# UNIMODAL SEQUENCES AND QUANTUM AND MOCK MODULAR FORMS 

JENNIFER BRYSON, KEN ONO, SARAH PITMAN AND ROBERT C. RHOADES


#### Abstract

We show that the rank generating function $U(t ; q)$ for strongly unimodal sequences lies at the interface of quantum modular forms and mock modular forms. We use $U(-1 ; q)$ to obtain a quantum modular form which is "dual" to the quantum form Zagier constructed from Kontsevich's "strange" function $F(q)$. As a result we obtain a new representation for a certain generating function for $L$-values. The series $U(i ; q)=U(-i ; q)$ is a mock modular form, and we use this fact to obtain new congruences for certain enumerative functions.


## 1. Introduction and Statement of Results

A sequence of integers $\left\{a_{i}\right\}_{i=1}^{s}$ is a strongly unimodal sequence of size $n$ if it satisfies

$$
0<a_{1}<a_{2}<\cdots<a_{k}>a_{k+1}>a_{k+2}>\cdots>a_{s}>0
$$

for some $k$ and $a_{1}+\cdots+a_{s}=n$. Let $u(n)$ be the number of such sequences. The rank of such a sequence is $s-2 k+1$, the number of terms after the maximal term minus the number of terms that precede it.

By letting $t$ (resp. $t^{-1}$ ) keep track of the terms after (resp. before) a maximal term, we find that $u(m, n)$, the number of size $n$ and rank $m$ sequences, satisfies ${ }^{1}$

$$
\begin{equation*}
U(t ; q):=\sum_{m, n} u(m, n) t^{m} q^{n}=\sum_{n=0}^{\infty}(-t q ; q)_{n}\left(-t^{-1} q ; q\right)_{n} q^{n+1}=q+q^{2}+\left(t+1+t^{-1}\right) q^{3}+\ldots, \tag{1.1}
\end{equation*}
$$

where $(x ; q)_{n}:=(1-x)(1-x q)\left(1-x q^{2}\right) \cdots\left(1-x q^{n-1}\right)$ for $n \geq 1$ and $(x ; q)_{0}:=1$.
Example. The strongly unimodal sequences of size 5 are: $\{5\},\{1,4\},\{4,1\},\{1,3,1\},\{2,3\}$, $\{3,2\}$, and so $u(5)=6$. Respectively, their ranks are $0,-1,1,0,-1,1$.

The $q$-series $U(-1 ; q)$, the generating function for the number of size $n$ sequences with even rank minus the number with odd rank, is intimately related to Kontsevich's strange function ${ }^{2}$

$$
\begin{equation*}
F(q):=\sum_{n=0}^{\infty}(q ; q)_{n}=1+(1-q)+(1-q)\left(1-q^{2}\right)+(1-q)\left(1-q^{2}\right)\left(1-q^{3}\right)+\ldots \tag{1.2}
\end{equation*}
$$

The authors thank the NSF and the Asa Griggs Candler Fund for their generous support.
${ }^{1}$ In [1] $u(n)$ is denoted $u^{*}(n)$ and $U(1 ; q)$ is denoted $U^{*}(q)$.
${ }^{2}$ Zagier credits Kontsevich for relating $F(q)$ to Feynmann integrals in a lecture at Max Planck in 1997.

It is strange because it does not converge on any open subset of $\mathbb{C}$, but is well-defined at all roots of unity. Zagier [2] proved that this function satisfies the even "stranger" identity

$$
\begin{equation*}
F(q)=-\frac{1}{2} \sum_{n=1}^{\infty} n \chi_{12}(n) q^{\frac{n^{2}-1}{24}}, \tag{1.3}
\end{equation*}
$$

where $\chi_{12}(\bullet)=\left(\frac{12}{\bullet}\right)$. Neither side of this identity makes sense simultaneously. Indeed, the right hand side ${ }^{3}$ converges in the unit disk $|q|<1$, but nowhere on the unit circle. The identity means that $F(q)$ at roots of unity agrees with the radial limit of the right hand side.

We prove that $U(-1 ; q)$, which converges in $|q|<1$, also gives $F\left(q^{-1}\right)$ at roots of unity.
Theorem 1.1. If $q$ is a root of unity, then $F\left(q^{-1}\right)=U(-1 ; q)$.
Example. Here are two examples: $U(-1 ;-1)=F(-1)=3$ and $U(-1 ; i)=F(-i)=8+3 i$.
Remark. Th. 1.1 is analogous to the result of Cohen [3, 4] that $\sigma(q)=-\sigma^{*}\left(q^{-1}\right)$ for roots of unity $q$, for the well-known $q$-series $\sigma(q)$ and $\sigma^{*}(q)$ that Andrews, Dyson, and Hickerson [5] defined in their work on partition ranks.

