (1990), 1-17.

[3] A. Granville and K. Ono, *Defect zero p-blocks for finite simple groups*, Trans. Amer. Math. Soc. **348** (1996), 331-347.

[4] P. Hagis, Jr., *Partitions with a restriction on the multiplicity of the summands*, Trans. Amer. Math. Soc. **155** (1971), 375-384.

[5] Y.-H. He, personal communication on May 7, 2022.

[6] G. James and A. Kerber, *Representation theory of the symmetric group*, Cambridge Univ. Press, 1984.

[7] A. Klyachko, *Modular forms and representations of symmetric groups, integral lattices, and finite linear groups*, Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov **116** (1982), 74-85.

[8] A. R. Miller, *On parity and characters of symmetric groups*, J. Comb. Th., Ser. A **162** (2019), 231-240.

[9] A. R. Miller, *The probability that a character value is zero for the symmetric group*, Math. Z. **277** (2014), 1011-1015.

[10] K. Ono and L. Sze, *4-core partitions and class numbers*, Acta Arith. **80** (1997), 249-272.

[11] S. Peluse, *On even entries in the character table of the symmetric group*, (arXiv: `https://arxiv.org/abs/2007.06652`), preprint.

[12] S. Peluse and K. Soundararajan, *Almost all entries in the character table of the symmetric group are multiples of any given prime*, (arXiv: `https://arxiv.org/abs/2010.12410`), J. Reine Angew. Math., to appear.

[13] B. E. Sagan, *The symmetric group: Representations, combinatorial algorithms, and symmetric functions*, Springer-Verlag, New York, 2001.

[14] R. P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Adv. Math., Cambridge Univ. Press, Cambridge, 1999.

Department of Mathematics, University of Virginia, Charlottesville, VA 22904
*Email address*: `egm3zq@virginia.edu`
*Email address*: `ken.ono691@virginia.edu`