## ON DIVISORS OF MODULAR FORMS

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In celebration of Don Zagier's 65th birthday.

ABSTRACT. The denominator formula for the Monster Lie algebra is the product expansion for the modular function  $j(z) - j(\tau)$  given in terms of the Hecke system of  $\mathrm{SL}_2(\mathbb{Z})$ -modular functions  $j_n(\tau)$ . It is prominent in Zagier's seminal paper on traces of singular moduli, and in the Duncan-Frenkel work on Moonshine. The formula is equivalent to the description of the generating function for the  $j_n(z)$  as a weight 2 modular form with a pole at z. Although these results rely on the fact that  $X_0(1)$  has genus 0, here we obtain a generalization, framed in terms of polar harmonic Maass forms, for all of the  $X_0(N)$  modular curves. We use these functions to study divisors of modular forms.

#### 1. Introduction and statement of results

As usual, let  $j(\tau)$  be the  $SL_2(\mathbb{Z})$ -modular function defined by

$$j(\tau) = \sum_{n=-1}^{\infty} c(n)e^{2\pi i n\tau} := \frac{E_4(\tau)^3}{\Delta(\tau)} = e^{-2\pi i \tau} + 744 + 196884e^{2\pi i \tau} + \cdots,$$

where  $E_k(\tau) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n \tau}$  is the weight  $k \in 2\mathbb{N}$  Eisenstein series,  $\sigma_{\ell}(n) := \sum_{d|\ell} d^{\ell}$ ,  $B_k$  is the kth Bernoulli number, and  $\Delta(\tau) := (E_4(\tau)^3 - E_6(\tau)^2)/1728$ . By Moonshine (for example, see [12]),  $j(\tau)$  is the McKay-Thompson series for the identity (i.e., its coefficients are the graded dimensions of the Monster module  $V^{\natural}$ ). Moonshine also offers the striking infinite product

$$j(z) - j(\tau) = e^{-2\pi i z} \prod_{m>0, n\in\mathbb{Z}} \left(1 - e^{2\pi i m z} e^{2\pi i n \tau}\right)^{c(mn)},$$

the denominator formula for the Monster Lie algebra. This formula is equivalent to the following identity of Asai, Kaneko, and Ninomiya (see Theorem 3 of [1])

(1.1) 
$$H_z(\tau) := \sum_{n=0}^{\infty} j_n(z) e^{2\pi i n \tau} = \frac{E_4(\tau)^2 E_6(\tau)}{\Delta(\tau)} \frac{1}{j(\tau) - j(z)} = -\frac{1}{2\pi i} \frac{j'(\tau)}{j(\tau) - j(z)}.$$

The functions  $j_n(\tau)$  form a Hecke system. Namely, if we let  $j_0(\tau) := 1$  and  $j_1(\tau) := j(\tau) - 744$ , then the others are obtained by applying the normalized Hecke operator T(n)

$$(1.2) j_n(\tau) := j_1(\tau) \mid T(n).$$

Remark. The functions  $H_z(\tau)$  and  $j_n(\tau)$  played central roles in Zagier's [18] seminal paper on traces of singular moduli and the Duncan-Frenkel work [11] on the Moonshine Tower. Carnahan [8] has obtained similar denominator formulas for completely replicable modular functions.

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If  $z \in \mathbb{H}$ , then  $H_z(\tau)$  is a weight 2 meromorphic modular form on  $SL_2(\mathbb{Z})$  with a single pole (modulo  $SL_2(\mathbb{Z})$ ) at the point z. Using these functions, modular form avatars for divisors of meromorphic modular forms were defined in [7]. More precisely, if  $f(\tau)$  is a normalized weight k meromorphic modular form on  $SL_2(\mathbb{Z})$ , then the divisor modular form<sup>1</sup> is

(1.3) 
$$f^{\operatorname{div}}(\tau) := \sum_{z \in \operatorname{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} e_z \operatorname{ord}_z(f) H_z(\tau),$$

where  $e_z := 2/\mathrm{Stab}_z(\mathrm{SL}_2(\mathbb{Z}))$ . With  $\Theta := \frac{1}{2\pi i} \frac{d}{d\tau}$ , Theorem 1 of [7] asserts that

(1.4) 
$$f^{\text{div}}(\tau) = -\frac{\Theta(f(\tau))}{f(\tau)} + \frac{kE_2(\tau)}{12}.$$

Although these results rely on the fact that  $X_0(1)$  has genus 0, there is a natural extension for congruence subgroups. This extension requires polar harmonic Maass forms (see Section 2). Here we consider the modular curves  $X_0(N)$ . For  $n \in \mathbb{N}$ , we define a Hecke system of  $\Gamma_0(N)$  harmonic Maass functions  $j_{N,n}(\tau)$  in Section 3 which generalize the  $j_n(\tau)$ . One key property that we note about the functions  $j_{N,n}(\tau)$  is their growth in n-aspect, which is not easily described in terms of Fourier expansions. Instead, we use "Ramanujan-like" expansions, sums of the form

(1.5) 
$$\sum_{\lambda \in \Lambda_z} \sum_{(c,d) \in S_\lambda} \frac{1}{\lambda^k} e\left(-\frac{n}{\lambda} r_z(c,d,k)\right) e^{\frac{2\pi n \operatorname{Im}(z)}{\lambda}},$$

for some real numbers  $r_z(c, d, k)$  (see (3.2)),  $\Lambda_z$  a lattice in  $\mathbb{R}$  (see (3.3)), and  $S_\lambda$  the set of solutions to  $Q_z(c, d) = \lambda$  for a certain positive-definite binary quadratic form  $Q_z$  (see (3.4)).

