RAMANUJAN'S MOCK THETA FUNCTIONS

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ABSTRACT. In his famous deathbed letter, Ramanujan introduced the notion of a *mock theta function*, and he offered some alleged examples. Recent work by Zwegers has elucidated the theory encompassing these examples. They are *holomorphic parts* of special harmonic weak Maass forms. Despite this understanding, little attention has been given to Ramanujan's original definition. Here we prove that Ramanujan's examples do indeed satisfy his original definition.

1. Introduction and Statement of Results

Ramanujan's deathbed letter [1] gave tantalizing hints of his theory of *mock theta functions*. Thanks to Zwegers [2, 3], it is now known that these functions are essentially the holomorphic parts of weight 1/2 *harmonic weak Maass forms*[1] whose nonholomorphic parts are period integrals of weight 3/2 unary theta functions. This realization has many applications (e.g. [5, 6]).

Here we revisit Ramanujan's original definition from his deathbed letter [1]. After a discussion of the asymptotics of certain modular forms which are given as *Eulerian series*, he writes:

"...Suppose there is a function in the Eulerian form and suppose that all or an infinity of points $q = e^{2i\pi m/n}$ are exponential singularities and also suppose that at these points the asymptotic form of the function closes as neatly as in the cases of (A) and (B). The question is: - is the function taken the sum of two functions one of which is an ordinary theta function and the other a (trivial) function which is $O(1)$ at <u>all</u> the points $e^{2i\pi m/n}$? The answer is <u>it is not necessarily so.</u> When it is not so I call the function Mock ϑ-function. I have not proved rigorously that <u>it is not necessarily so</u>. But I have constructed a number of examples in which it is inconceivable to construct a ϑ-function to cut out the singularities of the original function."

Remark. By *ordinary theta function*, Ramanujan meant a *weakly holomorphic modular form* with weight $k \in \frac{1}{2}\mathbb{Z}$ on some $\Gamma_1(N)$ (see [7] for background). Recall that a weakly holomorphic modular form is a meromorphic modular form whose poles (if any) are supported at cusps.

Little attention has been given to Ramanujan's original definition, prompting Berndt to remark [8] that "*it has not been proved that any of Ramanujan's mock theta functions are really mock theta functions according to his definition.*" The following fact fills in this gap.

Theorem 1.1. *Suppose that $f(z) = f^-(z) + f^+(z)$ is a harmonic weak Maass form of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\Gamma_1(N)$, where $f^-(z)$ (resp. $f^+(z)$) is the nonholomorphic (resp. holomorphic) part of $f(z)$. If $f^-(z)$ is nonzero and $g(z)$ is a weight k weakly holomorphic modular form on any $\Gamma_1(N')$, then $f^+(z) - g(z)$ has exponential singularities as q approaches infinitely many roots of unity ζ.*

Remark. Harmonic weak Maass forms in this paper have principal parts at all cusps.

[1]These forms were defined recently by Bruinier and Funke [4].

As a corollary, we obtain the following fitting conclusion to Ramanujan's enigmatic question by proving that his alleged examples indeed satisfy his original definition (Note. Throughout, we let $q := e^{2\pi iz}$). More precisely, we prove the following.

Corollary 1.2. *Suppose that $M(z)$ is one of Ramanujan's mock theta functions, and let γ and δ be integers for which $q^\gamma M(\delta z)$ is the holomorphic part of a weight 1/2 harmonic weak Maass form. Then there does not exist a weakly holomorphic modular form $g(z)$ of any weight $k \in \frac{1}{2}\mathbb{Z}$ on any congruence subgroup $\Gamma_1(N')$ such that for every root of unity ζ we have*

$$\lim_{q\to\zeta} \left(q^\gamma M(\delta z) - g(z)\right) = O(1).$$

Remark. The limits in Corollary 1.2 are *radial* limits taken from within the unit disk.

