CLASS NUMBERS AND SELF-CONJUGATE 7-CORES

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ABSTRACT. We investigate $\mathrm{sc}_7(n)$, the number of self-conjugate 7-core partitions of size n. It turns out that $\mathrm{sc}_7(n) = 0$ for $n \equiv 7 \pmod 8$. For $n \equiv 1, 3, 5 \pmod 8$, with $n \not\equiv 5 \pmod 7$, we find that $\mathrm{sc}_7(n)$ is essentially a Hurwitz class number. Using recent work of Gao and Qin, we show that

$$\mathrm{sc}_7(n) = 2^{-\varepsilon(n)-1} \cdot H(-D_n),$$

where $-D_n := -4^{\varepsilon(n)}(7n + 14)$ and $\varepsilon(n) := \frac{1}{2} \cdot (1 + (-1)^{\frac{n-1}{2}})$. This fact implies several corollaries which are of interest. For example, if $-D_n$ is a fundamental discriminant and $p \notin \{2, 7\}$ is a prime with $\mathrm{ord}_p(-D_n) \leq 1$, then for every positive integer k we have

$$\mathrm{sc}_7\left((n+2)p^{2k} - 2\right) = \mathrm{sc}_7(n) \cdot \left(1 + \frac{p^{k+1} - p}{p - 1} - \frac{p^k - 1}{p - 1} . \left(\frac{-D_n}{p}\right)\right), \tag{1}$$

where $\left(\frac{-D_n}{p}\right)$ is the Legendre symbol.

1. INTRODUCTION AND STATEMENT OF RESULTS

A *partition* of a non-negative integer n is any nonincreasing sequence of positive integers which sum to n. The partition function $p(n)$, which counts the number of partitions of n, has been studied extensively in number theory. In particular, Ramanujan proved that

$$\begin{aligned} p(5n + 4) &\equiv 0 \pmod 5, \\ p(7n + 5) &\equiv 0 \pmod 7, \\ p(11n + 6) &\equiv 0 \pmod{11}. \end{aligned}$$

Partitions also play a significant role in representation theory (for example, see [10]). Indeed, partitions of size n are used to define *Young tableaux*, and their combinatorial properties encode the representation theory of the symmetric group S_n. Moreover, the *t-core partitions* of size n play an important role in number theory (for example, see [6, 7, 12]) and the modular representation theory of S_n and A_n (for example, see Chapter 2 of [10], and [5, 7]). Recall that a partition is a t-core if none of the hook numbers of its Ferrers-Young diagram are multiples of t. If p is prime, then the existence of a p-core of size n is equivalent to the existence of a defect 0 p-block for both S_n and A_n.

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For positive integers t, we let $c_t(n)$ denote the number of t-core partitions of size n. If $t \in \{2,3\}$, then it is well-known that $c_t(n) = 0$ for *almost all* $n \in \mathbb{N}$. However, if $t \geq 4$, then $c_t(n) > 0$ for every positive integer n (see [7]).

The case where $t = 4$ is particularly interesting, as these partitions arise naturally in algebraic number theory. As usual, let $H(-D)$ denote the discriminant $-D < 0$ *Hurwitz class number*. For the discriminants considered here, $H(-D)$ is the number of inequivalent (not necessarily primitive) positive definite binary quadratic forms with discriminant $-D < 0$. In particular, if $-D \notin \{-3,-4\}$ is a fundamental discriminant, then $H(-D)$ is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$. Sze and the first author proved (see Theorem 2 of [12]) that

$$c_4(n) = \frac{1}{2}H(-32n-20).$$

Furthermore, for primes p and $N \in \mathbb{N}$ with $\mathrm{ord}_p(N) \leq 1$, they proved (see Corollary 2 of [12]) for positive integers k that

$$c_4\left(\frac{Np^{2k}-5}{8}\right) = c_4\left(\frac{N-5}{8}\right)\cdot\left(1+\frac{p^{k+1}-p}{p-1}-\frac{p^k-1}{p-1}\cdot\left(\frac{-N}{p}\right)\right), \tag{2}$$

where $\left(\frac{-N}{p}\right)$ is the Legendre symbol. These formulas implied earlier conjectures of Hirschhorn and Sellers [9].