Zagier [2] used (1.3) to obtain the following identity

$$
\begin{equation*}
e^{-\frac{t}{24}} \sum_{n=0}^{\infty}\left(1-e^{-t}\right)\left(1-e^{-2 t}\right) \ldots\left(1-e^{-n t}\right)=\sum_{n=0}^{\infty} \frac{T_{n}}{n!} \cdot\left(\frac{t}{24}\right)^{n}, \tag{1.4}
\end{equation*}
$$

where Glaisher's $T_{n}$ numbers (see (2.3) and A002439 in [6]) are the "algebraic factors" of $L\left(\chi_{12}, 2 n+2\right)$. As a companion to Th. 1.1, we use $U(-1 ; q)$ to give these same $L$-values.
Theorem 1.2. As a power series in $t$, we have that

$$
e^{\frac{t}{24}} \cdot U\left(-1 ; e^{-t}\right)=\sum_{n=0}^{\infty} \frac{T_{n}}{n!} \cdot\left(\frac{-t}{24}\right)^{n}=\frac{6 \sqrt{3}}{\pi^{2}} \cdot \sum_{n=0}^{\infty} \frac{(2 n+1)!}{n!} \cdot L\left(\chi_{12}, 2 n+2\right) \cdot\left(\frac{-3 t}{2 \pi^{2}}\right)^{n} .
$$

These results are related to the next theorem which gives a new quantum modular form. Following Zagier ${ }^{4}$ [4], a weight $k$ quantum modular form is a complex-valued function $f$ on $\mathbb{Q}$, or possibly $\mathbb{P}^{1}(\mathbb{Q}) \backslash S$ for some finite set $S$, such that for all $\gamma=\left(\begin{array}{cc}a & b \\ c & d\end{array}\right) \in \mathrm{SL}_{2}(\mathbb{Z})$ the function

$$
h_{\gamma}(x):=f(x)-\epsilon(\gamma)(c x+d)^{-k} f\left(\frac{a x+b}{c x+d}\right)
$$

satisfies a "suitable" property of continuity or analyticity. The $\epsilon(\gamma)$ are roots of unity, such as those in the theory of half-integral weight modular forms when $k \in \frac{1}{2} \mathbb{Z} \backslash \mathbb{Z}$. We prove that

$$
\begin{equation*}
\phi(x):=e^{-\frac{\pi i x}{12}} \cdot U\left(-1 ; e^{2 \pi i x}\right) \tag{1.5}
\end{equation*}
$$

is a weight $\frac{3}{2}$ quantum modular form. Since $\mathrm{SL}_{2}(\mathbb{Z})=\left\langle\left(\begin{array}{cc}1 & 1 \\ 0 & 1\end{array}\right),\left(\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right)\right\rangle$ and $\phi(x)-e^{\frac{\pi i}{12}} \cdot \phi(x+1)=0$, it suffices to consider $\left(\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right)$. The following theorem establishes the desired relationship on the larger domain $\mathbb{Q} \cup \mathbb{H}-\{0\}$, where $\mathbb{H}$ is the upper-half of the complex plane.

[^0]Theorem 1.3. If $x \in \mathbb{Q} \cup \mathbb{H}-\{0\}$, then

$$
\phi(x)+(-i x)^{-\frac{3}{2}} \phi(-1 / x)=h(x),
$$

where $(i x)^{-\frac{3}{2}}$ is the principal branch and

$$
h(x):=\frac{\sqrt{3}}{2 \pi i} \int_{0}^{i \infty} \frac{\eta(\tau)}{(-i(x+\tau))^{\frac{3}{2}}} d \tau-\frac{i}{2} e^{\frac{\pi i x}{6}}\left(e^{2 \pi i x} ; e^{2 \pi i x}\right)_{\infty}^{2} \cdot \int_{0}^{i \infty} \frac{\eta(\tau)^{3}}{(-i(x+\tau))^{\frac{1}{2}}} d \tau .
$$

Here $\eta(\tau):=e^{\frac{\pi i \tau}{12}}\left(e^{2 \pi i \tau} ; e^{2 \pi i \tau}\right)_{\infty}$ is Dedekind's eta-function. Moreover, taking $\eta(x)=0$ for $x \in \mathbb{R}, h: \mathbb{R} \rightarrow \mathbb{C}$ is a $C^{\infty}$ function which is real analytic everywhere except at $x=0$, and $h^{(n)}(0)=(-\pi i / 12)^{n} \cdot T_{n}$, where $T_{n}$ is the $n$th Glaisher number.

Remark. Zagier [2] proved that $e^{\frac{\pi i x}{12}} \cdot F\left(e^{2 \pi i x}\right)$ is a quantum modular form. Th. 1.3 gives a dual quantum modular form, one whose domain naturally extends beyond $\mathbb{Q}$ to include $\mathbb{H}$. This is somewhat analogous to the situation for $\sigma(q)$ and $\sigma^{*}(q)$ discussed above. Zagier constructed a quantum modular form from these $q$-series in Example 1 of [4].