As his letter indicates, Ramanujan was inspired by the intimate relationship between the exponential singularities of modular forms at roots of unity and the asymptotics of their corresponding Fourier coefficients. As a toy model of his question, we begin by considering the following question whose solution would have been clear to him: If $f(z)$ is a weight k_1 weakly holomorphic modular form which has some exponential singularities at cusps, then can there be another weakly holomorphic modular form of different weight k_2, say $g(z)$, that exactly cuts out its singularities at roots of unity? The answer is *no.* If such a $g(z)$ existed, then both $f(z)$ and $g(z)$ must have the same principal parts at all cusps, and at least one of these must be nonconstant. Without loss of generality, suppose that the principal part at the cusp infnity is nonconstant, and then consider the function $h(z) := f(z) - g(z)$. By hypothesis, $h(z)$ has bounded radial limits as q approaches every root of unity. Now, since $f(z)$ and $g(z)$ are modular on some common subgroup $\Gamma_1(N')$, then if we take $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N')$ with $cd \neq 0$, then we have

$$h\left(\tfrac{az+b}{cz+d}\right) = f\left(\tfrac{az+b}{cz+d}\right) - g\left(\tfrac{az+b}{cz+d}\right) = (cz+d)^{k_1} f(z) - (cz+d)^{k_2} g(z). \tag{1.1}$$

Letting $z \to i\infty$, we find that $f(z)$ and $g(z)$ cannot cut out the same exponential singularities at roots of unity because of the difference between the weights.

In the case of Ramanujan's examples, the situation is much more subtle, and this is the point of his last letter and this paper.

Example. Although a weakly holomorphic modular and a mock theta function cannot cut out each other's singularities, Ramanujan discusses a *near miss.* He considers his mock theta function

$$f(q) := 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots, \tag{1.2}$$

and he compares it to a q-series $b(q)$ which is essentially a weight 1/2 weakly holomorphic modular form. He then conjectures, as q approaches an even order $2k$ primitive root of unity ζ, that

$$\lim_{q\to\zeta} \left(f(q) - (-1)^k b(q)\right) = O(1).$$

Watson confirmed this in [9], and Folsom, Rhoades, and the second author went further by deriving formulas for the $O(1)$ numbers as explicit numbers in $\mathbb{Z}[\zeta]$.

Theorem 1.1 follows from recent developments in the theory of harmonic Maass forms; in particular we make use of the extended Petersson scalar product of Bruinier and Funke [4]. Their work implies the that a harmonic weak Maass form which is not a weakly holomorphic modular form must have a nonconstant principal part at some cusp. To obtain the corollary, we employ the theory of Poincaré series and the method of quadratic twists to first show that a putative modular form must have weight 1/2. Corollary 1.2 then follows by applying Theorem 1.1.

The paper is organized as follows. In §2 we recall the basic facts about harmonic weak Maass forms, and the pairing of Bruinier and Funke. In §3 we describe the construction of the Poincaré series. In §4 we conclude with the proof of Theorem 1.1 and Corollary 1.2.

2. Harmonic weak Maass forms

Here we recall some of the work of Bruinier and Funke [4] on harmonic weak Maass forms.

2.1. **Definitions.** Throughout we suppose that $k \in \frac{1}{2}\mathbb{Z}$. The usual weight k hyperbolic Laplacian operator is given by

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right), \tag{2.1}$$

where $z = x + iy$. For weights $k \in \frac{1}{2}+\mathbb{Z}$, we note that the level N of $\Gamma_1(N)$ must be a multiple of 4. In this case, we define ϵ_d for odd d by

$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases} \tag{2.2}$$

Definition 2.1. A *harmonic weak Maass form* of weight k on a congruence subgroup $\Gamma_1(N)$ is any smooth function $f\colon \mathbb{H} \to \mathbb{C}$ satisfying:

(1) For all $\gamma \in \Gamma_1(N)$,

$$f\left(\tfrac{az+b}{cz+d}\right) = \begin{cases} (cz+d)^k f(z) & \text{if } k \in \mathbb{Z} \\ \left(\tfrac{c}{d}\right)^{2k}\epsilon_d^{-2k}(cz+d)^k f(z) & \text{if } k \in \tfrac{1}{2}+\mathbb{Z}. \end{cases}$$

(2) We have that $\Delta_k f = 0$.

