UNIMODAL SEQUENCES AND QUANTUM AND MOCK MODULAR FORMS

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ABSTRACT. We show that the rank generating function U(t;q) for strongly unimodal sequences lies at the interface of quantum modular forms and mock modular forms. We use U(-1;q) to obtain a quantum modular form which is "dual" to the quantum form Zagier constructed from Kontsevich's "strange" function F(q). As a result we obtain a new representation for a certain generating function for L-values. The series U(i;q) = U(-i;q) is a mock modular form, and we use this fact to obtain new congruences for certain enumerative functions.

1. INTRODUCTION AND STATEMENT OF RESULTS

A sequence of integers $\{a_i\}_{i=1}^s$ is a strongly unimodal sequence of size n if it satisfies

$$0 < a_1 < a_2 < \dots < a_k > a_{k+1} > a_{k+2} > \dots > a_s > 0$$

for some k and $a_1 + \cdots + a_s = n$. Let u(n) be the number of such sequences. The rank of such a sequence is s - 2k + 1, the number of terms after the maximal term minus the number of terms that precede it.

By letting t (resp. t^{-1}) keep track of the terms after (resp. before) a maximal term, we find that u(m, n), the number of size n and rank m sequences, satisfies¹

(1.1)
$$U(t;q) := \sum_{m,n} u(m,n)t^m q^n = \sum_{n=0}^{\infty} (-tq;q)_n (-t^{-1}q;q)_n q^{n+1} = q + q^2 + (t+1+t^{-1})q^3 + \dots,$$

where $(x;q)_n := (1-x)(1-xq)(1-xq^2)\cdots(1-xq^{n-1})$ for $n \ge 1$ and $(x;q)_0 := 1$.

Example. The strongly unimodal sequences of size 5 are: $\{5\}$, $\{1,4\}$, $\{4,1\}$, $\{1,3,1\}$, $\{2,3\}$, $\{3,2\}$, and so u(5) = 6. Respectively, their ranks are 0, -1, 1, 0, -1, 1.

The q-series U(-1;q), the generating function for the number of size n sequences with even rank minus the number with odd rank, is intimately related to Kontsevich's strange function²

(1.2)
$$F(q) := \sum_{n=0}^{\infty} (q;q)_n = 1 + (1-q) + (1-q)(1-q^2) + (1-q)(1-q^2)(1-q^3) + \dots$$

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¹In [1] u(n) is denoted $u^*(n)$ and U(1;q) is denoted $U^*(q)$.

²Zagier credits Kontsevich for relating F(q) to Feynmann integrals in a lecture at Max Planck in 1997.

It is strange because it does not converge on any open subset of \mathbb{C} , but is well-defined at all roots of unity. Zagier [2] proved that this function satisfies the even "stranger" identity

(1.3)
$$F(q) = -\frac{1}{2} \sum_{n=1}^{\infty} n\chi_{12}(n) q^{\frac{n^2 - 1}{24}},$$

where $\chi_{12}(\bullet) = \left(\frac{12}{\bullet}\right)$. Neither side of this identity makes sense simultaneously. Indeed, the right hand side³ converges in the unit disk |q| < 1, but nowhere on the unit circle. The identity means that F(q) at roots of unity agrees with the radial limit of the right hand side.

We prove that U(-1;q), which converges in |q| < 1, also gives $F(q^{-1})$ at roots of unity.

Theorem 1.1. If q is a root of unity, then $F(q^{-1}) = U(-1;q)$.

Example. Here are two examples: U(-1; -1) = F(-1) = 3 and U(-1; i) = F(-i) = 8 + 3i.

Remark. Th. 1.1 is analogous to the result of Cohen [3, 4] that $\sigma(q) = -\sigma^*(q^{-1})$ for roots of unity q, for the well-known q-series $\sigma(q)$ and $\sigma^*(q)$ that Andrews, Dyson, and Hickerson [5] defined in their work on partition ranks.

