MULTIQUADRATIC FIELDS GENERATED BY CHARACTERS OF A_n

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ABSTRACT. For a finite group G, let K(G) denote the field generated over \mathbb{Q} by its character values. For n > 24, G. R. Robinson and J. G. Thompson [6] proved that

 $K(A_n) = \mathbb{Q}\left(\left\{\sqrt{p^*} : p \le n \text{ an odd prime with } p \ne n-2\right\}\right),$

where $p^* := (-1)^{\frac{p-1}{2}} p$. Confirming a speculation of Thompson [7], we show that arbitrary suitable multiquadratic fields are similarly generated by the values of A_n -characters restricted to elements whose orders are only divisible by ramified primes. To be more precise, we say that a π -number is a positive integer whose prime factors belong to a set of odd primes $\pi :=$ $\{p_1, p_2, \ldots, p_t\}$. Let $K_{\pi}(A_n)$ be the field generated by the values of A_n -characters for even permutations whose orders are π -numbers. If $t \geq 2$, then we determine a constant N_{π} with the property that for all $n > N_{\pi}$, we have

$$K_{\pi}(A_n) = \mathbb{Q}\left(\sqrt{p_1^*}, \sqrt{p_2^*}, \dots, \sqrt{p_t^*}\right).$$

1. INTRODUCTION AND STATEMENT OF RESULTS

If G is a finite group, then let K(G) denote the field generated over \mathbb{Q} by all of the Gcharacter values. In stark contrast to the case of symmetric groups, where $K(S_n) = \mathbb{Q}$, G. R. Robinson and J. G. Thompson [6] proved for alternating groups that the $K(A_n)$ are generally large multiquadratic extensions. In particular, for n > 24, they proved that

 $K(A_n) = \mathbb{Q}\left(\left\{\sqrt{p^*} : p \le n \text{ an odd prime with } p \ne n-2\right\}\right),$

where $m^* := (-1)^{\frac{m-1}{2}}m$ for any odd integer m.

In a letter [7] to the second author in 1994, Thompson asked for a refinement of this result that mirrors the Kronecker-Weber Theorem and the theory of complex multiplication, where abelian extensions are generated by the values of $e^{2\pi i x}$ and modular functions respectively, at arguments that determine the ramified primes. Instead of employing special analytic functions at designated arguments, which is the gist of *Hilbert's 12th Problem* [4], Thompson offered the characters of A_n evaluated at elements whose orders are only divisible by ramified primes.

To formulate this problem, we let $\pi := \{p_1, p_2, \ldots, p_t\}$ denote a set of $t \ge 2$ distinct odd primes¹ listed in increasing order. A π -number is a positive integer whose prime factors belong to π . The speculation is that $K_{\pi}(A_n)$, the field generated by the values of A_n -characters restricted to elements $\sigma \in A_n$ with π -number order, generally generates $\mathbb{Q}(\sqrt{p_1^*}, \sqrt{p_2^*}, \ldots, \sqrt{p_t^*})$. However, an inspection of the character tables for the first few A_n casts doubt on this speculation. For example, one easily finds that

$$\mathbb{Q} = K_{\{3,5\}}(A_{22}) = K_{\{5,7\}}(A_{1738}) = K_{\{7,11\}}(A_{159557})$$

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¹The phenomenon cannot hold when t = 1 for any $n \not\equiv 0, 1 \pmod{p}$.

by checking that the A_n -character values for even permutations with π -number order are in \mathbb{Z} .

Despite these discouraging examples, Thompson's speculation is indeed true for sufficiently large A_n . We let $\Omega_{\pi} := p_t^{4(p_{t-1})^2}$, and in turn we define