Further relationships between integer partitions and class numbers are expected to be extremely rare. In this note we find one more instance where t-cores and class numbers are intimately related. To this end, we let $\mathrm{sc}_t(n)$ denote number of the *self-conjugate* t-core partitions of size n. These are t-core partitions which are symmetric with respect to the operation which switches the rows and columns of a Ferrers-Young diagram.

To formulate these results, for a positive odd integer n we define the negative discriminant

$$-D_n := \begin{cases} -28n-56 & \text{if } n \equiv 1 \pmod 4, \\ -7n-14 & \text{if } n \equiv 3 \pmod 4. \end{cases} \tag{3}$$

Theorem 1. *If $n \not\equiv 5 \pmod 7$ is a positive odd integer, then we have*

$$\mathrm{sc}_7(n) = \begin{cases} \frac{1}{4}H(-D_n) & \text{if } n \equiv 1 \pmod 4 \\ \frac{1}{2}H(-D_n) & \text{if } n \equiv 3 \pmod 8 \\ 0 & \text{if } n \equiv 7 \pmod 8. \end{cases}$$

Example. *Here we consider the case where $n = 9$. According to Theorem 1, we have that $\mathrm{sc}_7(9) = H(-308)/4$. One readily finds that there are eight equivalence classes of discriminant -308 binary quadratic forms. The reduced forms representing these classes are:*

$$2X^2+2XY+39Y^2,\quad 3X^2-2XY+26Y^2,\quad 3X^2+2XY+26Y^2,\quad 6X^2-2XY+13Y^2,$$
$$6X^2+2XY+13Y^2,\quad 7X^2+11Y^2,\quad 9X^2-4XY+9Y^2,\quad 9X^2+4XY+9Y^2.$$

There are fourteen 7-core partitions of $n = 9$. However, only two of them are self-conjugate. They are (subscripts are the hook numbers):

$$\begin{array}{ccc} \bullet_5 & \bullet_4 & \bullet_3 \\ \bullet_4 & \bullet_3 & \bullet_2 \\ \bullet_3 & \bullet_2 & \bullet_1 \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} \bullet_9 & \bullet_4 & \bullet_3 & \bullet_2 & \bullet_1 \\ \bullet_4 & & & & \\ \bullet_3 & & & & \\ \bullet_2 & & & & \\ \bullet_1 & & & & \end{array}.$$

This example illustrates the conclusion that $\mathrm{sc}_7(9) = 2 = H(-308)/4$.

Example. *For $n = 25$, we have that $-D_n = -756 = -3^2 \cdot 84$ and $H(-756) = 16$. Therefore, Theorem 1 implies that* $\mathrm{sc}_7(25) = H(-756)/4 = 4$.

Theorem 1 implies simple short finite formulas for $\mathrm{sc}_7(n)$. To this end, we recall the standard Kronecker character for a discriminant D. Define the Kronecker character $\chi_D(n)$ for positive integers n by

$$\chi_D(n) = \left(\frac{D}{n}\right) := \prod \left(\frac{D}{p_i}\right)^{a_i},$$

where $n = \prod p_i^{a_i}$ and $\left(\frac{D}{p}\right)$ is the Legendre symbol when p is an odd prime and

$$\left(\frac{D}{2}\right) := \begin{cases} 0 & \text{if } D \text{ is even} \\ (-1)^{(D^2-1)/8} & \text{if } D \text{ is odd.} \end{cases}$$

Corollary 2. *If $n \not\equiv 5 \pmod 7$ is a non-negative odd integer for which $-D_n$ is a fundamental discriminant, then*

$$\mathrm{sc}_7(n) = \begin{cases} -\frac{1}{4D_n} \sum_{m=1}^{D_n} \left(\frac{-D_n}{m}\right) m & \text{if } n \equiv 1 \pmod 4, \\ -\frac{1}{2D_n} \sum_{m=1}^{D_n} \left(\frac{-D_n}{m}\right) m, & \text{if } n \equiv 3 \pmod 8, \\ 0 & \text{if } n \equiv 7 \pmod 8. \end{cases}$$

Example. *If $n = 11$, then $-D_{11} = -91$. We then find that*

$$\mathrm{sc}_7(11) = -\frac{1}{182} \sum_{m=1}^{91} \left(\frac{-91}{m}\right) m = 1.$$

Hurwitz class numbers enjoy a host of multiplicative properties which can be formulated in terms of the Möbius function $\mu(d)$ and the divisor function $\sigma_1(n) := \sum_{1 \le d \mid n} d$. These imply the following simple corollary which is analogous to (2).