Zagier [2] used (1.3) to obtain the following identity

(1.4)
$$e^{-\frac{t}{24}} \sum_{n=0}^{\infty} (1-e^{-t})(1-e^{-2t})\dots(1-e^{-nt}) = \sum_{n=0}^{\infty} \frac{T_n}{n!} \cdot \left(\frac{t}{24}\right)^n,$$

where Glaisher's T_n numbers (see (2.3) and A002439 in [6]) are the "algebraic factors" of $L(\chi_{12}, 2n+2)$. As a companion to Th. 1.1, we use U(-1;q) to give these same L-values.

Theorem 1.2. As a power series in t, we have that

$$e^{\frac{t}{24}} \cdot U(-1; e^{-t}) = \sum_{n=0}^{\infty} \frac{T_n}{n!} \cdot \left(\frac{-t}{24}\right)^n = \frac{6\sqrt{3}}{\pi^2} \cdot \sum_{n=0}^{\infty} \frac{(2n+1)!}{n!} \cdot L(\chi_{12}, 2n+2) \cdot \left(\frac{-3t}{2\pi^2}\right)^n.$$

These results are related to the next theorem which gives a new quantum modular form. Following Zagier⁴ [4], a weight k quantum modular form is a complex-valued function f on \mathbb{Q} , or possibly $\mathbb{P}^1(\mathbb{Q}) \setminus S$ for some finite set S, such that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ the function

$$h_{\gamma}(x) := f(x) - \epsilon(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right)$$

satisfies a "suitable" property of continuity or analyticity. The $\epsilon(\gamma)$ are roots of unity, such as those in the theory of half-integral weight modular forms when $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. We prove that

(1.5)
$$\phi(x) := e^{-\frac{\pi i x}{12}} \cdot U(-1; e^{2\pi i x})$$

is a weight $\frac{3}{2}$ quantum modular form. Since $\operatorname{SL}_2(\mathbb{Z}) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle$ and $\phi(x) - e^{\frac{\pi i}{12}} \cdot \phi(x+1) = 0$, it suffices to consider $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The following theorem establishes the desired relationship on the larger domain $\mathbb{Q} \cup \mathbb{H} - \{0\}$, where \mathbb{H} is the upper-half of the complex plane.

³As Zagier points out in Section 6 of [2], the right hand side of the identity is essentially the "half-derivative" of Dedekind's eta-function, which then suggests that the series may be related to a weight 3/2 modular object.

⁴Zagier's definition of a quantum modular form is intentionally vague with the idea that sufficient flexibility is required to allow for interesting examples. Here we modify his definition to include half-integral weights k and multiplier systems $\epsilon(\gamma)$.

Theorem 1.3. If $x \in \mathbb{Q} \cup \mathbb{H} - \{0\}$, then

$$\phi(x) + (-ix)^{-\frac{3}{2}}\phi(-1/x) = h(x)$$

where $(ix)^{-\frac{3}{2}}$ is the principal branch and

$$h(x) := \frac{\sqrt{3}}{2\pi i} \int_0^{i\infty} \frac{\eta(\tau)}{(-i(x+\tau))^{\frac{3}{2}}} d\tau - \frac{i}{2} e^{\frac{\pi i x}{6}} (e^{2\pi i x}; e^{2\pi i x})_\infty^2 \cdot \int_0^{i\infty} \frac{\eta(\tau)^3}{(-i(x+\tau))^{\frac{1}{2}}} d\tau.$$

Here $\eta(\tau) := e^{\frac{\pi i \tau}{12}} (e^{2\pi i \tau}; e^{2\pi i \tau})_{\infty}$ is Dedekind's eta-function. Moreover, taking $\eta(x) = 0$ for $x \in \mathbb{R}, h : \mathbb{R} \to \mathbb{C}$ is a C^{∞} function which is real analytic everywhere except at x = 0, and $h^{(n)}(0) = (-\pi i/12)^n \cdot T_n$, where T_n is the nth Glaisher number.

Remark. Zagier [2] proved that $e^{\frac{\pi ix}{12}} \cdot F(e^{2\pi ix})$ is a quantum modular form. Th. 1.3 gives a dual quantum modular form, one whose domain naturally extends beyond \mathbb{Q} to include \mathbb{H} . This is somewhat analogous to the situation for $\sigma(q)$ and $\sigma^*(q)$ discussed above. Zagier constructed a quantum modular form from these q-series in Example 1 of [4].