In Section 2 we construct weight 2 polar harmonic Maass forms  $H_{N,z}^*(\tau)$  which generalize the  $H_z(\tau)$ . We have two cases for the  $H_{N,z}^*(\tau)$  which we consider separately. The following theorem summarizes the essential properties of these functions when  $z \in \mathbb{H}$ .

**Theorem 1.1.** If  $z \in \mathbb{H}$ , then  $H_{N,z}^*(\tau)$  is a weight 2 polar harmonic Maass form on  $\Gamma_0(N)$  which vanishes at all cusps and has a single simple pole at z. Moreover, the following are true:

(1) If  $z \in \mathbb{H}$  and  $\operatorname{Im}(\tau) > \max\{\operatorname{Im}(z), \frac{1}{\operatorname{Im}(z)}\}$ , then we have that

$$H_{N,z}^*(\tau) = \frac{3}{\pi \left[ \text{SL}_2(\mathbb{Z}) : \Gamma_0(N) \right] \text{Im}(\tau)} + \sum_{n=1}^{\infty} j_{N,n}(z) e^{2\pi i n \tau}.$$

- (2) For (N, n) = 1, we have  $j_{N,n}(\tau) = j_{N,1}(\tau) \mid T(n)$ .
- (3) For  $n \mid N$ , we have  $j_{N,n}(\tau) = j_{\frac{N}{n},1}(n\tau)$ .
- (4) As  $n \to \infty$ , we have

$$j_{N,n}(\tau) = \sum_{\substack{\lambda \in \Lambda_{\tau} \\ \lambda \le n}} \sum_{(c,d) \in S_{\lambda}} e\left(-\frac{n}{\lambda} r_{\tau}(c,d)\right) e^{\frac{2\pi n \operatorname{Im}(\tau)}{\lambda}} + O_{\tau}(n).$$

Five Remarks.

- (1) In Theorem 1.1 (1), the inequality on  $\text{Im}(\tau)$  is required for convergence.
- (2) For N=1, we have that  $H_{1,z}^*(\tau)=H_z(\tau)-E_2^*(\tau)$ , where  $E_2^*(\tau):=-\frac{3}{\pi \text{Im}(\tau)}+E_2(\tau)$  is the usual weight 2 nonholomorphic Eisenstein series, and we have that  $j_{1,n}(\tau)=j_n(\tau)$ .
- (3) The sums (1.5) were introduced by Hardy and Ramanujan [13] (see also [2, 3]) in their study of  $1/E_6$ . These results have been generalized [5, 6] to negative weight meromorphic modular forms. Theorem 1.1 (4) extends these results to weight 0 where the series are not absolutely convergent.

<sup>&</sup>lt;sup>1</sup>Note that this summation does not include the cusp  $i\infty$ .

- (4) Theorem 1.1 (4) gives asymptotics for  $j_{N,n}(z)$  in n-aspect. Individual  $j_{N,n}(z)$  are easily computed. By replacing  $c > \sqrt{n}/y$  with c > C in (3.10) and (3.11), one finds that the terms decay like  $C^{-1/2+\varepsilon}$  times a power of n. For  $c \leq C$ , the expansions in Proposition 3.1 decay exponentially in m.
- (5) If  $y \ge \text{Im}(Mz)$  for all  $M \in \Gamma_0(N)$ , then

(1.6) 
$$j_{N,n}(z) \approx e^{-2\pi i n z} + \sum_{\substack{c \ge 1 \\ N \mid c \gcd(c,d) = 1 \\ |cz+d|^2 = 1}} e\left(n\frac{d-a}{c}\right) e^{2\pi i n \overline{z}}.$$

The second case we consider are those  $H_{N,\rho}^*(\tau)$  where  $\rho$  is a cusp of  $X_0(N)$ . These functions are compatible with the  $H_{N,z}^*(\tau)$  considered in Theorem 1.1. More precisely, since  $z \mapsto H_{N,z}^*(\tau)$  is continuous (even harmonic) and  $\Gamma_0(N)$ -invariant, it follows that

(1.7) 
$$H_{N,\rho}^*(\tau) := \lim_{z \to \rho} H_{N,z}^*(\tau)$$

is well-defined and only depends on the equivalence class of  $\rho$ . The next result summarizes these functions' properties. We use the Kloosterman sums  $K_{i\infty,\rho}(0,-n;c)$  of (2.4) and the weight 2 harmonic Eisenstein series  $E_{2,N,\rho}^*(\tau)$  with constant term 1 at  $\rho$  and vanishing at all other cusps.

**Theorem 1.2.** We have that  $H_{N,\rho}^*(\tau) = -E_{2,N,\rho}^*(\tau)$ . Moreover, the following are true:

(1) We have

$$H_{N,\rho}^{*}(\tau) = \frac{3}{\pi \left[ \operatorname{SL}_{2}(\mathbb{Z}) : \Gamma_{0}(N) \right] \operatorname{Im}(\tau)} - \delta_{\rho,\infty} + \sum_{n=1}^{\infty} j_{N,n}(\rho) e^{2\pi i n \tau}, \quad \text{with}$$
$$j_{N,n}(\rho) := \lim_{\tau \to \rho} j_{N,n}(\tau) = \frac{4\pi^{2} n}{\ell_{\rho}} \sum_{\substack{c \geq 1 \\ N \mid c}} \frac{K_{i\infty,\rho}(0, -n; c)}{c^{2}},$$

where  $\delta_{\rho,\infty} := 1$  if  $\rho = i\infty$  and 0 otherwise.

- (2) For gcd(N, n) = 1, we have  $j_{N,n}(\rho) = \lim_{\tau \to \rho} j_{N,1}(\tau) \mid T(n)$ .
- (3) For  $n \mid N$ , we have  $j_{N,n}(\rho) = \lim_{\tau \to \rho} j_{\frac{N}{\tau},1}(n\tau)$ .

Two Remarks.