Remark. Th. 1.3 implies that $\Phi(z):=\eta(z) \phi(z)$ behaves analogously to a weight 2 modular form for $\mathrm{SL}_{2}(\mathbb{Z})$ for $z \in \mathbb{H}$ with a suitable error function. Namely, $\Phi(z+1)=\Phi(z)$ and $\Phi(z)-z^{-2} \Phi\left(-\frac{1}{z}\right)=\eta(z) h(z)$, see also Th. 1.1 of $[7]$.

It turns out that $U(1 ; q)$ and $U( \pm i ; q)$ also possess deep properties. We have that $U(1 ; q)[1]$ is a mixed mock modular form, and $U( \pm i ; q)$ is a mock theta function (see $[8,9,10]$ ). We use these facts to study congruences for certain enumerative functions.

Theorem 1.4. If $3<\ell \not \equiv 23(\bmod 24)$ is prime, $\delta(\ell):=\left(\ell^{2}-1\right) / 24$ and $\ell \nmid k$, then for all $n$

$$
u\left(\ell^{2} n+k \ell-\delta(\ell)\right) \equiv 0 \quad(\bmod 2)
$$

Example. If $\ell=7$, then Th. 1.4 gives $u(49 n+a) \equiv 0(\bmod 2)$ for $a \in\{5,12,19,26,33,40\}$.
The nature of Th. 1.4 suggests the existence of a Hecke-type identity for $U(-1 ; q)$ analogous to those obtained for $\sigma(q)$ and $\sigma^{*}(q)$ in [5]. Here we obtain such an identity.

Theorem 1.5. We have that

$$
U(-1 ; q)=\sum_{n>0} \sum_{6 n \geq|6 j+1|}(-1)^{j+1} q^{2 n^{2}-\frac{j(3 j+1)}{2}}+2 \sum_{n, m>0} \sum_{6 n \geq|6 j+1|}(-1)^{j+1} q^{2 n^{2}+m n-\frac{j(3 j+1)}{2}} .
$$

These congruences appear to have refinements modulo 4. In analogy with the theory of partition ranks [11, 12, 13], we suspect that ranks also "explain" these congruences. Namely, let $u(a, b ; n)$ be the number of size $n$ strongly unimodal sequences with rank $\equiv a(\bmod b)$.

Conjecture 1.6. If $\ell \equiv 7,11,13,17(\bmod 24)$ is prime and $\left(\frac{k}{\ell}\right)=-1$, then for all $n$ we have

$$
\begin{equation*}
u\left(\ell^{2} n+k \ell-\delta(\ell)\right) \equiv 0 \quad(\bmod 4) \tag{1.6}
\end{equation*}
$$

Moreover, for $a \in\{0,1,2,3\}$ we have $u\left(a, 4 ; \ell^{2} n+k \ell-\delta(\ell)\right) \equiv 0(\bmod 2)$ and

$$
\begin{equation*}
u\left(0,4 ; \ell^{2} n+k \ell-\delta(\ell)\right) \equiv u\left(2,4 ; \ell^{2} n+k \ell-\delta(\ell)\right) \quad(\bmod 4) \tag{1.7}
\end{equation*}
$$

We have that $u(1,4 ; n)=u(3,4 ; n)$, and so the truth of (1.7) is a proposed explanation of (1.6). Therefore, it is natural to study $U( \pm 1 ; q)$ and the 3rd order mock theta function $[14,15,16]$

$$
U( \pm i ; q)=\Psi(q)=\sum_{n=1}^{\infty} \frac{q^{n^{2}}}{\left(q ; q^{2}\right)_{n}}=\sum_{n=0}^{\infty}\left(-q^{2} ; q^{2}\right)_{n} q^{n+1}=\frac{q}{\left(q^{4}\right)_{\infty}} \cdot \sum_{n \in \mathbb{Z}} \frac{(-1)^{n} q^{6 n(n+1)}}{1-q^{4 n+1}}
$$

Using this mock theta function we are able to obtain the following related congruences.
Theorem 1.7. If $(Q, 6)=1$, then there are arithmetic progressions $A n+B$ such that

$$
u(0,4 ; A n+B) \equiv u(2,4 ; A n+B) \quad(\bmod Q)
$$

Example. For $Q=5$ the cusp form in the proof of Th. 1.7 is annihilated by $T\left(11^{2}\right)$, and so if

$$
a(24 n-1):=u(0,4 ; n)-u(2,4 ; n) \quad(\bmod 5)
$$

$($ note. $a(n)=0$ if $n \not \equiv 23(\bmod 24))$, then for every $n \equiv 23,47(\bmod 120)$ we have that

$$
a(121 n)-\left(\frac{n}{11}\right) a(n)+a(n / 121) \equiv 0 \quad(\bmod 5)
$$

Since $\left(\frac{n}{11}\right)=0$ and $a(n / 121)=0$ when $11 \| n$, this gives congruences such as