Remark. Th. 1.3 implies that $\Phi(z) := \eta(z)\phi(z)$ behaves analogously to a weight 2 modular form for $\operatorname{SL}_2(\mathbb{Z})$ for $z \in \mathbb{H}$ with a suitable error function. Namely, $\Phi(z+1) = \Phi(z)$ and $\Phi(z) - z^{-2}\Phi\left(-\frac{1}{z}\right) = \eta(z)h(z)$, see also Th. 1.1 of [7].

It turns out that U(1;q) and $U(\pm i;q)$ also possess deep properties. We have that U(1;q) [1] is a mixed mock modular form, and $U(\pm i;q)$ is a mock theta function (see [8, 9, 10]). We use these facts to study congruences for certain enumerative functions.

Theorem 1.4. If $3 < \ell \not\equiv 23 \pmod{24}$ is prime, $\delta(\ell) := (\ell^2 - 1)/24$ and $\ell \nmid k$, then for all n

$$u(\ell^2 n + k\ell - \delta(\ell)) \equiv 0 \pmod{2}.$$

Example. If $\ell = 7$, then Th. 1.4 gives $u(49n + a) \equiv 0 \pmod{2}$ for $a \in \{5, 12, 19, 26, 33, 40\}$.

The nature of Th. 1.4 suggests the existence of a Hecke-type identity for U(-1;q) analogous to those obtained for $\sigma(q)$ and $\sigma^*(q)$ in [5]. Here we obtain such an identity.

Theorem 1.5. We have that

$$U(-1;q) = \sum_{n>0} \sum_{6n \ge |6j+1|} (-1)^{j+1} q^{2n^2 - \frac{j(3j+1)}{2}} + 2\sum_{n,m>0} \sum_{6n \ge |6j+1|} (-1)^{j+1} q^{2n^2 + mn - \frac{j(3j+1)}{2}}.$$

These congruences appear to have refinements modulo 4. In analogy with the theory of partition ranks [11, 12, 13], we suspect that ranks also "explain" these congruences. Namely, let u(a, b; n) be the number of size n strongly unimodal sequences with rank $\equiv a \pmod{b}$.

Conjecture 1.6. If $\ell \equiv 7, 11, 13, 17 \pmod{24}$ is prime and $\left(\frac{k}{\ell}\right) = -1$, then for all n we have

(1.6)
$$u(\ell^2 n + k\ell - \delta(\ell)) \equiv 0 \pmod{4}.$$

Moreover, for $a \in \{0, 1, 2, 3\}$ we have $u(a, 4; \ell^2 n + k\ell - \delta(\ell)) \equiv 0 \pmod{2}$ and

(1.7)
$$u(0,4;\ell^2 n + k\ell - \delta(\ell)) \equiv u(2,4;\ell^2 n + k\ell - \delta(\ell)) \pmod{4}.$$

We have that u(1,4;n) = u(3,4;n), and so the truth of (1.7) is a proposed explanation of (1.6). Therefore, it is natural to study $U(\pm 1;q)$ and the 3rd order mock theta function [14, 15, 16]

$$U(\pm i;q) = \Psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n} = \sum_{n=0}^{\infty} (-q^2;q^2)_n q^{n+1} = \frac{q}{(q^4)_{\infty}} \cdot \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{6n(n+1)}}{1 - q^{4n+1}}.$$

Using this mock theta function we are able to obtain the following related congruences.

Theorem 1.7. If (Q, 6) = 1, then there are arithmetic progressions An + B such that $u(0, 4; An + B) \equiv u(2, 4; An + B) \pmod{Q}$.