- (1) Recall that the Fourier expansion in Theorem 1.1 (1) is not valid as  $z \to i\infty$ .
- (2) The  $j_{N,n}(\rho)$  are explicit divisor sums, which we leave to the interested reader to verify. From a generalization of the Weil bound (3.9) one can obtain  $j_{N,n}(\rho) = O(n^{\frac{3}{2}})$ .

We turn to the task of extending (1.4) to generic  $\Gamma_0(N)$ . Suppose that f is a weight k meromorphic modular form on  $\Gamma_0(N)$ . In analogy with (1.3), we define the divisor polar harmonic Maass form

(1.8) 
$$f^{\text{div}}(\tau) := \sum_{z \in X_0(N)} e_{N,z} \operatorname{ord}_z(f) H_{N,z}^*(\tau),$$

where  $e_{N,z} := 2/\# \operatorname{Stab}_z(\Gamma_0(N))$  (we let  $e_{N,\rho} := 1$ ). Generalizing (1.4), we show the following.

**Theorem 1.3.** If  $S_2(\Gamma_0(N))$  denotes the space of weight 2 cusp forms on  $\Gamma_0(N)$ , then

$$f^{\mathrm{div}}(\tau) \equiv \frac{k}{4\pi \operatorname{Im}(\tau)} - \frac{\Theta(f(\tau))}{f(\tau)} \pmod{S_2(\Gamma_0(N))}.$$

Three Remarks.

- (1) The coefficient of  $1/\operatorname{Im}(\tau)$  in  $H_{N,z}^*(\tau)$  is independent of z. By the valence formula, summing over every element of  $X_0(N)$  in the definition of  $f^{\operatorname{div}}(\tau)$  multiplies this constant by  $\frac{k}{12} [\operatorname{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$ , giving the nonholomorphic correction term on the right-hand side of Theorem 1.3.
- (2) At first glance, definitions (1.3) and (1.8) might appear different for N=1. Indeed,  $H_{1,z}^*(\tau)=H_z(\tau)-E_2^*(\tau)$ , and the sum in (1.8) includes the cusp  $i\infty$  whereas (1.3) omits it. The quasimodular Eisenstein series  $E_2(\tau)$  in (1.4) and the valence formula guarantees that they coincide.
- (3) The formula in Theorem 1.3 has already been obtained by Choi using a regularized inner product due to Petersson, but without relating the Fourier coefficients of  $f^{\text{div}}$  to the polar harmonic Maass forms  $H_{N_z}^*$  (see Theorem 1.4 of [9]).

Theorem 1.3 can be used to numerically compute divisors of meromorphic modular forms  $f(\tau)$ , which, in general, is a difficult task (for example, see [10]). The idea is simple. The series  $-\frac{\Theta(f(\tau))}{f(\tau)}$  is the logarithmic derivative of  $f(\tau)$ , and this fact converts the points  $z \in \mathbb{H}$  in the divisor of  $f(\tau)$  into simple poles. These can be identified by the asymptotic properties of the coefficients of  $H_{N,z}^*(\tau)$  given in Theorem 1.1. This follows from Theorem 1.3 and the fact that coefficients of cusp forms satisfy Deligne's bound. This method is based on the following immediate corollary to Theorems 1.1–1.3.

Corollary 1.4. Suppose that  $f(\tau)$  is a meromorphic modular form of weight k on  $\Gamma_0(N)$  whose divisor is not supported at cusps. Let  $y_1$  be the largest imaginary part of any points in the divisor of  $f(\tau)$  lying in  $\mathbb{H}$ . Then if  $-\frac{\Theta(f(\tau))}{f(\tau)} =: \sum_{n \gg -\infty} a(n)q^n$   $(q = e^{2\pi i \tau})$ , we have that

$$y_1 = \limsup_{n \to \infty} \frac{\log |a(n)|}{2\pi n}.$$

Two Remarks.

- (1) We require  $\limsup$  in Corollary 1.4 because the a(n) can vanish on arithmetic progressions.
- (2) It would be interesting to develop a practical algorithm for numerically computing modular form divisors. The idea would be to carefully peel away poles of  $f^{\text{div}}(\tau)$  in descending order until one is left with a linear combination of functions  $E_{N,\rho}^*(\tau)$ .

**Example 1.** For the Eisenstein series  $E_4(\tau)$ , we have

$$-\frac{\Theta(E_4(\tau))}{E_4(\tau)} = -240q + 53280q^2 - 12288960q^3 + 2835808320q^4 - 654403831200q^5 + \cdots$$

The sequence  $\{b(n)\}_{n\geq 1} = \{\log |a(n)|/(2\pi n)\}_{n\geq 1}$  converges rapidly. Indeed, b(2) = 0.866066794..., and b(10) = 0.866025404... matches the first 16 digits of the limiting value. The divisor of  $E_4(\tau)$  is supported on a zero at  $\omega := (-1 + \sqrt{-3})/2$ . By (1.6), since  $\omega$  lies on the unit circle (implying that the second term on the right-hand side of (1.6) appears) for large n, a(n) should very nearly be  $\frac{1}{3}\left(e^{-2\pi i n\omega} + 2e^{2\pi i n\overline{\omega}}\right) = e^{-2\pi i n\omega}$ , which is very easily seen numerically.

**Example 2.** We consider  $f(\tau) := E_4(2\tau) + \frac{\eta^{16}(2\tau)}{\eta^8(\tau)}$ , where  $\eta(\tau)$  is Dedekind's eta-function. By the valence formula for  $\Gamma_0(2)$ , it has a single zero, say  $z_0$ , in  $X_0(2)$ . We find that

$$-\frac{\Theta(f(\tau))}{f(\tau)} = -q - 495q^2 + 659q^3 + 113233q^4 - 261211q^5 + \cdots$$

After the first 3000 terms the sequence  $\log |a(n)|/(2\pi n)$  stabilizes and offers  $\operatorname{Im}(z_0) \approx 0.4357$ . As  $f(\tau)$  has real coefficients and there is only one zero,  $-\overline{z}_0$  must be  $\Gamma_0(2)$ -equivalent to  $z_0$ . Thus, either  $\operatorname{Re}(z) = 0$ , 1/2 or z lies on the arc |2z - 1| = 1. The first two cases are easily excluded by the sign patterns of a(n), and the zero on the arc is easily approximated as  $z_0 \approx 0.2547 + 0.4357i$ .