(3) There is a polynomial $P_f = \sum_{n\le 0} C_F(n)q^n \in \mathbb{C}[q^{-1}]$ such that $f(z) - P_f = O(e^{-\epsilon y})$ for some $\epsilon > 0$ as $y \to +\infty$. We require the analogous condition at all the cusps of $\Gamma_1(N)$.

We denote the space of weight k harmonic weak Maass forms on $\Gamma_1(N)$ by $H_k(\Gamma_1(N))$.

Three remarks.

1) The polynomials in Definition 2.1 (3) are the *principal parts* of $f(z)$ at cusps.

2) This space corresponds to $H_k^+(\Gamma_1(N))$ in the notation of [4].

3) Weakly holomorphic modular forms are harmonic weak Maass forms, however in this paper we will primarily be interested in Maass forms which are non-holomorphic.

2.2. **Fourier expansions.** Harmonic weak Maass forms have two components (see [4]), a holomorphic piece and a nonholomorphic piece. If we let $e(\alpha) := e^{2\pi i\alpha}$ and let $H_k(w) := e^{-w}\int_{-2w}^{\infty} e^{-t}t^{-k}dt$, then every $f(z) \in H_k(\Gamma_1(N))$ decomposes as $f(z) = f^-(z) + f^+(z)$, where

$$f^+(z) = \sum_{n\gg-\infty} c_f^+(n)q^n \quad \text{and} \quad f^-(z) = \sum_{n<0} c_f^-(n)H_k(2\pi ny)e(nx).$$

We refer to $f^+(z)$ as the *holomorphic part* and $f^-(z)$ as the *nonholomorphic part*.

Fact 2.2. *Suppose that $M(z)$ is one of Ramanujan's alleged examples of a mock theta function. Thanks to Zwegers [2, 3], there are integers γ and δ for which $q^\gamma M(\delta z) =: f^+(z)$ is the holomorphic part of a weight 1/2 harmonic weak Maass form $f(z)$ on a congruence subgroup $\Gamma_1(N)$. Moreover, the nonholomorphic part of this form is the period integral of a weight 3/2 unary theta function. In particular, there are finitely many positive integers $\delta_1, \ldots, \delta_s$ for which $c_f^-(n) = 0$ unless $n = -\delta_i m^2$ for some $1 \le i \le s$ and some integer m.*

2.3. **The Bruinier-Funke Pairing.** Here we recall the Bruinier and Funke pairing, defined using the operator $\xi_k := 2iy^k \cdot \overline{\frac{\partial}{\partial \bar{z}}}$, which induces a surjective map $\xi_{2-k} : H_{2-k}(\Gamma_1(N)) \to S_k(\Gamma_1(N))$ onto the space of weight k cusp forms on $\Gamma_1(N)$. The image $\xi_{2-k}(f)$ is nonzero if and only if f has a nonzero nonholomorphic part. Bruinier and Funke [4] used ξ_{2-k} to define a bilinear pairing $\{\cdot,\cdot\} : M_k(\Gamma_1(N)) \times H_{2-k}(\Gamma_1(N)) \to \mathbb{C}$ by

$$\{g, f\} := (g, \xi_{2-k} f)_k\,, \tag{2.3}$$

where $(\cdot,\cdot)_k$ is the usual Petersson scalar product. Here $M_k(\Gamma_1(N))$ denotes the space of weight k holomorphic modular forms on $\Gamma_1(N)$. Proposition 3.5 of [4] gives this pairing in terms of the coefficients of $g(z)$ and $f^+(z)$. In particular, suppose at a cusp ρ, that $g(z)$ has an expansion $\sum_n a(\rho,n)q^n$ and $f^+(z)$ has the expansion $\sum_n b(\rho,n)q^n$. They prove that

$$\{g, f\} = \sum_\rho \sum_{n \le 0} a(\rho, -n) b(\rho, n). \tag{2.4}$$

The first sum is over the components of a vector-valued form. In their work all forms have level 1, and higher level forms may be viewed as level 1 vector-valued forms organized by cusps.