Example. For Q = 5 the cusp form in the proof of Th. 1.7 is annihilated by $T(11^2)$, and so if

$$a(24n-1) := u(0,4;n) - u(2,4;n) \pmod{5}$$

(note. a(n) = 0 if $n \not\equiv 23 \pmod{24}$), then for every $n \equiv 23, 47 \pmod{120}$ we have that

$$a(121n) - \left(\frac{n}{11}\right)a(n) + a(n/121) \equiv 0 \pmod{5}.$$

Since $\left(\frac{n}{11}\right) = 0$ and a(n/121) = 0 when 11||n, this gives congruences such as

 $u(0,2;73205n+721) \equiv u(2,4;73205n+721) \pmod{5}.$

2. Quantum properties of U(-1;q)

Here we prove the quantum properties of U(-1;q). We first prove Th. 1.1 relating the values of Kontsevich's F(q) and U(-1;q) at roots of unity. We then prove Th. 1.2 giving a new representation of Zagier's *L*-value generating function, and we conclude with a proof of Th. 1.3.

2.1. Proof of Theorem 1.1. For ξ a fixed kth root of unity, define the polynomial

$$C(X) = \sum_{n=0}^{k-1} (X - \xi^{-1}) \cdots (X - \xi^{-n}).$$

We have the identity

(2.1)
$$C(\xi^{-1}X) = (X-1)^2 C(X) - X(X^k - 1) + X$$

Define the functions $u_a(X)$ for $a \ge 1$ by

$$(2 - X^k)u_a(\xi^{-a}X) = C(\xi^{-a}X) - (1 - X)^2 \cdots (1 - \xi^{-(a-1)}X)^2 C(X).$$

Hence for a = k we have

(2.2)
$$X^k C(X) = u_k(X)$$

Then we have

$$(2-X^k)(u_{a+1}(X)-u_a(X)) = (1-\xi X)^2 \cdots (1-\xi^a X)^2 (C(\xi^a X)-(1-\xi^{a+1})^2 C(\xi^{a+1} X)).$$

By (2.1), we have

$$C(\xi^{a}X) = (1 - \xi^{a+1}X)^{2}C(\xi^{a+1}X) + \xi^{a+1}.$$

Letting X = 1 gives $u_{a+1}(1) - u_a(1) = \xi^{a+1}(1-\xi)^2 \cdots (1-\xi^a)^2$. Induction and (2.2) gives $C(1) = \sum_{n=0}^{k-1} \xi^{n+1}(1-\xi)^2 \cdots (1-\xi^n)^2.$

2.2. **Proof of Theorem 1.2.** By the results of Andrews, Zwegers and the fourth author [7] (see (9.2) and Prop. 9.2 and 9.3) with $q = e^{-2\pi z}$, we have

$$qv(q) = \sum_{n=0}^{\infty} \frac{q^{n+1}}{(q^{n+1};q)_{\infty}^2} = \frac{e^{\frac{\pi}{6}(\frac{1}{z} - \frac{3}{2}z)}}{\sqrt{3z}} \int_{-\infty}^{\infty} x e^{-\frac{\pi x^2}{3z}} \cdot \frac{\sinh(\frac{2\pi x}{3})}{\cos(\pi x)} \, dx \cdot (1 + O(z^N))$$

for any positive N where $v(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^n;q)_{\infty}^2}$. Since we have $U(-1;q) = (q;q)_{\infty}^2 qv(q)$ and $(q;q)_{\infty}^2 = e^{-\frac{\pi}{6}(\frac{1}{z}-z)}z^{-1}(1+O(z^N))$ for any positive N, we have

$$q^{-\frac{1}{24}}U(-1;q) = \frac{1}{\sqrt{3}z^{\frac{3}{2}}} \int_{\mathbb{R}} x e^{-\frac{\pi x^2}{3z}} \cdot \frac{\sinh(\frac{2\pi x}{3})}{\cos(\pi x)} \, dx \left(1 + O\left(z^N\right)\right)$$

for any N. The Glaisher's T-numbers are given by

(2.3)
$$\frac{\sinh(\frac{2\pi x}{3})}{\cosh(\pi x)} = \frac{2}{i} \sum_{n=0}^{\infty} \frac{T_n}{(2n+1)!} \left(\frac{i\pi x}{3}\right)^{2n+1}$$

We also have the identity

$$\int_{\mathbb{R}} x^{2j} e^{-\frac{\pi x^2}{3z}} dx = \frac{(2j)!}{2^j j!} \left(\frac{3}{2\pi}\right)^j \sqrt{3z} z^j.$$

Combining these identities and then setting $t = 2\pi z$ completes the proof.