This paper is organized as follows. In Section 2 we construct the weight 2 polar harmonic Maass forms  $H_{N,z}^*(\tau)$ . In Section 3 we relate its Fourier coefficients to the values of the weight 0 weak Maass forms at  $\tau = z$ . In other words, we prove Theorems 1.1 and 1.2. In Section 4 we show Theorem 1.3.

## 2. Weight 2 Polar Harmonic Maass forms

2.1. The  $H_{N,z}^*(\tau)$  when  $z \in \mathbb{H}$ . Define for  $z, \tau \in \mathbb{H}$  and  $s \in \mathbb{C}$  with Re(s) > 0

(2.1) 
$$P_{N,s}(\tau,z) := \sum_{M \in \Gamma_0(N)} \frac{\varphi_s\left(M\tau,z\right)}{j\left(M,\tau\right)^2 |j\left(M,\tau\right)|^{2s}}$$

with  $j(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau) := (c\tau + d)$  and

$$\varphi_s(\tau, z) := (\operatorname{Im}(z))^{1+s} (\tau - z)^{-1} (\tau - \overline{z})^{-1} |\tau - \overline{z}|^{-2s}.$$

These functions were introduced and investigated in the z-variable in [4], where it was shown that these are polar harmonic Maass forms. These functions are allowed to have poles in the upper half plane instead of only at the cusps. In this paper, we are interested in properties of  $P_{N,s}(\tau,z)$  as functions of  $\tau$ . A direct calculation shows that for  $L \in \Gamma_0(N)$ 

$$P_{N,s}(L\tau,z) = j(L,\tau)^2 |j(L,\tau)|^{2s} P_{N,s}(\tau,z).$$

In [4] it was shown, by a lengthy calculation, that the function  $P_{N,s}(\tau,z)$  has an analytic continuation to s=0, which we denote by  $\operatorname{Im}(z)\Psi_{2,N}(\tau,z)$ . Let  $\mathcal{H}_k(\Gamma_0(N))$  be the space of polar harmonic Maass forms with respect to  $\Gamma_0(N)$ . Lemma 4.4 of [4] then states that  $z\mapsto \operatorname{Im}(z)\Psi_{2,N}(\tau,z)\in \mathcal{H}_0(\Gamma_0(N))$ . In the  $\tau$  variable, these functions are also polar harmonic Maass forms, as the next proposition shows. For this, for  $w\in\mathbb{C}$ , let  $e(w):=e^{2\pi i w}$ , and

$$K(m, n; c) := \sum_{\substack{a, d \pmod{c} \\ ad \equiv 1 \pmod{c}}} e\left(\frac{md + na}{c}\right).$$

Moreover,  $I_k$  and  $J_k$  denote the usual I- and J-Bessel functions. The following proposition can be obtained by a careful inspection of the proof of Theorem 3.1 of [4].

**Proposition 2.1.** We have that  $\tau \mapsto \operatorname{Im}(z)\Psi_{2,N}(\tau,z) \in \mathcal{H}_2(\Gamma_0(N))$ . For  $\operatorname{Im}(\tau) > \max\{\operatorname{Im}(z), \frac{1}{\operatorname{Im}(z)}\}$ , its Fourier expansion (in  $\tau$ ) has the form

$$\begin{split} y\Psi_{2,N}(\tau,z) &= -\frac{6}{[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]\,\mathrm{Im}(\tau)} - 2\pi \sum_{m\geq 1} \left(e^{-2\pi i m z} - e^{-2\pi i m \overline{z}}\right) e^{2\pi i m \tau} \\ &- 4\pi^2 \sum_{m\geq 1} \sum_{\substack{n,c\geq 1\\N|c}} \sqrt{\frac{m}{n}} \frac{K(m,-n;c)}{c} I_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{2\pi i n z} e^{2\pi i m \tau} \\ &- 4\pi^2 \sum_{m\geq 1} \sum_{\substack{n,c\geq 1\\N|c}} \sqrt{\frac{m}{n}} \frac{K(m,n;c)}{c} J_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{-2\pi i n \overline{z}} e^{2\pi i m \tau} - 8\pi^3 \sum_{m\geq 1} m \sum_{\substack{c\geq 1\\N|c}} \frac{K(m,0;c)}{c^2} e^{2\pi i m \tau}. \end{split}$$

We then set

(2.2) 
$$H_{N,z}^*(\tau) := -\frac{\operatorname{Im}(z)}{2\pi} \Psi_{2,N}(\tau,z).$$

Remark. We have, as  $\tau \to z$ ,

(2.3) 
$$H_{N,z}^*(\tau) = \frac{1}{2\pi i e_{N,z}} \frac{1}{\tau - z} + O(1).$$

2.2. The  $H_{N,z}^*(\tau)$  for cusps. We require the Fourier expansion of the functions  $H_{N,\rho}^*(\tau)$  defined in (1.7). For any cusp  $\rho$  of  $\Gamma_0(N)$ , denote by  $\ell_\rho$  the cusp width and let  $M_\rho$  be a matrix in  $\mathrm{SL}_2(\mathbb{Z})$  with  $\rho = M_\rho i \infty$ . For two cusps  $\mathfrak{a}, \mathfrak{b}$  of  $\Gamma_0(N)$ , the generalized Kloosterman sums are

(2.4) 
$$K_{\mathfrak{a},\mathfrak{b}}(m,n;c) := \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{\infty}^{\ell_{\mathfrak{a}}} \setminus M_{\mathfrak{a}}^{-1} \Gamma_{0}(N) M_{\mathfrak{b}} / \Gamma_{\infty}^{\ell_{\mathfrak{b}}}} e\left(\frac{md}{\ell_{\mathfrak{b}}c} + \frac{na}{\ell_{\mathfrak{a}}c}\right)$$

with  $\Gamma_{\infty} := \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \}$ . Note that we have  $K_{i\infty,i\infty}(m,n;c) = K(m,n;c)$ .