$$
u(0,2 ; 73205 n+721) \equiv u(2,4 ; 73205 n+721) \quad(\bmod 5)
$$

## 2. Quantum properties of $U(-1 ; q)$

Here we prove the quantum properties of $U(-1 ; q)$. We first prove Th. 1.1 relating the values of Kontsevich's $F(q)$ and $U(-1 ; q)$ at roots of unity. We then prove Th. 1.2 giving a new representation of Zagier's $L$-value generating function, and we conclude with a proof of Th. 1.3.
2.1. Proof of Theorem 1.1. For $\xi$ a fixed $k$ th root of unity, define the polynomial

$$
C(X)=\sum_{n=0}^{k-1}\left(X-\xi^{-1}\right) \cdots\left(X-\xi^{-n}\right)
$$

We have the identity

$$
\begin{equation*}
C\left(\xi^{-1} X\right)=(X-1)^{2} C(X)-X\left(X^{k}-1\right)+X \tag{2.1}
\end{equation*}
$$

Define the functions $u_{a}(X)$ for $a \geq 1$ by

$$
\left(2-X^{k}\right) u_{a}\left(\xi^{-a} X\right)=C\left(\xi^{-a} X\right)-(1-X)^{2} \cdots\left(1-\xi^{-(a-1)} X\right)^{2} C(X)
$$

Hence for $a=k$ we have

$$
\begin{equation*}
X^{k} C(X)=u_{k}(X) \tag{2.2}
\end{equation*}
$$

Then we have

$$
\left(2-X^{k}\right)\left(u_{a+1}(X)-u_{a}(X)\right)=(1-\xi X)^{2} \cdots\left(1-\xi^{a} X\right)^{2}\left(C\left(\xi^{a} X\right)-\left(1-\xi^{a+1}\right)^{2} C\left(\xi^{a+1} X\right)\right)
$$

By (2.1), we have

$$
C\left(\xi^{a} X\right)=\left(1-\xi^{a+1} X\right)^{2} C\left(\xi^{a+1} X\right)+\xi^{a+1}
$$

Letting $X=1$ gives $u_{a+1}(1)-u_{a}(1)=\xi^{a+1}(1-\xi)^{2} \cdots\left(1-\xi^{a}\right)^{2}$. Induction and (2.2) gives

$$
C(1)=\sum_{n=0}^{k-1} \xi^{n+1}(1-\xi)^{2} \cdots\left(1-\xi^{n}\right)^{2}
$$

2.2. Proof of Theorem 1.2. By the results of Andrews, Zwegers and the fourth author [7] (see (9.2) and Prop. 9.2 and 9.3 ) with $q=e^{-2 \pi z}$, we have

$$
q v(q)=\sum_{n=0}^{\infty} \frac{q^{n+1}}{\left(q^{n+1} ; q\right)_{\infty}^{2}}=\frac{e^{\frac{\pi}{6}\left(\frac{1}{z}-\frac{3}{2} z\right)}}{\sqrt{3 z}} \int_{-\infty}^{\infty} x e^{-\frac{\pi x^{2}}{3 z}} \cdot \frac{\sinh \left(\frac{2 \pi x}{3}\right)}{\cos (\pi x)} d x \cdot\left(1+O\left(z^{N}\right)\right)
$$

for any positive $N$ where $v(q)=\sum_{n=0}^{\infty} \frac{q^{n}}{\left(q^{n} ; q\right)_{\infty}^{2}}$. Since we have $U(-1 ; q)=(q ; q)_{\infty}^{2} q v(q)$ and $(q ; q)_{\infty}^{2}=e^{-\frac{\pi}{6}\left(\frac{1}{z}-z\right)} z^{-1}\left(1+O\left(z^{N}\right)\right)$ for any positive $N$, we have

$$
q^{-\frac{1}{24}} U(-1 ; q)=\frac{1}{\sqrt{3} z^{\frac{3}{2}}} \int_{\mathbb{R}} x e^{-\frac{\pi x^{2}}{3 z}} \cdot \frac{\sinh \left(\frac{2 \pi x}{3}\right)}{\cos (\pi x)} d x\left(1+O\left(z^{N}\right)\right)
$$

for any $N$. The Glaisher's $T$-numbers are given by

$$
\begin{equation*}
\frac{\sinh \left(\frac{2 \pi x}{3}\right)}{\cosh (\pi x)}=\frac{2}{i} \sum_{n=0}^{\infty} \frac{T_{n}}{(2 n+1)!}\left(\frac{i \pi x}{3}\right)^{2 n+1} \tag{2.3}
\end{equation*}
$$

We also have the identity

$$
\int_{\mathbb{R}} x^{2 j} e^{-\frac{\pi x^{2}}{3 z}} d x=\frac{(2 j)!}{2^{j} j!}\left(\frac{3}{2 \pi}\right)^{j} \sqrt{3 z} z^{j}
$$

