# A NOTE ON NON-ORDINARY PRIMES 

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$$
\begin{aligned}
& \text { AbSTRACT. Suppose that } O_{L} \text { is the ring of integers of a number field } L \text {, and suppose that } \\
& \qquad f(z)=\sum_{n=1}^{\infty} a_{f}(n) q^{n} \in S_{k} \cap O_{L}[[q]]
\end{aligned}
$$

(note: $q:=e^{2 \pi i z}$ ) is a normalized Hecke eigenform for $\mathrm{SL}_{2}(\mathbb{Z})$. We say that $f$ is non-ordinary at a prime $p$ if there is a prime ideal $\mathfrak{p} \subset O_{L}$ above $p$ for which

$$
a_{f}(p) \equiv 0 \quad(\bmod \mathfrak{p})
$$

For any finite set of primes $S$, we prove that there are normalized Hecke eigenforms which are non-ordinary for each $p \in S$. The proof is elementary and follows from a generalization of work of Choie, Kohnen and the third author [1].

## 1. Introduction and Statement of Results

If $k \geq 4$ is even, then let $M_{k}$ (resp. $S_{k}$ ) denote the finite dimensional $\mathbb{C}$-vector space of weight $k$ holomorphic modular forms (resp. cusp forms) on $\mathrm{SL}_{2}(\mathbb{Z})$. Furthermore, let $M_{k}^{!}$ denote the infinite dimensional space of weakly holomorphic modular forms of weight $k$ with respect to $\mathrm{SL}_{2}(\mathbb{Z})$. Recall that a meromorphic modular form is weakly holomorphic if its poles (if any) are supported at cusps. We shall identify a modular form on $\mathrm{SL}_{2}(\mathbb{Z})$ by its Fourier expansion at infinity

$$
f(z)=\sum_{n \gg-\infty} a_{f}(n) q^{n},
$$

where $q:=e^{2 \pi i z}$.
Suppose that $O_{L}$ is the ring of integers of a number field $L$, and suppose that

$$
f(z)=\sum_{n=1}^{\infty} a_{f}(n) q^{n} \in S_{k} \cap O_{L}[[q]]
$$

is a normalized Hecke eigenform for $\mathrm{SL}_{2}(\mathbb{Z})$. We say that $f$ is non-ordinary at a prime $p$ if there is a prime ideal $\mathfrak{p} \subset O_{L}$ above $p$ for which

$$
a_{f}(p) \equiv 0 \quad(\bmod \mathfrak{p})
$$

Very little is known about the distribution of non-ordinary primes. We recall the following well-known open problem (see Gouvêa's expository article [2]).

[^0]Problem. Are there infinitely many non-ordinary primes for a generic normalized Hecke eigenform $f(z)$ ?

We do not solve this problem here. It remains open. However, we establish the following related result.

Theorem 1.1. If $S$ is a finite set of primes, then there are infinitely many normalized Hecke eigenforms for $\mathrm{SL}_{2}(\mathbb{Z})$ which are non-ordinary for each $p \in S$.

Remark. The proof of Theorem 1.1 relies on a general theorem about the Fourier coefficients of weakly holomorphic modular forms modulo $p$ (see Theorem [2.5). For normalized Hecke eigenforms, this general result incorporates classical results of Hatada 3] (in the case where $p=2$ and 3) and Hida [4, 5, 6] (for primes $p \geq 5$ ) on non-ordinary primes.

Remark. The proof of Theorem 1.1 is constructive. Suppose that $S=\left\{p_{1}, p_{2}, \ldots, p_{m}\right\}$ is a finite set of primes. Suppose that $k \geq 12$ is an even integer. If for each $p \in S$ there is a choice of $t \in A=\{4,6,8,10,14\}$ for which $(p-1) \mid(k-t)$, then every prime in $S$ is non-ordinary for every normalized Hecke eigenform $f \in S_{k}$. The earlier work of Choie, Kohnen and the third author [1] is eclipsed by this result thanks to the flexibility in the choice of $t$ above.