**Lemma 2.2** (Lemma 5.4 of [4]). We have

$$H_{N,\rho}^*(\tau) = \frac{3}{\pi \left[ \operatorname{SL}_2(\mathbb{Z}) : \Gamma_0(N) \right] \operatorname{Im}(\tau)} - \delta_{\rho,\infty} + \frac{4\pi^2}{\ell_\rho} \sum_{n \ge 1} n \sum_{c \ge 1} \frac{K_{\rho,i\infty}(n,0;c)}{c^2} e^{2\pi i n \tau}.$$

The Fourier expansions in Lemma 2.2 yield a connection with the harmonic weight 2 Eisenstein series  $E_{2,N,\rho}^*(\tau)$  for  $\Gamma_0(N)$ . For Re(s) > 0, define

(2.5) 
$$E_{2,N,\rho,s}^{*}(\tau) := \sum_{M \in \Gamma_{\rho} \backslash \Gamma_{0}(N)} j(M_{\rho}M,\tau)^{-2} |j(M_{\rho}M,\tau)|^{-2s}.$$

Using the Hecke trick, it is well-known (cf. Satz 6 of [14]) that  $E_{2,N,\rho,s}^*(\tau)$  has an analytic continuation to s=0, denoted by  $E_{2,N,\rho}^*(\tau)$ . From the Fourier expansion given in Theorem 1 of [17], we see

(2.6) 
$$H_{N,\rho}^*(\tau) = -E_{2,N,\rho}^*(\tau).$$

- 3. The  $j_{N,n}(z)$  and the proofs of Theorems 1.1 and 1.2
- 3.1. The  $j_{N,n}(z)$ . The functions  $j_{N,n}(z)$  are constructed as analytic continuations of Niebur's Poincaré series [16]. To be more precise, set for  $n \in \mathbb{N}$  and Re(s) > 1

$$F_{N,-n,s}(z) := \sum_{M \in \Gamma_{\infty} \backslash \Gamma_{0}(N)} e\left(-n \operatorname{Re}(Mz)\right) \operatorname{Im}(Mz)^{\frac{1}{2}} I_{s-\frac{1}{2}}\left(2\pi n \operatorname{Im}(Mz)\right).$$

These functions are weak Maass forms of weight 0; instead of being annihilated by  $\Delta_0$ , they have eigenvalue s(1-s). To obtain an analytic continuation to s=1, one computes the Fourier expansion of  $F_{N,-n,s}(z)$  and uses

$$\lim_{s\to 1} y^{\frac{1}{2}} I_{s-\frac{1}{2}}\left(2\pi ny\right) = y^{\frac{1}{2}} I_{\frac{1}{2}}\left(2\pi ny\right) = \frac{1}{\pi\sqrt{n}} \sinh\left(2\pi ny\right) = \frac{e^{2\pi ny} - e^{-2\pi ny}}{2\pi\sqrt{n}}.$$

**Proposition 3.1** (Theorem 1 of [16]). The function  $F_{N,-n,s}(z)$  has an analytic continuation  $F_{N,-n}(z)$  to s=1 and  $F_{N,-n}(z) \in \mathcal{H}_0(\Gamma_0(N))$ . We have the Fourier expansion

$$F_{N,-n}(z) = \frac{e^{-2\pi i n z} - e^{-2\pi i n \overline{z}}}{2\pi \sqrt{n}} + c_N(n,0) + \sum_{m \ge 1} \left( c_N(n,m) e^{2\pi i m z} + c_N(n,-m) e^{-2\pi i m \overline{z}} \right),$$

where the coefficients are given by

$$c_N(n,m) := \sum_{\substack{c \ge 1 \\ N|c}} \frac{K(m,-n;c)}{c} \times \begin{cases} \frac{1}{\sqrt{m}} I_1\left(\frac{4\pi\sqrt{mn}}{c}\right) & \text{if } m > 0, \\ \frac{2\pi\sqrt{n}}{c} & \text{if } m = 0, \\ \frac{1}{\sqrt{|m|}} J_1\left(\frac{4\pi\sqrt{|m|n}}{c}\right) & \text{if } m < 0. \end{cases}$$

We then define the functions  $j_{N,n}(z)$  by

(3.1) 
$$j_{N,n}(z) := 2\pi \sqrt{n} F_{N,-n}(z).$$

For N=1, we recover the  $j_n(z)$  from the introduction up to the constant  $2\pi\sqrt{n}c_1(n,0)=24\sigma_1(n)$ .

3.2. **Proofs of Theorems 1.1 and 1.2.** In order to formally state Theorem 1.1 (4), for an arbitrary solution  $a, b \in \mathbb{Z}$  to ad - bc = 1, we define

(3.2) 
$$r_z(c,d) := ac|z|^2 + (ad + bc)\operatorname{Re}(z) + bd,$$

(3.3) 
$$\Lambda_z := \left\{ \alpha^2 |z|^2 + \beta \operatorname{Re}(z) + \gamma^2 > 0 : \alpha, \beta, \gamma \in \mathbb{Z} \right\},\,$$

(3.4) 
$$S_{\lambda} := \{ (c, d) \in N \mathbb{N}_0 \times \mathbb{Z} : \gcd(c, d) = 1 \text{ and } Q_z(c, d) = \lambda \},$$
$$Q_z(c, d) := c^2 |z|^2 + 2cd \operatorname{Re}(z) + d^2.$$

Note that although  $r_z(c,d)$  is not uniquely determined,  $e(-nr_z(c,d)/Q_z(c,d))$  is well-defined.