Combining these identities and then setting $t=2 \pi z$ completes the proof.
2.3. Proof of Theorem 1.3. Define $G(z):=\left(e^{2 \pi i z} ; e^{2 \pi i z}\right)_{\infty} U\left(-1 ; e^{2 \pi i z}\right)$. Th. 1.1 of [7] gives

$$
G(z)-\frac{i}{2} \eta(z)^{3} \int_{-\bar{z}}^{i \infty} \frac{\eta(\tau)^{3}}{(-i(z+\tau))^{\frac{1}{2}}} d \tau+\frac{\sqrt{3}}{2 \pi i} \eta(z) \int_{-\bar{z}}^{i \infty} \frac{\eta(\tau)}{(-i(\tau+z))^{\frac{3}{2}}} d \tau
$$

$$
\begin{equation*}
=z^{-2}\left(G\left(-\frac{1}{z}\right)-\frac{i}{2} \eta\left(-\frac{1}{z}\right)^{3} \int_{\frac{1}{\bar{z}}}^{i \infty} \frac{\eta(\tau)^{3}}{\left(-i\left(-\frac{1}{z}+\tau\right)\right)^{\frac{1}{2}}} d \tau+\frac{\sqrt{3}}{2 \pi i} \eta\left(-\frac{1}{z}\right) \int_{\frac{1}{\bar{z}}}^{i \infty} \frac{\eta(\tau)}{\left(-i\left(\tau-\frac{1}{z}\right)\right)^{\frac{3}{2}}} d \tau\right) \tag{2.4}
\end{equation*}
$$

Note that using $\eta\left(-\frac{1}{z}\right)=\sqrt{-i z} \eta(z)$ we have

$$
\begin{align*}
\eta\left(-\frac{1}{z}\right)^{3} \int_{\frac{1}{\bar{z}}}^{i \infty} \frac{\eta(\tau)^{3}}{\left(-i\left(\tau-\frac{1}{z}\right)\right)^{\frac{1}{2}}} d \tau & =(\sqrt{-i z})^{3} \eta(z)^{3} \int_{-\bar{z}}^{0} \frac{\eta\left(-\frac{1}{\tau}\right)^{3}}{\left(-i\left(-\frac{1}{z}-\frac{1}{\tau}\right)\right)^{\frac{1}{2}}} \tau^{-2} d \tau  \tag{2.5}\\
& =(\sqrt{-i z})^{3} \eta(z)^{3} \int_{-\bar{z}}^{0} \frac{(\sqrt{-i \tau} \eta(\tau))^{3}(-z \tau)^{\frac{1}{2}}}{(-i(z+\tau))^{\frac{1}{2}}} \tau^{-2} d \tau \\
& =-z^{2} \eta(z)^{3} \int_{0}^{-\bar{z}} \frac{\eta(\tau)^{3}}{(-i(z+\tau))^{\frac{1}{2}}} d \tau
\end{align*}
$$

Similarly, we have

$$
\begin{equation*}
\eta\left(-\frac{1}{z}\right) \int_{\frac{1}{\bar{z}}}^{i \infty} \frac{\eta(\tau)}{\left(-i\left(\tau-\frac{1}{z}\right)\right)^{\frac{3}{2}}} d \tau=-z^{2} \eta(z) \int_{0}^{-\bar{z}} \frac{\eta(\tau)}{(-i(z+\tau))^{\frac{3}{2}}} d \tau \tag{2.6}
\end{equation*}
$$

Combining (2.4)-(2.6) gives

$$
G(z)-z^{-2} G\left(-\frac{1}{z}\right)=\frac{\sqrt{3}}{2 \pi i} \eta(z) \int_{0}^{i \infty} \frac{\eta(\tau)}{(-i(z+\tau))^{\frac{3}{2}}} d \tau-\frac{i}{2} \eta(z)^{3} \int_{0}^{i \infty} \frac{\eta(\tau)^{3}}{(-i(z+\tau))^{\frac{1}{2}}} d \tau .
$$

Dividing by $\eta(z)$ and using its modular transformation property give the result for $x \in \mathbb{H}$.
For $x \in \mathbb{Q}$, note that $\left(e^{2 \pi i x} ; e^{2 \pi i x}\right)_{\infty}=0$. Moreover, Zagier, in the discussion after the theorem of Section 6 of [2] explains how the integral $\int_{0}^{\infty} \eta(z)(z+x)^{-\frac{3}{2}} d z$ is real analytic for real $x$.

## 3. Congruence properties and the Hecke-type identity

We first prove Th. 1.4 on the parity of $u(n)$, and we then prove Th. 1.5 giving the Hecke-type identity for $U(-1 ; q)$. We then conclude this section with the proof of Th. 1.7.
3.1. Proof of Theorem 1.4. By Th. 1 of [14] (see equation (1.2)), we have that

$$
U(-1 ; q)=\frac{1}{(q ; q)_{\infty}} \cdot\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(1+q^{n}\right) q^{\frac{3 n^{2}+n}{2}}}{\left(1-q^{n}\right)^{2}}-\sum_{n=1}^{\infty} \frac{q^{n}}{\left(1-q^{n}\right)^{2}}+2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n q^{\frac{n^{2}+n}{2}}}{1-q^{n}}\right)
$$