This pairing has the important property that $\{\xi_{2-k} f, f\} = (\xi_{2-k} f, \xi_{2-k} f)_k \neq 0$ when $f^-(z) \neq 0$. However, since $\xi_k f$ is a cusp form, (2.4) immediately gives the following.

Theorem 2.3 (Bruinier, Funke)**.** *If $f(z) \in H_{2-k}(\Gamma_1(N))$ has nonzero nonholomorphic part, then $f(z)$ must have a nonconstant principal part at some cusp.*

3. Poincaré series

We require Maass-Poincaré series, which were considered previously in work of Niebur [10, 11]. Their principal parts will serve as a basis for the principal parts of the mock theta functions. For $s \in \mathbb{C}$ and $y \in \mathbb{R} - \{0\}$ we let $\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\operatorname{sgn}(y), s-\frac{1}{2}}(|y|)$, where $M_{\nu,\mu}$ is the usual M-Whittaker function which satisfies

$$\frac{\partial^2 u}{\partial z^2} + \left(-\frac{1}{4} + \frac{\nu}{z} + \frac{\frac{1}{4} - \mu^2}{z^2}\right) u = 0.$$

Since spaces of forms on $\Gamma_1(N)$ are a direct sum over the spaces of Maass forms on $\Gamma_0(N)$ with Nebentypus, it suffices to construct Poincaré series on $\Gamma_0(N)$ with arbitrary Nebentypus χ. For a positive integer m, we define $\phi_{-m,s}(z) := \mathcal{M}_s(-4\pi m y)e(-mx)$, and we define the Poincaré series on $\Gamma_0(N)$ with Nebentypus χ and weight $k \in \frac{1}{2} + \mathbb{Z}$ by

$$\mathcal{F}_k(-m, s, z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \left(\frac{c}{d}\right)^{-2k} \epsilon_d^{2k} \chi(d)^{-1} (\phi_{-m,s}|_k \gamma)(z). \tag{3.1}$$

It turns out that $\phi_{-m,s}(z)$ is an eigenfunction of Δ_k with eigenvalue $s(1-s)+(k^2-2k)/4$. Therefore $\mathcal{F}_k(-m,s,z)$ is a weak Maass form of weight k on Γ with character χ whenever the series is absolutely convergent. This is clear if $\Re(s)>1$ as $\phi_{-m,s}(z)=O(y^{\Re(s)-\frac{k}{2}})$ as $y\to 0$. To obtain a *harmonic* Maass form, we choose $s=\frac{k}{2}$ (or $s=1-\frac{k}{2}$ if $k<1$). Convergence for this choice of s for weight $k\in\frac{1}{2}+\mathbb{Z}$ Poincaré series is only questionable if $k=1/2$ or $k=3/2$. We are primarily interested in the case when $k=1/2$.

The Fourier expansion of such series is well known (for example, see [10, 12, 13, 14, 15]). We recall the *Kloosterman sum* of weight $k\in\frac{1}{2}+\mathbb{Z}$ for $\Gamma_0(N)$ with Nebentypus χ.

$$K_k(m,n,c,\chi):=\sum_{d \pmod{c}^\times}\left(\frac{c}{d}\right)^{-2k}\epsilon_d^{2k}\chi(d)e\left(\frac{m\overline{d}+nd}{c}\right), \tag{3.2}$$

where d runs through primitive residue classes mod c and $\overline{d}$ is the multiplicative inverse of d mod c. We then have the following.