Corollary 3. *If $n \not\equiv 5 \pmod 7$ is a positive odd integer, and $-D_n$ is a fundamental discriminant, then for all odd integers f coprime to 7, we have*

$$\mathrm{sc}_7((n+2)f^2 - 2) = \mathrm{sc}_7(n) \sum_{1 \le d \mid f} \mu(d) \left(\frac{-D_n}{d}\right) \sigma_1(f/d).$$

Remark. *The cases where $f = p^k$ is a prime power coincide with (1).*

Example. *If $n = 11$, then we have that $-D_{11} = -91$. Now suppose that $f = 15$. Corollary 3 implies that*

$$\mathrm{sc}_7(2923) = \mathrm{sc}_7(11) \sum_{1\leq d|15} \mu(d) \left(\frac{-91}{d}\right) \sigma_1(15/d).$$

By direct calculation, since $\mathrm{sc}_7(11) = 1$, the right hand side of this expression equals

$$\sigma_1(15) + \sigma_1(5) - \sigma_1(3) - \sigma_1(1) = 25.$$

Using the q-series identities in the next section (for example (4)), one can indeed check directly that $\mathrm{sc}_7(2923) = 25$.

2. Proofs

Here we prove Theorem 1 and Corollaries 2 and 3.

Proof of Theorem 1. If t is a positive odd integer, then the generating function for $\mathrm{sc}_t(n)$ (see (2) of [8]) is

$$\sum_{n=0}^{\infty} \mathrm{sc}_t(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{2tn})^{\frac{t-1}{2}}(1+q^{2n-1})}{(1+q^{t(2n-1)})}. \tag{4}$$

Therefore, in terms of Dedekind's eta-function $\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty}(1-q^n)$, where $q := e^{2\pi i\tau}$, we have that

$$S(\tau) := \sum_{n=0}^{\infty} \mathrm{sc}_7(n) q^{n+2} = \frac{\eta(2\tau)^2\eta(14\tau)\eta(7\tau)\eta(28\tau)}{\eta(4\tau)\eta(\tau)}.$$

In particular, by the standard theory of modular forms (for example, see Chapter 1.4 of [11]), it follows that $S(\tau)$ is a holomorphic modular form of weight $3/2$ on $\Gamma_0(28)$ with Nebentypus $\chi_7(n) := \left(\frac{7}{\bullet}\right)$. In terms of theta functions, Alpoge found that (see Theorem 9 of [1])

$$S(\tau) = \frac{1}{14}\Theta_1(\tau) - \frac{1}{7}\Theta_2(\tau) + \frac{1}{14}\Theta_3(\tau),$$

where

$$\Theta_1(\tau) = \sum_{(x,y,z)\in\mathbb{Z}^3} q^{Q_1(x,y,z)} = \sum_{(x,y,z)\in\mathbb{Z}^3} q^{x^2+y^2+2z^2-yz},$$

$$\Theta_2(\tau) = \sum_{(x,y,z)\in\mathbb{Z}^3} q^{Q_2(x,y,z)} = \sum_{(x,y,z)\in\mathbb{Z}^3} q^{x^2+4y^2+8z^2-4yz}$$

and

$$\Theta_3(\tau) = \sum_{(x,y,z)\in\mathbb{Z}^3} q^{Q_3(x,y,z)} = \sum_{(x,y,z)\in\mathbb{Z}^3} q^{2x^2+2y^2+3z^2+2yz+2xz+2xy}.$$

As a result, we get that

$$\mathrm{sc}_7(n) = \frac{1}{14}R(Q_1; n+2) - \frac{1}{7}R(Q_2; n+2) + \frac{1}{14}R(Q_3; n+2), \tag{5}$$

where $R(Q_i; n+2)$ is the number of integral representations of $n+2$ by $Q_i(x,y,z)$.