2.3. **Proof of Theorem 1.3.** Define $G(z) := (e^{2\pi i z}; e^{2\pi i z})_{\infty} U(-1; e^{2\pi i z})$. Th. 1.1 of [7] gives $G(z) - \frac{i}{2} \eta(z)^3 \int_{-\overline{z}}^{i\infty} \frac{\eta(\tau)^3}{(-i(z+\tau))^{\frac{1}{2}}} d\tau + \frac{\sqrt{3}}{2\pi i} \eta(z) \int_{-\overline{z}}^{i\infty} \frac{\eta(\tau)}{(-i(\tau+z))^{\frac{3}{2}}} d\tau$ (2.4) $= z^{-2} \left(G\left(-\frac{1}{z}\right) - \frac{i}{2} \eta\left(-\frac{1}{z}\right)^3 \int_{\frac{1}{\overline{z}}}^{i\infty} \frac{\eta(\tau)^3}{(-i(-\frac{1}{z}+\tau))^{\frac{1}{2}}} d\tau + \frac{\sqrt{3}}{2\pi i} \eta\left(-\frac{1}{z}\right) \int_{\frac{1}{\overline{z}}}^{i\infty} \frac{\eta(\tau)}{(-i(\tau-\frac{1}{z}))^{\frac{3}{2}}} d\tau \right).$ Note that using $\eta\left(-\frac{1}{z}\right) = \sqrt{-iz} \eta(z)$ we have

$$(2.5) \qquad \eta \left(-\frac{1}{z}\right)^3 \int_{\frac{1}{z}}^{i\infty} \frac{\eta(\tau)^3}{\left(-i\left(\tau-\frac{1}{z}\right)\right)^{\frac{1}{2}}} d\tau = (\sqrt{-iz})^3 \eta(z)^3 \int_{-\overline{z}}^0 \frac{\eta \left(-\frac{1}{\tau}\right)^3}{\left(-i\left(-\frac{1}{z}-\frac{1}{\tau}\right)\right)^{\frac{1}{2}}} \tau^{-2} d\tau \\ = (\sqrt{-iz})^3 \eta(z)^3 \int_{-\overline{z}}^0 \frac{\left(\sqrt{-i\tau}\eta(\tau)\right)^3 (-z\tau)^{\frac{1}{2}}}{(-i(z+\tau))^{\frac{1}{2}}} \tau^{-2} d\tau \\ = -z^2 \eta(z)^3 \int_0^{-\overline{z}} \frac{\eta(\tau)^3}{(-i(z+\tau))^{\frac{1}{2}}} d\tau.$$

Similarly, we have

(2.6)
$$\eta\left(-\frac{1}{z}\right)\int_{\frac{1}{z}}^{i\infty}\frac{\eta(\tau)}{(-i\left(\tau-\frac{1}{z}\right))^{\frac{3}{2}}}d\tau = -z^{2}\eta(z)\int_{0}^{-\overline{z}}\frac{\eta(\tau)}{(-i(z+\tau))^{\frac{3}{2}}}d\tau.$$

Combining (2.4)-(2.6) gives

$$G(z) - z^{-2}G\left(-\frac{1}{z}\right) = \frac{\sqrt{3}}{2\pi i}\eta(z)\int_0^{i\infty}\frac{\eta(\tau)}{(-i(z+\tau))^{\frac{3}{2}}}d\tau - \frac{i}{2}\eta(z)^3\int_0^{i\infty}\frac{\eta(\tau)^3}{(-i(z+\tau))^{\frac{1}{2}}}d\tau.$$

Dividing by $\eta(z)$ and using its modular transformation property give the result for $x \in \mathbb{H}$.

For $x \in \mathbb{Q}$, note that $(e^{2\pi i x}; e^{2\pi i x})_{\infty} = 0$. Moreover, Zagier, in the discussion after the theorem of Section 6 of [2] explains how the integral $\int_0^\infty \eta(z)(z+x)^{-\frac{3}{2}}dz$ is real analytic for real x.