Proposition 3.1. *If m is a positive integer, then the Poincaré series $\mathcal{F}_k(-m,z,s)$ for $\Gamma_0(N)$ with Nebentypus χ has the Fourier expansion*

$$\mathcal{F}_k(-m,z,s)=\mathcal{M}_s(-4\pi my)e(-mx)+\sum_{n\in\mathbb{Z}}c(n,y,s)e(nx),$$

where the coefficients $c(n,y,s)$ are given by

$$\begin{cases}\frac{2\pi i^{-k}\Gamma(2s)}{\Gamma(s-k/2)}\left|\frac{n}{m}\right|^{\frac{k-1}{2}}\sum\limits_{c>0,N|c}\dfrac{K_k(-m,n,c,\chi)}{c}J_{2s-1}\left(\dfrac{4\pi\sqrt{|mn|}}{c}\right)\mathcal{W}_s(4\pi ny), & n<0\\ \frac{2\pi i^{-k}\Gamma(2s)}{\Gamma(s+k/2)}\left|\frac{n}{m}\right|^{\frac{k-1}{2}}\sum\limits_{c>0,N|c}\dfrac{K_k(-m,n,c,\chi)}{c}I_{2s-1}\left(\dfrac{4\pi\sqrt{|mn|}}{c}\right)\mathcal{W}_s(4\pi ny), & n>0\\ \frac{4^{1-k/2}\pi^{1+s-k/2}i^{-k}|m|^{s-k/2}y^{1-s-k/2}\Gamma(2s-1)}{\Gamma(s+k/2)\Gamma(s-k/2)}\sum\limits_{c>0,N|c}\dfrac{K_k(-m,0,c,\chi)}{c^{2s}}, & n=0\end{cases}$$

In the proposition above, I_k is the usual modified Bessel function and J_k is the Bessel function of the first kind. If $s\geq 1$ and equals $k/2$ or $1-k/2$, then these Poincaré series converge and are harmonic weak Maass forms. For $k=1/2$ it is known that the formulas still hold. For completeness, we shall give brief remarks below concerning the convergence.

Before we discuss the weight 1/2 case, we stress that this proposition allows us to easily determine the asymptotics of the coefficients of holomorphic parts of harmonic weak Maass forms. This follows from the well-known asymptotic

$$I_k(x)\sim\frac{e^x}{\sqrt{2\pi x}}\left(1-\tfrac{4k^2-1}{8x}+\dots\right). \tag{3.3}$$

The Poincaré series constructed above have nonconstant principal parts only at the cusp infinity. We may similarly construct Poincaré series at any cusp h. We let $\mathcal{F}_k(-m,s,z,h)$ denote the Poincaré series which is defined by modifying (3.1) as

$$\mathcal{F}_k(-m,s,z,h):=\sum_{\gamma\in\Gamma_h\backslash\Gamma_0(N)}\left(\frac{c}{d}\right)^{-2k}\epsilon_d^{2k}\chi(d)^{-1}(\phi_{-m,s}|_k\gamma)(z),$$

where Γ_h is the stabilizer of h. As in the case of the cusp at infinity, we obtain a weak Maass form with order $-m$ principal part at the cusp h and constant principal parts at all other cusps.

These facts allow us to conclude with the following crucial fact.

Fact 3.2. *Suppose that $f(z)$ is a weight 1/2 harmonic weak Maass form with a nonconstant principal part at some cusp. Let $f_P(z)$ be the weight 1/2 harmonic weak Maass form that is a linear combination of Maass-Poincaré series which matches, up to constants, the principal parts of $f(z)$ at all cusps. By Theorem 1.1, it follows that $f(z) - f_P(z)$ is a weight 1/2 holomorphic modular form, which, by the Serre-Stark Basis Theorem (for example, see* [7]*), implies that $f(z) - f_P(z)$ is a linear combination of weight 1/2 unary theta functions. Therefore, the subexponential growth of the I-Bessel function, combined with the periodicity of the Kloosterman sums in n, when m and c are fixed, then implies that a positive proportion of the coefficients of the holomorphic part of $f^+(z)$ are nonzero. Indeed, this gives arithmetic progressions of coefficients with smooth asymptotic subexponential growth.*