If $\operatorname{spt}(n)$ is the smallest parts partition function of Andrews, then by Th. 4 of [17] we have:

$$
S(q):=\sum_{n=0}^{\infty} s p t(n) q^{n}=\frac{1}{(q ; q)_{\infty}}\left(\sum_{n=1}^{\infty} \frac{q^{n}}{\left(1-q^{n}\right)^{2}}+\sum_{n=1}^{\infty} \frac{(-1)^{n} q^{\frac{3 n^{2}+n}{2}}\left(1+q^{n}\right)}{\left(1-q^{n}\right)^{2}}\right)
$$

We have used the elementary fact that

$$
\begin{equation*}
\sum_{n=1}^{\infty} \sum_{d \mid n} d q^{n}=\sum_{n=1}^{\infty} \frac{q^{n}}{\left(1-q^{n}\right)^{2}}=\sum_{n=1}^{\infty} \frac{n q^{n}}{1-q^{n}} \tag{3.1}
\end{equation*}
$$

We have $U(-1 ; q) \equiv S(q)(\bmod 2)$, and so the theorem follows from Th. 1.2 in $[18]^{5}$.
3.2. Proof of Theorem 1.5. We prove Th. 1.5 using the method of Bailey pairs. As usual, we let $(a)_{n}:=(a ; q)_{n}$. Two sequences $\left(\alpha_{n}, \beta_{n}\right)$ form a Bailey pair for $a$ if

$$
\begin{aligned}
& \beta_{n}=\sum_{r=0}^{n} \frac{\alpha_{r}}{(q)_{n-r}(a q)_{n+r}} \\
& \alpha_{n}=\frac{\left(1-a q^{2 n}\right)(a)_{n}(-1)^{n} q^{\frac{n(n-1)}{2}}}{(1-a)(q)_{n}} \sum_{j=0}^{n}\left(q^{-n} ; q\right)_{j}\left(a q^{n} ; q\right)_{j} q^{j} \beta_{j} .
\end{aligned}
$$

The following Bailey pair is central to the proof of Th. 1.5.

[^1]Lemma 3.1. If $\beta_{n}=1$ and $\alpha_{0}=1$ and for $n>0$

$$
\alpha_{n}=\left(1-q^{2 n}\right) q^{2 n^{2}-n}\left(\frac{q-2}{1-q}+\sum_{j=2}^{n}(-1)^{j} \frac{\left(1-q^{2 j-1}\right)}{\left(1-q^{j}\right)\left(1-q^{j-1}\right)} q^{-\frac{3 j(j-1)}{2}}\right)
$$

then $\left(\alpha_{n}, \beta_{n}\right)$ is a Bailey pair with respect to 1.
Proof. We apply Th. 8 of [20] with $\beta_{n}=1$ for all $n$. By letting $b, c, d \rightarrow 0$, and then letting $a=1$, one obtains the lemma. Some care is required for the $j=0$ and $j=1$ terms.

The following is Bailey's Lemma (for example, see [20]).
Lemma 3.2 (Bailey's Lemma). If $\alpha_{n}$ and $\beta_{n}$ form a Bailey pair relative to $a$, then

$$
\sum_{n \geq 0} \frac{\left(\rho_{1}\right)_{n}\left(\rho_{2}\right)_{n}\left(a q / \rho_{1} \rho_{2}\right)^{n}}{\left(a q / \rho_{1}\right)_{n}\left(a q / \rho_{2}\right)_{n}} \alpha_{n}=\frac{(a q)_{\infty}\left(a q / \rho_{1} \rho_{2}\right)_{\infty}}{\left(a q / \rho_{1}\right)_{\infty}\left(a q / \rho_{2}\right)_{\infty}} \sum_{n \geq 0}\left(\rho_{1}\right)_{n}\left(\rho_{2}\right)_{n}\left(a q / \rho_{1} \rho_{2}\right)^{n} \beta_{n}
$$

Proof of Theorem 1.5. By Lemma 3.2 with $\rho_{1}=x, \rho_{2}=x^{-1}$ and $a=1$, Lemma 3.1 gives

$$
\sum_{n \geq 0}(x)_{n}\left(x^{-1}\right)_{n} q^{n}=\frac{(x q)_{\infty}\left(x^{-1} q\right)_{\infty}}{(q)_{\infty}^{2}}+\frac{(x)_{\infty}\left(x^{-1}\right)_{\infty}}{(q)_{\infty}^{2}} \sum_{n \geq 1} \frac{q^{n}}{\left(1-x q^{n}\right)\left(1-x^{-1} q^{n}\right)} \cdot \alpha_{n}
$$