Proof of Theorem 1.1. (1) For  $m \in \mathbb{N}$ , inspecting the expansions in Propositions 2.1 and 3.1 yields that  $2\pi\sqrt{m}F_{N,-m}(z)$  is the coefficient of  $e^{2\pi im\tau}$  in  $-\operatorname{Im}(z)\Psi_{2,N}(\tau,z)/(2\pi)$ , yielding the claim. (2) Since  $\gcd(N,n)=1$ , T(n) commutes with the action of  $\Gamma_0(N)$ , and so it suffices to show that (by analytic continuation)  $f_n(z)=f_1(z)\mid T(n)$ , where

$$f_n(z) = f_{n,s}(z) := e(-n \operatorname{Re}(z)) (n \operatorname{Im}(z))^{\frac{1}{2}} I_{s-\frac{1}{2}} (2\pi n \operatorname{Im}(z)).$$

Since  $f_n(z)$  is a (nonholomorphic) Fourier coefficient, we use the formula  $(\gcd(n, N) = 1)$ 

(3.5) 
$$f(z) \mid T(n) = n \sum_{m \in \mathbb{Z}} \sum_{d \mid \gcd(m,n)} \frac{a_{\frac{d^2}{n} \operatorname{Im}(z)} \left(\frac{mn}{d^2}\right)}{d} e^{2\pi i m z},$$

if  $f(z) = \sum_{m \in \mathbb{Z}} a_{\text{Im}(z)}(m) e^{2\pi i m z}$  is a nonholomorphic modular function of weight 0. Write  $f_n(z) = f_n^*(\text{Im}(z)) e^{-2\pi i n z}$  with  $f_n^*(y) := (ny)^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi n y) e^{-2\pi n y}$ . The *m*th coefficient in (3.5) vanishes unless m = -n. Moreover, only d = n contributes, giving

$$f_1(z) \mid T(n) = f_n^* \left( n \operatorname{Im}(z) \right) e^{-2\pi i n z} = \left( n \operatorname{Im}(z) \right)^{\frac{1}{2}} I_{s-\frac{1}{2}} \left( 2\pi n \operatorname{Im}(z) \right) e^{-2\pi i n \operatorname{Re}(z)} = f_n(z).$$

(3) For  $n \mid N$ , we rewrite

$$\sum_{M \in \Gamma_{\infty} \backslash \Gamma_{0}(N)} f_{n}(Mz) = \sum_{M \in \Gamma_{\infty} \backslash \Gamma_{0}(N)} f_{1}(nMz).$$

Now, with  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ , we have  $nMz = \frac{anz+bn}{\frac{c}{n}nz+d}$  and  $\begin{pmatrix} a & bn \\ \frac{c}{n} & d \end{pmatrix}$  runs through  $\Gamma_{\infty} \setminus \Gamma_0(\frac{N}{n})$  if M runs through  $\Gamma_{\infty} \setminus \Gamma_0(N)$ , implying the claim for  $n \mid N$ .

(4) We first rewrite the claimed asymptotic formula in terms of the corresponding points Mz with  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\infty} \backslash \Gamma_0(N)$ . Directly plugging in and simplifying yields  $r_z(c,d)/Q_z(c,d) = \text{Re}(Mz)$  and  $\text{Im}(z)/Q_z(c,d) = \text{Im}(Mz)$ , so the claim in Theorem 1.1 (4) is equivalent to

(3.6) 
$$j_{N,n}(z) = \sum_{\substack{M \in \Gamma_{\infty} \backslash \Gamma_{0}(N) \\ n \operatorname{Im}(Mz) \ge \operatorname{Im}(z)}} e^{-2\pi i n M z} + O_{z}(n).$$

In order to show (3.6), we only expand the Fourier expansion for large c. That is to say, we write

$$j_{N,n}(z) = 2 \sum_{\substack{1 \le c \le \frac{\sqrt{n}}{\operatorname{Im}(z)} \gcd(c,d) = 1 \\ N|c}} \sum_{\substack{d \in \mathbb{Z} \\ N|c}} e(-n\operatorname{Re}(Mz)) \sinh(2\pi n\operatorname{Im}(Mz))$$

$$+ 2\pi \sqrt{n} \sum_{\substack{c > \frac{\sqrt{n}}{\operatorname{Im}(z)} \\ N|c}} \sum_{\substack{m \ge 1}} \frac{K(m, -n; c)}{\sqrt{m}c} I_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{2\pi i m z} + 4\pi^2 n \sum_{\substack{c > \frac{\sqrt{n}}{\operatorname{Im}(z)} \\ N|c}} \frac{K(0, -n; c)}{c^2}$$

$$+ 2\pi \sqrt{n} \sum_{\substack{c > \frac{\sqrt{n}}{\operatorname{Im}(z)} \\ N|c}} \sum_{\substack{m \ge 1 \\ N|c}} \frac{K(-m, -n; c)}{\sqrt{m}c} J_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{-2\pi i m \overline{z}}.$$

In order to obtain (3.6), we split the main terms with  $n \operatorname{Im}(Mz) \geq \operatorname{Im}(z)$  off and rewrite

(3.8) 
$$2\sinh(2\pi n\operatorname{Im}(Mz)) = e^{2\pi n\operatorname{Im}(Mz)} - e^{-2\pi n\operatorname{Im}(Mz)}.$$

The second term above is obviously bounded. Since

$$\operatorname{Im}(z) \le n \operatorname{Im}(Mz) = \frac{n \operatorname{Im}(z)}{c^2 \operatorname{Im}(z)^2 + (d + c \operatorname{Re}(z))^2}$$

implies that  $c \leq \sqrt{n}/\operatorname{Im}(z) \ll_z \sqrt{n}$  and  $|d| \leq |c\operatorname{Re}(z)| + \sqrt{n\operatorname{Im}(z)} \ll_z \sqrt{n}$ , the contribution to the error from the sum of the second terms in (3.8) yields an error of at most  $O_z(n)$ .