Remark. We briefly discuss the convergence in Proposition 3.1 for weight 1/2 harmonic weak Maass forms. To show this, we need similar estimates for sums of the Kloosterman sums as in Theorem 4.1 of [14]. In that work the Kloosterman sums were rewritten as Salie-type sums, which were then estimated using the equidistibution of CM-points (similar results may also be found in [16]). It is clear that the shape of the Salie-type sums do not depend on the multiplier system in a crucial way. Alternatively, the more general case, results of Goldfeld and Sarnak in [17] and the spectral theory of automorphic forms apply. By the asymptotics for Bessel functions, it suffices to consider the continuation of the Selberg-Kloosterman zeta function

$$Z_{n,m}(s,\chi) := \sum_{c>0} \frac{K_k(-m,n,c,\chi)}{c^{2s}}. \tag{3.4}$$

Namely, for $k = 1/2$ we need to show convergence at $s = 3/4$. The convergence we require was shown for a special case in Theorem 2.1 of [18]. The general case follows *mutatis mutandis.*

Theorem 3.3. *If m is a positive integer, then $Z_{n,m}(s,\chi)$ is convergent at $s = 3/4$.*

4. The proof of Theorem 1.1 and Corollary 1.2

Here we prove Theorem 1.1 and Corollary 1.2.

4.1. **Proof of Theorem 1.1.** Suppose that $g(z)$ is a weakly holomorphic modular form on $\Gamma_1(N')$, for some N', which cuts out the exponential singularities of $f(z)$ as q approaches roots of unity. Then $h(z) := f(z) - g(z)$ is a harmonic weak Maass form of weight k on $\Gamma_1(\mathrm{lcm}(N, N'))$ with nonconstant nonholomorphic part. By Theorem 2.3, $h(z)$ has a nonconstant principal part at some cusp. Since the nonholomorphic part $f^-(z)$ exhibits exponential decay at cusps, it follows that $h(z)$ is also $O(1)$ as cusps. Suppose that $h(z)$ has a nonconstant principal part at infinity (a similar argument applies at other cusps). By choosing matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(\mathrm{lcm}(N, N'))$, combined with the fact that

$$\lim_{z \to i\infty} h\left(\tfrac{az+b}{cz+d}\right) = \lim_{z \to i\infty} (cz+d)^k h(z),$$

we find that infinitely many roots of unity are exponential singularities for $h(z)$.

4.2. **Proof of Corollary 1.2.** Suppose that $M(z)$ is one of Ramanujan's alleged examples of a mock theta function. Then there are integers γ and δ for which $q^{\gamma}M(\delta z) =: f^{+}(z)$ is the holomorphic part of the weight 1/2 harmonic weak Maass form. Now suppose that $g(z)$ is a weakly holomorphic modular form of some weight k which cuts out the exponential singularities of $f(z)$. Following the proof of Theorem 1.4 of [19], we can use Fact 2.2, Fact 3.2, and the theory of quadratic (and trivial) twists to obtain a weight 1/2 weakly holomorphic modular form $\widehat{f}(z)$. By Fact 3.2, this can be done so that $\widehat{f}(z)$ is nontrivial and has nonconstant principal parts at some cusp. Applying the same procedure to $g(z)$ gives a weakly holomorphic modular form $\widehat{g}(z)$. We then have that $\widehat{f}(z)$ and $\widehat{g}(z)$ cut out exactly the same exponential singularities at all roots of unity. By the discussion after Corollary 1.2, it then follows that $k = 1/2$. Therefore, if there is such a $g(z)$, then $f(z) - g(z)$ is a weight 1/2 harmonic weak Maass form which has a nonvanishing nonholomorphic part, which also has the property that $f^{+}(z) - g(z)$ has no exponential singularities at any roots of unity. This contradicts Theorem 1.1.

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