Dividing by $(1-x)\left(1-x^{-1}\right)$ and collecting the $n=0$ terms give

$$
\begin{align*}
\sum_{n>0}(x q)_{n-1}\left(x^{-1} q\right)_{n-1} q^{n}= & \frac{1}{(1-x)\left(1-x^{-1}\right)} \cdot\left(\frac{(x q)_{\infty}\left(x^{-1} q\right)_{\infty}}{(q)_{\infty}^{2}}-1\right) \\
& +\frac{(x q)_{\infty}\left(x^{-1} q\right)_{\infty}}{(q)_{\infty}^{2}} \sum_{n \geq 1} \frac{q^{n}}{\left(1-x q^{n}\right)\left(1-x^{-1} q^{n}\right)} \cdot \alpha_{n} \tag{3.2}
\end{align*}
$$

To simplify the $\alpha_{n}$, we have that

$$
\frac{1-q^{2 j-1}}{\left(1-q^{j}\right)\left(1-q^{j-1}\right)}=\frac{1}{2} \cdot\left(\frac{1+q^{j}}{1-q^{j}}+\frac{1+q^{j-1}}{1-q^{j-1}}\right)
$$

which in turn implies that

$$
\begin{aligned}
\sum_{j=2}^{n}(-1)^{j} \frac{\left(1-q^{2 j-1}\right)}{\left(1-q^{j}\right)\left(1-q^{j-1}\right)} q^{-\frac{3 j(j-1)}{2}} & =\frac{1}{2} \cdot \frac{1+q}{1-q} \cdot q^{-3}+\frac{(-1)^{n}}{2} \cdot \frac{1+q^{n}}{1-q^{n}} \cdot q^{-\frac{3 n(n-1)}{2}} \\
+ & \frac{1}{2} \sum_{j=2}^{n-1}(-1)^{j+1}\left(1+q^{j}\right) q^{-\frac{3 j(j-1)}{2}} \frac{1-q^{3 j}}{1-q^{j}}
\end{aligned}
$$

Thus $\alpha_{0}=1$, and for $n \geq 1$ we have

$$
\begin{aligned}
\alpha_{n}= & \left(1-q^{2 n}\right) q^{2 n^{2}-n}\left(\frac{q-2}{1-q}+\frac{1}{2}\left(\frac{1+q}{1-q} \cdot q^{-3}+\frac{(-1)^{n}\left(1+q^{n}\right) q^{-\frac{3 n(n-1)}{2}}}{1-q^{n}}\right.\right. \\
& \left.\left.\quad+\sum_{j=2}^{n-1}(-1)^{j+1}\left(1+q^{j}\right)\left(1+q^{j}+q^{2 j}\right) q^{-\frac{3 j(j+1)}{2}}\right)\right) \\
= & \left(1-q^{2 n}\right) q^{2 n^{2}-n}\left(-1+\sum_{j=1}^{n-1}(-1)^{j+1}\left(1+q^{j}\right) q^{-\frac{j(3 j+1)}{2}}+\frac{(-1)^{n} q^{-\frac{3 n(n-1)}{2}+n}}{1-q^{n}}\right) \\
= & \left(1-q^{2 n}\right) q^{2 n^{2}-n}\left(\sum_{j=-n+1}^{n-1}(-1)^{j+1} q^{-\frac{j(3 j+1)}{2}}\right)+(-1)^{n}\left(1+q^{n}\right) q^{\frac{n(n+3)}{2}} \\
= & \left(1-q^{2 n}\right) q^{2 n^{2}-n}\left(\sum_{j=-n}^{n-1}(-1)^{j+1} q^{-\frac{j(3 j+1)}{2}}\right)+(-1)^{n}\left(1+q^{n}\right) q^{\frac{n(n+1)}{2}}
\end{aligned}
$$

We note that

$$
\lim _{x \rightarrow 1} \frac{1}{(1-x)\left(1-x^{-1}\right)}\left(\frac{(x q)_{\infty}\left(x^{-1} q\right)_{\infty}}{(q)_{\infty}^{2}}-1\right)=\sum_{n>0} \frac{q^{n}}{\left(1-q^{n}\right)^{2}}=\sum_{n>0} \frac{(-1)^{n+1} q^{\frac{n(n+1)}{2}}\left(1+q^{n}\right)}{\left(1-q^{n}\right)^{2}}
$$

Now insert these facts in (3.2), let $x \rightarrow 1$, and use the identity $\frac{1+q^{n}}{1-q^{n}}=1+2 \sum_{m \geq 1} q^{m n}$.
3.3. Proof of Theorem 1.7. We give a sketch since it is analogous to Th. 1.5 of [12] and Th. 1 of [21]. We have

$$
U( \pm i ; q)=\Psi(q)=\sum_{n=0}^{\infty}(u(0,4 ; n)-u(2,4 ; n)) q^{n}
$$

where $\Psi(q)$ is one of Ramanujan's 3rd order mock theta functions. We have that $q^{-1} \Psi\left(q^{24}\right)$ is the holomorphic part of a weight $1 / 2$ harmonic Maass form whose shadow is a unary theta function. Using quadratic and trivial twists modulo $Q$, one obtains a weight $1 / 2$ weakly holomorphic modular form. By work of Treneer, [22], one obtains weakly holomorphic forms of half-integer weight which are congruent to cusp forms modulo $Q$. By the Shimura correspondence, we obtain even integer weight cusp forms, which by Lemma 3.30 of [23], are annihilated modulo $Q$ by infinitely many Hecke operators $T(p)$. Since the Shimura correspondence is Hecke equivariant, it follows that infinitely many half-integral weight Hecke operators $T\left(p^{2}\right)$ annihilate these cusp forms modulo $Q$. The proof follows from the formula for the action of these operators.