For the second, third, and fourth sums in (3.7), we use the Weil bound for Kloosterman sums

$$|K(m, -n; c)| \le \sqrt{\gcd(m, n, c)} \sigma_0(c) \sqrt{c} \ll \begin{cases} \sqrt{n} c^{\frac{1}{2} + \varepsilon} & \text{if } m = 0, \\ \sqrt{|m|} c^{\frac{1}{2} + \varepsilon} & \text{if } m \ne 0. \end{cases}$$

For the third sum in (3.7), this gives

(3.10) 
$$2\pi\sqrt{n} \sum_{\substack{c > \frac{\sqrt{n}}{\text{Im}(z)} \\ N|c}} \frac{K(0, -n; c)}{c^2} \ll n \sum_{\substack{c > \frac{\sqrt{n}}{\text{Im}(z)} \\ N|c}} c^{-\frac{3}{2} + \varepsilon} \ll_z n^{\frac{3}{4} + \varepsilon}.$$

Next note that for  $x \geq 0$  we have  $|J_1(x)| \leq I_1(x)$  by their series expansions. Since  $x \mapsto \frac{I_1(x)}{x}$  is monotonically increasing and grows at most exponentially, the contribution from the second and fourth terms in (3.7) may be bounded by, using (3.9),

$$\ll \sum_{\substack{c > \frac{\sqrt{n}}{\text{Im}(z)}}} \sum_{m \ge 1} \frac{|K(\pm m, -n; c)|}{\sqrt{m}c} I_1\left(\frac{4\pi\sqrt{mn}}{c}\right) e^{-2\pi m \operatorname{Im}(z)} \\
\ll \sqrt{n} \sum_{\substack{c > \frac{\sqrt{n}}{\text{Im}(z)}}} \sum_{m \ge 1} \frac{|K(\pm m, -n; c)|}{c^2} \frac{I_1\left(4\pi \operatorname{Im}(z)\sqrt{m}\right)}{4\pi \operatorname{Im}(z)\sqrt{m}} e^{-2\pi m \operatorname{Im}(z)} \\
\ll \sqrt{n} \sum_{m \ge 1} I_1\left(4\pi \operatorname{Im}(z)\sqrt{m}\right) e^{-2\pi m \operatorname{Im}(z)} \ll \sqrt{n}.$$

It remains to bound the terms in the first sum in (3.7) with  $|cz + d|^2 > n$ . Since each term gives a constant contribution, the terms with  $|d| < \sqrt{n} + |c\operatorname{Re}(z)|$  give an error term of at most  $O_z(n)$ .

We finally assume that  $|d| \ge \sqrt{n} + |c\operatorname{Re}(z)|$ . Since  $x \mapsto \frac{\sinh(x)}{x}$  is monotonically increasing and  $|cz + d|^2 > n$ , the remaining terms contribute

$$\left| \sum_{\substack{c \leq \frac{\sqrt{n}}{\operatorname{Im}(z)} \\ N \mid c}} \sum_{\substack{|d| \geq \sqrt{n} + |c\operatorname{Re}(z)| \\ \gcd(c,d) = 1,}} e\left( -n\operatorname{Re}\left(Mz\right) \right) \sinh\left(2\pi n\operatorname{Im}\left(Mz\right) \right) \right| \leq \sum_{\substack{c \leq \frac{\sqrt{n}}{\operatorname{Im}(z)} \\ \log d}} \sum_{\substack{|d| \geq \sqrt{n} + |cx| \\ \gcd(c,d) = 1,}} \sinh\left(\frac{2\pi n\operatorname{Im}(z)}{|cz + d|^2}\right)$$

$$\leq \sum_{\substack{c \leq \frac{\sqrt{n}}{\operatorname{Im}(z)} \\ \log d}} \sum_{\substack{|d| \geq \sqrt{n} \\ \gcd(c,d) = 1,}} \sinh\left(\frac{2\pi n\operatorname{Im}(z)}{|cz|}\right) \leq 2\pi \sqrt{n} \sum_{\substack{d \geq \sqrt{n} \\ d \geq \sqrt{n}}} \frac{n}{d^2} \frac{\sinh\left(2\pi\operatorname{Im}(z)\right)}{2\pi\operatorname{Im}(z)} = O_z(n),$$

This implies that the terms in the first sum in (3.7) with  $|cz+d|^2 > n$  contribute  $O_z(n)$ .

Proof of Theorem 1.2. (1) Let  $K_s$  denote the usual K-Bessel function. Expanding  $F_{N,-n,s}(z)$  at the cusp  $\rho$  as in Section 3.4 of [15], we obtain

$$F_{N,-n,s}(M_{\rho}z) = \frac{c_{\rho,s}(n,0)}{2s-1} (\operatorname{Im}(z))^{1-s} + \sum_{m \in \mathbb{Z} \setminus \{0\}} c_{\rho,s}(n,m) e^{2\pi i m \frac{\operatorname{Re}(z)}{\ell_{\rho}}} (\operatorname{Im}(z))^{\frac{1}{2}} K_{s-\frac{1}{2}} \left( \frac{2\pi |m| \operatorname{Im}(z)}{\ell_{\rho}} \right),$$
with  $c_{\rho,s}(n,m) := \sum_{c \geq 1} K_{i\infty,\rho}(m,-n;c) \times \begin{cases} \frac{2}{c\sqrt{\ell_{\rho}}} I_{2s-1} \left( \frac{4\pi\sqrt{mn}}{\ell_{\rho}c} \right) & \text{if } m > 0, \\ \frac{2\pi^{s} n^{s-\frac{1}{2}}}{\ell_{\rho}^{s} c^{2s} \Gamma(s)} & \text{if } m = 0, \\ \frac{2}{c\sqrt{\ell_{\rho}}} J_{2s-1} \left( \frac{4\pi\sqrt{|m|n}}{\ell_{\rho}c} \right) & \text{if } m < 0, \end{cases}$ 