In Section 2 we recall certain facts about modular forms, and we prove Theorem 2.5, The proof is elementary. In Section 3 we obtain Theorem 1.1 as a simple consequence when $p \geq 5$, combining with the known result on $p=2,3$, and in Section 4 we offer some numerical examples.

## 2. Preliminaries

2.1. Nuts and Bolts. As usual, let $\Delta(z) \in S_{12}$ be the cusp form

$$
\begin{equation*}
\Delta(z):=q \prod_{n=1}^{\infty}\left(1-q^{n}\right)^{24}=q-24 q^{2}+\ldots \tag{2.1}
\end{equation*}
$$

and, for even $k \geq 4$, let $E_{k}(z) \in M_{k}$ be the normalized Eisenstein series

$$
\begin{equation*}
E_{k}(z):=1-\frac{2 k}{B_{k}} \sum_{n=1}^{\infty}\left(\sum_{1 \leq d \mid n} d^{k-1}\right) q^{n} \tag{2.2}
\end{equation*}
$$

where the rational numbers $B_{k}$ are the usual Bernoulli numbers given by the generating function

$$
\sum_{k=0}^{\infty} B_{k} \cdot \frac{t^{k}}{k!}=\frac{t}{e^{t}-1}=1-\frac{1}{2} t+\frac{1}{12} t^{2}-\ldots
$$

For convenience, we let $E_{0}(z):=1$. Finally, we let $j(z)$ be the usual modular function

$$
\begin{equation*}
j(z):=\frac{E_{4}(z)^{3}}{\Delta(z)}=q^{-1}+744+196884 q+\ldots \tag{2.3}
\end{equation*}
$$

Finally, for convenience, if $k \in 2 \mathbb{Z}$, then throughout we define $\delta(k) \in\{0,4,6,8,10,14\}$ so that

$$
\begin{equation*}
\delta(k) \equiv k \quad(\bmod 12) \tag{2.4}
\end{equation*}
$$

In the proof, we need the following propositions.
Proposition 2.1. A normalized Hecke eigenform is non-ordinary at $p$ if there is an $m \geq 1$ such that $a_{f}\left(p^{m}\right) \equiv 0(\bmod p)$.

Proof. This follows from the fact that $T_{p} f(z)=a_{f}(p) f(z)$ for every prime $p$ when $f(z)$ is a normalized Hecke eigenform of weight $k$. Here $T_{p}$ is the $p$-th Hecke operator. In particular, on prime power exponents, we have

$$
a_{f}(p) a_{f}\left(p^{m}\right)=a_{f}\left(p^{m+1}\right)+p^{k-1} a_{f}\left(p^{m-1}\right) \equiv a_{f}\left(p^{m+1}\right)(\bmod p)
$$

for every non-negative integer $n$. By induction, we find that

$$
a_{f}\left(p^{m}\right) \equiv a_{f}(p)^{m} \quad(\bmod p) .
$$

This proves the proposition.
The following well-known propositions play a central role in the proof of Theorem 2.5.
Proposition 2.2. If $p \geq 5$ is prime, then as a $q$-series, $E_{p-1}(z) \equiv 1(\bmod p)$.
Proof. This can be found on page 38 of [7].
Proposition 2.3. If $f(z)=\sum_{n \gg-\infty} a_{f}(n) q^{n} \in M_{2}^{!}$, then $a_{f}(0)=0$.
Proof. By a simple generalization of Lemma 2.34 of [7, it is known that every weaklyholomorphic modular form $h(z)$ of weight 2 may be represented as $P(j(z)) E_{14}(z) \Delta(z)^{-1}$, where $P(x)$ is a polynomial of $x$. Dropping the dependence on $z$ for convenience, we have the following well-known identities

$$
\begin{gathered}
-\frac{1}{2 \pi i} \frac{d}{d z} j=\frac{E_{14}}{\Delta}, \\
j^{w} \frac{d}{d z} j=\frac{1}{w+1} \frac{d}{d z} j^{w+1},
\end{gathered}
$$

where $w \in \mathbb{Z}_{\geq 0}$. Therefore, it follows that $h$ is the derivative of a polynomial in $j$, and so its constant term in the Fourier expansion is zero.