## References

[1] Rhoades, R.C., Strongly Unimodal Sequences and Mixed Mock Modular Forms, preprint.
[2] Zagier, D., (2001) Vassiliev invariants and a strange identity related to the Dedekind eta-function, Topology 40, 945-960.
[3] Cohen, H., (1988) q-identities for Maass waveforms, Invent. Math. 91, 409-422.
[4] Zagier, D., (2010) Quantum modular forms, In Quanta of Maths: Conference in honor of Alain Connes, Clay Math. Proc. 11, Amer. Math. Soc., Providence, pages 659-675.
[5] Andrews, G. E., Dyson, F., Hickerson, D., (1988) Partitions and indefinite quadratic forms, Invent. Math. 91, 391-407.
[6] N. J. A. Sloane and S. Plouffe, The Encyclopedia of Integer Sequences, Academic Press, San Diego, 1995. On-line version: http://oeis.org/A002439
[7] Andrews, G. E., Rhoades, R. C., Zwegers, Z., Modularity of the concave composition generating function, preprint.
[8] Ono, K., (2009) Unearthing the visions of a master: harmonic Maass forms and number theory, Proc. 2008 Harvard-MIT Current Developments in Mathematics Conf., Somerville, Ma., 347-454.
[9] Zagier, D., (2006) Ramanujan's mock theta functions and their applications [d'aprés Zwegers and Bringmann-Ono], Séminaire Bourbaki, no. 986.
[10] Zwegers, S., (2002) Mock theta functions, Ph.D. Thesis (Advisor: D. Zagier), Universiteit Utrecht.
[11] Atkin, A. O. L., Swinnerton-Dyer, H. P. F., (1954) Some properties of partitions, Proc. London Math. Soc. 66 No. 4, 84-106.
[12] Bringmann, K., Ono, K., (2010) Dyson's ranks and Maass forms, Ann. of Math., 171, 419-449.
[13] Dyson, F., (1944) Some guesses in the theory of partitions, Eureka (Cambridge) 8, 10-15.
[14] Andrews, G. E., Concave and convex compositions, Ramanujan J., to appear.
[15] Fine, N. J., (1988) Basic hypergeometric series and applications, Math. Surveys and Monographs, no. 27, Amer. Math. Soc., Providence.
[16] Gordon, B., McIntosh, R., A survey of mock theta functions, I, preprint.
[17] Andrews, G. E., (2008) The number of smallest parts in the partitions of n, J. reine Angew. Math. 624, 133-142.
[18] Folsom, A., Ono, K., (2008) The spt-function of Andrews, Proc. Natl. Acad. Sci., USA, 105, no.51, 20152-20156.
[19] Andrews, G. E., Garvan, F., Liang, J., Combinatorial interpretations of congruences for the spt-function, Ramanujan J., to appear.
[20] Lovejoy, J., (2002) Lacunary Partition Functions, Math. Res. Lett. 9, 191-198.
[21] Ono, K., (2000) The partition function modulo m, Ann. of Math. 151, 293-307.
[22] Treneer, S., (2006) Congruences for the coefficients of weakly holomorphic modular forms, Proc. London Math. Soc., (3) 93, 304-324.
[23] Ono, K., (2004) The web of modularity: arithmetic of the coefficients of modular forms and q-series, CBMS Regional Conf. Series in Math., Amer. Math. Soc., Providence, vol. 102, 56.

Department of Mathematics, Texas A \& M University, College Station, TX. 77843
E-mail address: j.bryson@tamu.edu
Department of Mathematics, Emory University, Atlanta, GA. 30322
E-mail address: ono@mathcs.emory.edu
E-mail address: spitman@emory.edu
Department of Mathematics, Stanford University, Stanford, CA. 94305
E-mail address: rhoades@math.stanford.edu


[^0]:    ${ }^{3}$ As Zagier points out in Section 6 of [2], the right hand side of the identity is essentially the "half-derivative" of Dedekind's eta-function, which then suggests that the series may be related to a weight $3 / 2$ modular object.
    ${ }^{4}$ Zagier's definition of a quantum modular form is intentionally vague with the idea that sufficient flexibility is required to allow for interesting examples. Here we modify his defintion to include half-integral weights $k$ and multiplier systems $\epsilon(\gamma)$.

[^1]:    ${ }^{5}$ Th. 1.2 in [18] is not stated correctly in [18]. One must replace $p m^{2}$ by $p^{4 a+1} m^{2}$ where $\operatorname{gcd}(p, m)=1$. Recent work by Andrews, Garvan, and Liang [19] gives a new proof of this result.