To prove the theorem, we must relate these three theta functions to the Eisenstein series in the space $M_{\frac{3}{2}}(28, \chi_7)$. Thankfully, Gao and Qin [4] have already carried out these calculations. They produce three Eisenstein series which form a basis of this space (see Theorem 3.2 of [4]). These series are given by their Fourier expansions:

$$\begin{aligned}
g_1(\tau) &= 1 + 2\pi\sqrt{7}\sum_{n=1}^{\infty}\lambda(7n,28)\alpha(7n)\left(A(7,7n)-\frac{1}{7}\right)\sqrt{n}q^n,\\
g_2(\tau) &= \frac{2}{49}\pi\sqrt{7}\sum_{n=1}^{\infty}\lambda(7n,28)\alpha(7n)\sqrt{n}q^n,\\
g_3(\tau) &= 2\pi\sqrt{7}\sum_{n=1}^{\infty}\lambda(7n,28)\left(A(7,7n)-\frac{1}{7}\right)\sqrt{n}q^n.
\end{aligned}$$

The quantities $\alpha(m), A(p,m), \lambda(7m,28)$, $h_p(m)$ and $h'_p(m)$ are defined by

$$\alpha(m) := \begin{cases} 3\cdot 2^{-\frac{1+h_2(m)}{2}} & \text{if } h_2(m) \text{ is odd,}\\ 3\cdot 2^{-1-\frac{h_2(m)}{2}} & \text{if } h_2(m) \text{ is even and } h'_2(m)\equiv 1 \pmod 4,\\ 0, & \text{if } h_2(m) \text{ is even and } h'_2(m)\equiv 3 \pmod 8,\\ 2^{-\frac{h_2(m)}{2}} & \text{if } h_2(m) \text{ is even and } h'_2(m)\equiv 7 \pmod 8,\end{cases}$$

$$A(p,m) := \begin{cases} p^{-1}-(1+p)p^{-\frac{3+h_p(m)}{2}} & \text{if } h_p(m) \text{ is odd,}\\ p^{-1}-2p^{-1-\frac{h_p(m)}{2}} & \text{if } h_p(m) \text{ is even and } \left(\frac{-h'_p(m)}{p}\right)=-1,\\ p^{-1}, & \text{if } h_p(m) \text{ is even and } \left(\frac{-h'_p(m)}{p}\right)=1,\end{cases}$$

where $h_p(m)$ is the non-negative integer for which $p^{h_p(m)}||m$ and $h'_p(m) := \frac{m}{p^{h_p(m)}}$, and

$$\lambda(7m,28) = \begin{cases} \frac{49}{4\pi\sqrt{7m}}\cdot H(-7m), & \text{if } m\equiv 5 \pmod 8,\\ \frac{49}{12\pi\sqrt{7m}}\cdot H(-7m), & \text{otherwise.}\end{cases}$$

The formula for $\alpha(m)$, when $h'_2(m)\equiv 3,7 \pmod 8$, corrects a typographical error in [4].

Since the three Eisenstein series form a basis of this space, it is trivial to deduce that

$$\Theta_1(\tau) = g_1(\tau) - 3g_2(\tau),\ \Theta_2(\tau) = g_1(\tau) - \frac{3}{2}g_3(\tau),\ \Theta_3(\tau) = g_1(\tau) + 14g_2(\tau).$$

Using $h_7(7n) = 1$, $h'_7(7n) = n$, $h_2(7n) = 0$ and $h'_2(7n) = 7n$ for $(n,14) = 1$, we find that

$$A(7,7n) = \frac{1}{7} - \frac{8}{49},$$

and

$$\alpha(7n) := \begin{cases} \frac{3}{2} & \text{if } n \equiv 3 \pmod 4, \\ 1 & \text{if } n \equiv 1 \pmod 8, \\ 0 & \text{if } n \equiv 5 \pmod 8. \end{cases}$$