(1.1)
$$\mathcal{N}_{\pi}^{+} := p_t + 2^{2^{2^{t2\pi}}}$$

The formula for \mathcal{N}_{π}^+ provides a bound for this phenomenon. It can generally be improved when $t \geq 3$. To make this precise, we let $\mathbb{N}_{\pi}^{\square} = \{a_1, a_2, \ldots\}$ be the subset of π -numbers that are perfect squares in increasing order, and we let $S_{\pi}(n)$ denote the number of elements in $\mathbb{N}_{\pi}^{\square}$ not exceeding n. Choose a positive real number $B_{\pi} \geq 2^{2^{40}}$ for which

$$S_{\pi}(2n) - S_{\pi}(n) > 2^{20} \log_2 \log_2 n$$

for all $n \geq B_{\pi}$. Then choose a positive real number $C_{\pi} \geq B_{\pi}$ for which

$$\sum_{s=1}^{S_{\pi}(C_{\pi})} \|a_s\theta\|^2 > 2\log_2 B_{\pi} + 50$$

for all $\frac{1}{4B_{\pi}} \leq \theta \leq 1 - \frac{1}{4B_{\pi}}$, where ||x|| is the distance between x and the nearest integer. In terms of C_{π} , we can choose

(1.2)
$$\mathcal{N}_{\pi}^{-} := p_t + \max\left\{C_{\pi}^{10}, 2^{2^{2^{400}}}\right\}$$

We obtain the following confirmation of Thompson's speculation in terms of the bound $N_{\pi} := \min\{\mathcal{N}_{\pi}^+, \mathcal{N}_{\pi}^-\}$.

Theorem 1.1. Assuming the notation above, if $n > N_{\pi}$, then

$$K_{\pi}(A_n) = \mathbb{Q}\left(\sqrt{p_1^*}, \sqrt{p_2^*}, \dots, \sqrt{p_t^*}\right).$$

Remark. The proof of Theorem 1.1 uses the fact that $n - p_i$, for each $1 \le i \le t$, is the sum of distinct squares of π -numbers provided $n > N_{\pi}$. These Diophantine conditions guarantee that $\mathbb{Q}(\sqrt{p_i^*}) \subset K_{\pi}(A_n)$. However, these conditions are not necessary for these inclusions.

Remark. Although Theorem 1.1 requires that $t \ge 2$, it is simple to generate $\mathbb{Q}(\sqrt{p^*})$ for an odd prime p using permutations with cycle lengths that are π -numbers with $\pi = \{p, q\}$, where $q \ne p$ is an odd prime. As mentioned above, if $n > N_{\pi}$, then n - p is a sum of distinct squares of π -numbers. Then $K_{\pi_p}(A_n)$, the field generated by the values of the A_n -characters restricted to permutations with a single cycle of length p and other cycle lengths of such odd squares, satisfies $K_{\pi_p}(A_n) = \mathbb{Q}(\sqrt{p^*})$.

To prove Theorem 1.1, we follow the straightforward approach of Robinson and Thompson in [6]. In Section 2.1, we recall standard facts about the characters of A_n . Theorem 2.1 is the key device for relating cycle types to the surds $\sqrt{p^*}$. Then, in Section 2.2, we recall a classical result of J. W. Cassels [2] on partitions. We also recall recent work by J.-H. Fang and Y.-G. Chen [3] on a problem of Erdös which implies that every large positive integer is the sum of distinct squares of π -numbers when $\pi = \{p_1, p_2\}$. Theorem 1.1 follows immediately from these results.

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2. NUTS AND BOLTS AND THE PROOF OF THEOREM 1.1

2.1. Character theory for A_n . It is well known that the representation theory of S_n and A_n can be completely described using the partitions of n. In particular, a permutation $\sigma \in S_n$ has a cycle type that can be viewed as a partition of n, say $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. The cycle type determines the conjugacy class of the permutation, and so the irreducible representations of S_n can be indexed by (and constructed from) the partitions of n. It is also well known that the only conjugacy classes which split in A_n are those corresponding to partitions into distinct odd parts. There is a bijection between the set of partitions λ of n into distinct odd parts and the set of self-conjugate partitions γ of n which is realized by identifying the parts of each λ with the main hook lengths of some γ . Theorem 2.5.13 of [5] characterizes those A_n -character values that are not \mathbb{Z} -integral.

Theorem 2.1. Let $\sigma \in A_n$ be a permutation with cycle type given by a partition $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n, and let $d_{\lambda} := \prod_{i=1}^k \lambda_i$.