3. Congruence properties and the Hecke-type identity

We first prove Th. 1.4 on the parity of u(n), and we then prove Th. 1.5 giving the Hecke-type identity for U(-1;q). We then conclude this section with the proof of Th. 1.7.

3.1. Proof of Theorem 1.4. By Th. 1 of [14] (see equation (1.2)), we have that

$$U(-1;q) = \frac{1}{(q;q)_{\infty}} \cdot \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1+q^n)q^{\frac{3n^2+n}{2}}}{(1-q^n)^2} - \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} + 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}nq^{\frac{n^2+n}{2}}}{1-q^n} \right).$$

If spt(n) is the smallest parts partition function of Andrews, then by Th. 4 of [17] we have:

$$S(q) := \sum_{n=0}^{\infty} spt(n)q^n = \frac{1}{(q;q)_{\infty}} \left(\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{\frac{3n^2+n}{2}}(1+q^n)}{(1-q^n)^2} \right).$$

We have used the elementary fact that

(3.1)
$$\sum_{n=1}^{\infty} \sum_{d|n} dq^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} = \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}.$$

We have $U(-1;q) \equiv S(q) \pmod{2}$, and so the theorem follows from Th. 1.2 in $[18]^5$.

3.2. **Proof of Theorem 1.5.** We prove Th. 1.5 using the method of Bailey pairs. As usual, we let $(a)_n := (a; q)_n$. Two sequences (α_n, β_n) form a Bailey pair for a if

$$\beta_n = \sum_{r=0}^n \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}$$
$$\alpha_n = \frac{(1 - aq^{2n})(a)_n(-1)^n q^{\frac{n(n-1)}{2}}}{(1 - a)(q)_n} \sum_{j=0}^n (q^{-n};q)_j (aq^n;q)_j q^j \beta_j.$$

The following Bailey pair is central to the proof of Th. 1.5.

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⁵Th. 1.2 in [18] is not stated correctly in [18]. One must replace pm^2 by $p^{4a+1}m^2$ where gcd(p,m) = 1. Recent work by Andrews, Garvan, and Liang [19] gives a new proof of this result.

Lemma 3.1. If $\beta_n = 1$ and $\alpha_0 = 1$ and for n > 0

$$\alpha_n = (1 - q^{2n})q^{2n^2 - n} \left(\frac{q - 2}{1 - q} + \sum_{j=2}^n (-1)^j \frac{(1 - q^{2j-1})}{(1 - q^j)(1 - q^{j-1})} q^{-\frac{3j(j-1)}{2}} \right),$$

then (α_n, β_n) is a Bailey pair with respect to 1.

Proof. We apply Th. 8 of [20] with $\beta_n = 1$ for all n. By letting $b, c, d \to 0$, and then letting a = 1, one obtains the lemma. Some care is required for the j = 0 and j = 1 terms.

The following is Bailey's Lemma (for example, see [20]).

Lemma 3.2 (Bailey's Lemma). If α_n and β_n form a Bailey pair relative to a, then

Proof of Theorem 1.5. By Lemma 3.2 with $\rho_1 = x$, $\rho_2 = x^{-1}$ and a = 1, Lemma 3.1 gives

$$\sum_{n\geq 0} (x)_n (x^{-1})_n q^n = \frac{(xq)_\infty (x^{-1}q)_\infty}{(q)_\infty^2} + \frac{(x)_\infty (x^{-1})_\infty}{(q)_\infty^2} \sum_{n\geq 1} \frac{q^n}{(1-xq^n)(1-x^{-1}q^n)} \cdot \alpha_n.$$

Dividing by $(1-x)(1-x^{-1})$ and collecting the n = 0 terms give

(3.2)
$$\sum_{n>0} (xq)_{n-1} (x^{-1}q)_{n-1} q^n = \frac{1}{(1-x)(1-x^{-1})} \cdot \left(\frac{(xq)_{\infty}(x^{-1}q)_{\infty}}{(q)_{\infty}^2} - 1\right) + \frac{(xq)_{\infty}(x^{-1}q)_{\infty}}{(q)_{\infty}^2} \sum_{n\geq 1} \frac{q^n}{(1-xq^n)(1-x^{-1}q^n)} \cdot \alpha_n.$$