Combining these facts, for positive integers n we deduce that

$$R(Q_1; n) = 2\pi\sqrt{7n}\lambda(7n, 28)\left(A(7, 7n) - \frac{1}{7}\right)(\alpha(7n) - 3),$$

$$R(Q_2; n) = 2\pi\sqrt{7n}\lambda(7n, 28)\left(A(7, 7n) - \frac{1}{7}\right)(\alpha(7n) - \frac{3}{2}),$$

and

$$R(Q_3; n) = 2\pi\sqrt{7n}\lambda(7n, 28)\alpha(7n)\left(A(7, 7n) - \frac{1}{7} + \frac{14}{49}\right).$$

Finally, these expressions, for positive odd $n \not\equiv 5 \pmod 7$, simplify to

$$R(Q_1; n+2) = \begin{cases} 2H(-D_n) & \text{if } n \equiv 1 \pmod 4, \\ 8H(-D_n) & \text{if } n \equiv 3 \pmod 8, \\ 4H(-D_n) & \text{if } n \equiv 7 \pmod 8, \end{cases}$$

$$R(Q_2; n+2) = \begin{cases} 0 & \text{if } n \equiv 1 \pmod 4, \\ 2H(-D_n) & \text{if } n \equiv 3 \pmod 8, \\ 2H(-D_n) & \text{if } n \equiv 7 \pmod 8, \end{cases}$$

$$R(Q_3; n+2) = \begin{cases} \frac{3}{2}H(-D_n) & \text{if } n \equiv 1 \pmod 4, \\ 3H(-D_n) & \text{if } n \equiv 3 \pmod 8, \\ 0 & \text{if } n \equiv 7 \pmod 8. \end{cases}$$

The claimed formulas now follow from (5). ☐

Proof of Corollary 2. If $-D < 0$ is a fundamental discriminant, then it is well-known (for example, see (7.29) of [3]) that

$$H(-D) = -\frac{|O_K^\times|}{2D}\sum_{m=1}^{D}\left(\frac{-D}{m}\right)m,$$

where $K = \mathbb{Q}(\sqrt{-D})$ and $O_K^\times$ denotes the units in its corresponding ring of integers. The claim now follows from Theorem 1. ☐

Proof of Corollary 3. If $-D < 0$ is a fundamental discriminant of an imaginary quadratic field, then for every integer f it is known (p. 273, [2]) that

$$H(-Df^2) = \frac{H(-D)}{w(-D)} \cdot \sum_{1 \le d | f} \mu(d)\left(\frac{-D}{d}\right)\sigma_1(f/d),$$

where $w(-D)$ denotes half of the number of roots of unity in $\mathbb{Q}(\sqrt{-D})$. The claim now follows from Theorem 1.

□

References

[1] L. Alpoge, *Self-conjugate core partitions and modular forms*, J. Numb. Th. **140** (2014), 60-92.

[2] H. Cohen, *Sums involving the values at negative integers of L-functions of quadratic characters*, Math. Ann. **217** (1975), 271-285.

[3] D. Cox, *Primes of the Form $x^2 + ny^2$*. Wiley, New York, 1989.

[4] L. Gao and H. Qin, *Ternary quadratic forms and the class numbers of imaginary quadratic fields*, Comm. Algebra, **47**:11, 4605-4640.

[5] P. Fong and B. Srinivasan, *The blocks of finite general linear groups and unitary groups*, Invent. Math. **69** (1982), 109–153.

[6] F. Garvan, D. Kim and D. Stanton, *Cranks and t-cores*, Invent. Math. **101** (1990), 1-17.

[7] A. Granville and K. Ono, *Defect zero p-blocks for finite simple groups*, Trans. Amer. Math. Soc. **348** (1996), 331-347.

[8] C. Hanusa and R. Nath, *The number of self-conjugate core partitions*, J. Numb. Th. **133** (2013), 751-768.

[9] M. Hirschhorn and J. Sellers, *Some amazing facts about 4-cores*, J. Numb. Th. **60** (1996), 51-69.

[10] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Addison-Wesley, Reading, 1979.

[11] K. Ono, *The web of modularity: Arithmetic of the coefficients of modular forms and q-series*, Amer. Math. Soc., Providence, 2004.

[12] K. Ono and L. Sze, *4-core partitions and class numbers*, Acta Arith. 80 (1997), 249-272.

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