The right-hand side is analytic at s=1, which gives the expansion of  $F_{N,-n}(z)$  at  $\rho$ . Plugging in  $K_{\frac{1}{2}}(y)=\sqrt{\frac{\pi}{2y}}e^{-y}$  and taking the limit  $z\to i\infty$ , we obtain

(3.12) 
$$j_{N,n}(\rho) = 2\pi\sqrt{n} \lim_{s \to 1^+} c_{\rho,s}(n,0) = \frac{4\pi^2 n}{\ell_\rho} \sum_{c>1} \frac{K_{i\infty,\rho}(0,-n;c)}{c^2}.$$

We have  $K_{i\infty,\rho}(0,-n;c) = K_{\rho,i\infty}(n,0;c)$ , since  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  runs through  $\Gamma_0(N)M_\rho/\Gamma_\infty^{\ell_\rho}$  iff  $-M^{-1} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$  runs through  $\Gamma_\infty^{\ell_\rho}\backslash M_\rho^{-1}\Gamma_0(N)$  in (2.4). Hence (2.6) yields the claim. (2) and (3) follow by taking limits  $\tau \to \rho$  in Theorem 1.1 (2) and (3), respectively. Using the growth in n of  $j_{N,n}(\rho)$  from (3.12), these limits may be taken termwise.

# 4. Proof of Theorem 1.3

Proof of Theorem 1.3. We show that the difference of both sides has no poles in  $\mathbb{H}$  and decays towards the cusps. We start by considering the points in  $\mathbb{H}$ . One easily computes that the residue of  $-\frac{\Theta(f(\tau))}{f(\tau)}$  at  $\tau = z$  equals  $\frac{1}{2\pi i} \operatorname{ord}_z(f)$ . Using (2.3) gives that the principal part at z agrees. In a cusp  $\rho$  one similarly sees that  $\frac{\Theta(f(\tau))}{f(\tau)}$  has no pole and its constant term equals  $\operatorname{ord}_{\rho}(f)$ . Using that the constant term of  $H_{N,z}^*(\tau)$  at  $\rho$  is -1 then gives the claim.

## References

[1] T. Asai, M. Kaneko, and H. Ninomiya, Zeros of certain modular functions and an application, Comm. Math. Univ. Sancti Pauli 46 (1997), 93–101.

- [2] B. Berndt, P. Bialek, and A. Yee, Formulas of Ramanujan for the power series coefficients of certain quotients of Eisenstein series, Int. Math. Res. Not. **2002** (2002), 1077–1109.
- [3] P. Bialek, Ramanujan's formulas for the coefficients in the power series expansions of certain modular forms, Ph. D. thesis, University of Illinois at Urbana-Champaign, 1995.
- [4] K. Bringmann and B. Kane, A problem of Petersson about weight 0 meromorphic modular forms, Research in Mathematical Sciences, accepted for publication.
- [5] K. Bringmann and B. Kane, Ramanujan and coefficients of meromorphic modular forms, J. Math. Pures Appl., accepted for publication.
- [6] K. Bringmann and B. Kane, Ramanujan-like formulas for Fourier coefficients of all meromorphic cusp forms, submitted for publication.
- [7] J. Bruinier, W. Kohnen, and K. Ono, The arithmetic of the values of modular functions and the divisors of modular forms, Compositio Math. 130 (2004), 552–566.
- [8] S. Carnahan, Generalized moonshine, II: Borcherds products, Duke Math. J. 161 (2012), no. 5, 893–950.
- [9] D. Choi, Poincaré series and the divisors of modular forms, Proc. Amer. Math. Soc. 138 (2010), no 10, 3393–3403
- [10] C. Delaunay, Critical and ramification points of the modular parametrization of an elliptic curve, J. Théor. Nombres Bordeaux 17 (2005), no. 1, 109–124.
- [11] J. Duncan and I. Frenkel, Rademacher sums, moonshine and gravity, Comm. Numb. Th. Phys. 5 (2011), 1–128.
- [12] J. Duncan, M. Griffin, and K. Ono, Moonshine, Res. Math. Sci. 2 (2015), A11.
- [13] G. Hardy and S. Ramanujan, On the coefficients in the expansions of certain modular functions, Proc. Royal Soc. A 95 (1918), 144–155.
- [14] E. Hecke, Analytische Funktionen und algebraische Zahlen, zweiter Teil, Abh. Math. Sem. Hamburg Univ. 3 (1924) 213–236.
- [15] H. Iwaniec, Spectral Methods of Automorphic Forms, Graduate Studies in Mathematics 53 (2002), ed. 2, American Mathematical Society, Providence, RI; Revista
- [16] D. Niebur, A class of nonanalytic automorphic functions, Nagoya Math. J. 52 (1973), 133–145 Matemática Iberoamericana, Madrid.
- [17] J. Smart, On modular forms of dimension -2, J Trans. Amer. Math. Soc. 116 (1965), 86-107.
- [18] D. Zagier, *Traces of singular moduli*, Motives, Polylogarithms, and Hodge Theory (Ed. F. Bogomolov and L. Katzarkov), Lect. Ser. **3** Intl. Press, Somerville, 2002, 209–244.P

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