Remark. For more standard facts about modular forms the reader may see [7].
2.2. Our main technical result. In 2005 Choie, Kohnen and the third author proved the following (see Corollary 1.3 of [1]). This result recovered earlier aforementioned results of Hatada and Hida.

Theorem 2.4. Let $p$ be a prime, and suppose that $f(z)=\sum_{n=1}^{\infty} a_{f}(n) q^{n} \in S_{k}$ is a normalized Hecke eigenform. Let $L_{f}$ be the number field generated by the coefficients of $f(z)$, and let $\mathfrak{p} \in O_{L_{f}}$ be any prime ideal above $p$.
(1) If $p=2,3$, then

$$
a_{f}(p) \equiv 0 \quad(\bmod \mathfrak{p})
$$

(2) If $p \geq 5, \delta(k) \in\{4,6,8,10,14\}$ and $k \equiv \delta(k)(\bmod p-1)$, then

$$
a_{f}(p) \equiv 0 \quad(\bmod \mathfrak{p})
$$

Here we strengthen this result for primes $p \geq 5$ by extending it to all the $k$ without any condition on $\delta(k)$.
Theorem 2.5. Let $p \geq 5$ be prime, and suppose that $f(z)=\sum_{n \gg-\infty}^{\infty} a_{f}(n) q^{n} \in M_{k}^{!} \cap O_{L}[[q]]$, where $k \in 2 \mathbb{Z}$ and $O_{L}$ is the ring of algebraic integers of a number field $L$.
(1) Suppose that $a \geq 0$ and $m \in A=\{4,6,8,10,14\}$ are integers for which

$$
k-2 \leq(m-2) p^{a} .
$$

If $\operatorname{ord}_{\infty}(f)>-p^{a}$ and $(p-1) \mid(k-m)$, then for any integer $b \geq a$, we have

$$
a_{f}\left(p^{b}\right) \equiv-\frac{2 m}{B_{m}} a_{f}(0) \quad(\bmod p)
$$

(2) Suppose that $k \leq 2, r, s \in \mathbb{Z}_{\geq 0}$ and $t, u \in \mathbb{Z}_{>0}$ are integers for which

$$
2-k=r(p-1)+s p^{t}
$$

where $s \neq 2$. If $\operatorname{ord}_{\infty}(f)>-p^{u}, u \leq t$, then for any integer $v$ such that $u \leq v \leq t$, we have

$$
a_{f}\left(p^{v}\right) \equiv a_{f}(0) \equiv 0 \quad(\bmod p)
$$

Proof. The proofs in both cases begin with the construction of suitable weakly-holomorphic modular forms of weight $2-k$. The product of such forms with $f$ have weight 2 , and so Proposition 2.3 implies that their constant terms vanish.

For the case (1), first note that $(k-2)-(m-2) p^{b} \equiv k-m(\bmod p-1)$. As we have $(p-1) \mid(k-m)$ and $k-2 \leq(m-2) p^{b}$, we may find a non-negative integer $c$ such that

$$
2-k=c(p-1)-(m-2) p^{b} .
$$

Let $g_{m}$ be the function

$$
g_{m}:=j \frac{E_{6}^{\left(1+i^{m}\right) / 2}}{E_{4}^{\left(m+1+3 i^{m}\right) / 4}}=\left\{\begin{array}{ll}
j \frac{E_{6}}{E_{4}^{2}} & \text { for } m=4 \\
j \frac{1}{E_{4}} & \text { for } m=6 \\
j \frac{E_{6}}{E_{6}^{4}} & \text { for } m=8 \\
j \frac{1}{E_{1}^{2}} & \text { for } m=10 \\
j \frac{1}{E_{4}^{3}} & \text { for } m=14
\end{array} \in M_{2-m}^{!} .\right.
$$