- (1) If λ does not have distinct odd parts, then the A_n -character values of σ are all in \mathbb{Z} .
- (2) If λ has distinct odd parts, then let γ be the self-conjugate partition of n with main hook lengths $\lambda_1, \lambda_2, \ldots, \lambda_k$, and let χ_{γ} be the A_n -character associated to γ . We have that

$$\chi_{\gamma}(\sigma) = \frac{1}{2} \left((-1)^{\frac{d_{\lambda}-1}{2}} \pm \sqrt{d_{\lambda}^*} \right),$$

where $d_{\lambda}^* := (-1)^{\frac{d_{\lambda}-1}{2}} \prod_{i=1}^k \lambda_i$. Moreover, every A_n -character χ which is not algebraically conjugate to χ_{γ} has $\chi(\sigma) \in \mathbb{Z}$.

2.2. Some facts about partitions into distinct parts. In 1959, Birch [1] proved a conjecture of Erdös on representations of sufficiently large integers by sums of distinct numbers of the form $p^a q^b$, where p, q are coprime and a, b are positive integers. In 2017, this result was quantified by Fang and Chen (see Theorem 1.1 of [3]).

Theorem 2.2 (Fang-Chen). For any coprime integers p, q > 1, there exist positive integers K and B with

$$K < 2^{2^{q^{2p}}}, \quad B < 2^{2^{2^{q^{2}}}}$$

such that every integer $n \geq B$ can be expressed as the sum of distinct terms taken from

$$\{p^a q^b \mid a \ge 0, \ 0 \le b \le K, \ a+b > 0, \ a, b \in \mathbb{Z}\}.$$

Motivated by Birch's earlier work, Cassels (see Theorem 1 of [2]) studied the problem of representing sufficiently large integers as sums of distinct elements from suitable integer sequences. A careful inspection of his paper gives the following theorem, which was proved using the "circle method".

Theorem 2.3 (Cassels). Suppose that $\mathbb{N}_T = \{a_1, a_2, ...\}$ is a subset of positive integers in increasing order. Let $S_T(n)$ denote the number of integers in \mathbb{N}_T not exceeding n. Choose $B_T \geq 2^{2^{40}}$ to be a positive real number for which

$$S_T(2n) - S_T(n) > 2^{20} \log_2 \log_2 n$$

for all $n \ge B_T$, and choose $C_T \ge B_T$ to be a positive real number for which

$$\sum_{s=1}^{S_T(C_T)} ||a_s\theta||^2 > 2\log_2 B_T + 50$$

for all $\frac{1}{4B_T} \leq \theta \leq 1 - \frac{1}{4B_T}$. If such B_T and C_T exist, then every positive integer $n > \max\left\{C_T^{10}, 2^{2^{2^{400}}}\right\}$ is expressible as a sum of distinct elements of \mathbb{N}_T .

2.3. **Proof of Theorem 1.1.** Let $\pi := \{p_1, \ldots, p_t\}$ be a set of distinct odd primes in increasing order. For $n > N_{\pi}$, we will establish, for each $1 \le i \le t$, that $n - p_i$ is a sum of distinct squares of π -numbers. These representations imply that $n = p_i + M_i$, where M_i is a sum of distinct squares of π -numbers. Theorem 2.1 then implies the theorem.

By Theorem 2.2, for each pair $1 \le i < j \le t$, we have that every integer $n \ge N_{i,j} := 2^{2^{\Omega_{\pi}(i,j)}}$, where $\Omega_{\pi}(i,j) := p_j^{4p_i^2}$, is a sum of distinct terms taken from the set

$$\left\{ p_i^{2a} p_j^{2b} : a, b \ge 0, a+b > 0, a, b \in \mathbb{Z} \right\}.$$

Obviously, the maximum bound occurs with p_{t-1} and p_t , and so the theorem follows for $n > \mathcal{N}_{\pi}^+$. Finally, Cassels' result, where \mathbb{N}_T denotes the set of squares of π -numbers, guarantees that Theorem 1.1 holds for all $n > \mathcal{N}_{\pi}^-$.

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