Then we have

$$
g_{m}^{p^{b}} E_{p-1}^{c} \in M_{2-k}^{!} .
$$

That is to say, the constant term of $g_{m}^{p^{b}} E_{p-1}^{c} f$ is zero. From Proposition 2.2 we know that

$$
E_{p-1} \equiv 1(\bmod p)
$$

Then we have that constant term of $g_{m}^{p^{b}} f$ is zero modulo $p$. By using Fermat's little theorem to compute the multinomials, we get

$$
\begin{aligned}
g_{m}^{p^{b}} f & =\left(q^{-1}+744+O(q)\right)^{p^{b}}\left(1-504 q+O\left(q^{2}\right)\right)^{\frac{p^{b}\left(1+i^{m}\right)}{2}}\left(1+(-240) q+O\left(q^{2}\right)\right)^{\frac{p^{b}\left(m+1+3 i^{m}\right)}{4}} f \\
& \equiv\left(q^{-p^{b}}+744+O\left(q^{p^{b}}\right)\right)\left(1-252\left(1+i^{m}\right) q^{p^{b}}+O\left(q^{2 p^{b}}\right)\right) \\
& \left(1-60\left(m+1+3 i^{m}\right) q^{p^{b}}+O\left(q^{2 p^{b}}\right)\right) \sum_{n \gg-\infty}^{\infty} a_{f}(n) q^{n} \\
& \equiv\left(q^{-p^{b}}+432-60 m-432 i^{m}+O\left(q^{p^{b}}\right)\right) \sum_{n \gg-\infty}^{\infty} a_{f}(n) q^{n}(\bmod p)
\end{aligned}
$$

We already know $\operatorname{ord}_{\infty}(f)>-p^{a} \geq-p^{b}$, then we know the constant term $c_{m, p}$ of $g_{m}^{p^{b}} f$ must satisfy the following congruence

$$
c_{m, p} \equiv a_{f}\left(p^{b}\right)+\left(432-60 m-432 i^{m}\right) a_{f}(0)(\bmod p)
$$

As $c_{m, p}$ is known to be zero modulo $p$ and for $m \in A$,

$$
\frac{2 m}{B_{m}}=432-60 m-432 i^{m}
$$

we get the conclusion.
For the case (2), as we have $2-k=r(p-1)+s p^{t}$ and $s p^{t-u} \neq 2$, we can find $c_{1}, c_{2} \in \mathbb{Z}_{\geq 0}$ such that $4 c_{1}+6 c_{2}=s p^{t-u}$. Then we have

$$
\left(E_{4}^{c_{1}} E_{6}^{c_{2}}\right)^{p^{u}} E_{p-1}^{r} f \in M_{2}^{!}
$$

Hence we have that the constant term of $\left(E_{4}^{c_{1}} E_{6}^{c_{2}}\right)^{p^{u}} E_{p-1}^{r} f$ is zero. As

$$
\left(E_{4}^{c_{1}} E_{6}^{c_{2}}\right)^{p^{u}} E_{p-1}^{r} f \equiv\left(1+O\left(q^{p^{u}}\right)\right) f(\bmod p)
$$

and $\operatorname{ord}_{\infty}(f)>-p^{u}$, we know $a_{f}(0) \equiv 0(\bmod p)$. To prove the case of $a_{f}\left(p^{v}\right)$, for $u \leq v \leq t$, we may find $c_{1}^{\prime}, c_{2}^{\prime} \in \mathbb{Z}_{\geq 0}$ such that $4 c_{1}^{\prime}+6 c_{2}^{\prime}=s p^{t-v}$. Then we have

$$
j^{p^{v}}\left(E_{4}^{c_{1}^{\prime}} E_{6}^{c_{2}^{\prime}}\right)^{p^{v}} E_{p-1}^{r} f \in M_{2}^{!}
$$

Hence the constant term of $j^{p^{v}}\left(E_{4}^{c_{1}^{\prime}} E_{6}^{c_{2}^{\prime}}\right)^{p^{v}} E_{p-1}^{r} f$ is zero. As

$$
\left(j E_{4}^{c_{1}^{\prime}} E_{6}^{c_{2}^{\prime}}\right)^{p^{v}} E_{p-1}^{r} f \equiv\left(q^{-p^{v}}+744+240 c_{1}^{\prime}-504 c_{2}^{\prime}+O\left(q^{p^{v}}\right)\right) f(\bmod p)
$$

and $\operatorname{ord}_{\infty}(f)>-p^{u} \geq-p^{v}$, we get

$$
a_{f}\left(p^{v}\right)+\left(744+240 c_{1}^{\prime}-504 c_{2}^{\prime}\right) a_{f}(0) \equiv 0(\bmod p)
$$

Knowing that $a_{f}(0) \equiv 0(\bmod p)$, we get the conclusion.

## 3. Proof of Theorem 1.1

By Theorem [2.4, $p=2$ and 3 are non-ordinary for every normalized Hecke eigenform on $\mathrm{SL}_{2}(\mathbb{Z})$. Therefore, we may assume that $S$ consists only of primes $p \geq 5$.

For the given finite set of primes $S$, let $k_{S}(j, m):=j \prod_{p \in S}(p-1)+m$, where $j$ is an arbitrary non-negative integer, $m \in A$. For each $j$ and $m$ let $b_{S}(j, m)$ be any integer for which

$$
k_{S}(j, m)-2<(m-2) p^{b_{S}(j, m)}
$$

for all $p \in S$. Let $f=\sum_{n=1}^{\infty} a_{f}(n) q^{n}$ be any Hecke eigenform of weight $k_{S}(j, m)$. By Theorem $2.5(1)$, since $a_{f}(0)=0$, we have

$$
a_{f}\left(p^{b_{S}(j, m)}\right) \equiv 0 \quad(\bmod p)
$$

for all $p \in S$. Applying Proposition 2.1, we know that $f$ is non-ordinary for each $p \in S$. As $j$ can be chosen freely, we get the conclusion.

## 4. Examples

Example. Let $S=\{2,3,5,7,11,13,17,19\}$. In the following table we list some of the weights $k$ for which Hecke eigenforms are non-ordinary at each prime $p$.


In particular, we consider the case $k=26$ and check its non-ordinariness. We have the following $q$-expansion of the normalized weight 26 Hecke eigenform $f_{26}=\Delta E_{6} E_{4}^{2}$,

$$
\begin{aligned}
f_{26}(z)= & q-48 q^{2}-195804 q^{3}-33552128 q^{4}-741989850 q^{5}+9398592 q^{6}+39080597192 q^{7} \\
& +3221114880 q^{8}-808949403027 q^{9}+35615512800 q^{10}+8419515299052 q^{11} \\
& +6569640870912 q^{12}-81651045335314 q^{13}-1875868665216 q^{14} \\
& +145284580589400 q^{15}+1125667983917056 q^{16}-2519900028948078 q^{17} \\
& +38829571345296 q^{18}-6082056370308940 q^{19}+O\left(q^{20}\right) .
\end{aligned}
$$

We can easily check that $a_{f_{26}}(p) \equiv 0(\bmod p)$ for each $p \in S$. Of course we can also choose weights $k$ of the form $k=26+720 j$, for every $j \in \mathbb{N}$. Note that $720=[5-1,7-1,11-$ $1,13-1,17-1,19-1]$.

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