To simplify the α_n , we have that

$$\frac{1-q^{2j-1}}{(1-q^j)(1-q^{j-1})} = \frac{1}{2} \cdot \left(\frac{1+q^j}{1-q^j} + \frac{1+q^{j-1}}{1-q^{j-1}}\right).$$

which in turn implies that

$$\sum_{j=2}^{n} (-1)^{j} \frac{(1-q^{2j-1})}{(1-q^{j})(1-q^{j-1})} q^{-\frac{3j(j-1)}{2}} = \frac{1}{2} \cdot \frac{1+q}{1-q} \cdot q^{-3} + \frac{(-1)^{n}}{2} \cdot \frac{1+q^{n}}{1-q^{n}} \cdot q^{-\frac{3n(n-1)}{2}} + \frac{1}{2} \sum_{j=2}^{n-1} (-1)^{j+1} (1+q^{j}) q^{-\frac{3j(j-1)}{2}} \frac{1-q^{3j}}{1-q^{j}}.$$

Thus $\alpha_0 = 1$, and for $n \ge 1$ we have

$$\begin{aligned} \alpha_n &= (1-q^{2n})q^{2n^2-n} \left(\frac{q-2}{1-q} + \frac{1}{2} \left(\frac{1+q}{1-q} \cdot q^{-3} + \frac{(-1)^n (1+q^n) q^{-\frac{3n(n-1)}{2}}}{1-q^n} \right. \\ &+ \sum_{j=2}^{n-1} (-1)^{j+1} (1+q^j) (1+q^j+q^{2j}) q^{-\frac{3j(j+1)}{2}} \right) \right) \\ &= (1-q^{2n})q^{2n^2-n} \left(-1 + \sum_{j=1}^{n-1} (-1)^{j+1} (1+q^j) q^{-\frac{j(3j+1)}{2}} + \frac{(-1)^n q^{-\frac{3n(n-1)}{2}+n}}{1-q^n} \right) \\ &= (1-q^{2n})q^{2n^2-n} \left(\sum_{j=-n+1}^{n-1} (-1)^{j+1} q^{-\frac{j(3j+1)}{2}} \right) + (-1)^n (1+q^n) q^{\frac{n(n+3)}{2}} \\ &= (1-q^{2n})q^{2n^2-n} \left(\sum_{j=-n}^{n-1} (-1)^{j+1} q^{-\frac{j(3j+1)}{2}} \right) + (-1)^n (1+q^n) q^{\frac{n(n+1)}{2}} \end{aligned}$$

We note that

$$\lim_{x \to 1} \frac{1}{(1-x)(1-x^{-1})} \left(\frac{(xq)_{\infty}(x^{-1}q)_{\infty}}{(q)_{\infty}^2} - 1 \right) = \sum_{n>0} \frac{q^n}{(1-q^n)^2} = \sum_{n>0} \frac{(-1)^{n+1}q^{\frac{n(n+1)}{2}}(1+q^n)}{(1-q^n)^2}.$$

Now insert these facts in (3.2), let $x \to 1$, and use the identity $\frac{1+q^n}{1-q^n} = 1 + 2\sum_{m\geq 1} q^{mn}$.

3.3. **Proof of Theorem 1.7.** We give a sketch since it is analogous to Th. 1.5 of [12] and Th. 1 of [21]. We have

$$U(\pm i;q) = \Psi(q) = \sum_{n=0}^{\infty} (u(0,4;n) - u(2,4;n))q^n,$$

where $\Psi(q)$ is one of Ramanujan's 3rd order mock theta functions. We have that $q^{-1}\Psi(q^{24})$ is the holomorphic part of a weight 1/2 harmonic Maass form whose shadow is a unary theta function. Using quadratic and trivial twists modulo Q, one obtains a weight 1/2 weakly holomorphic modular form. By work of Treneer, [22], one obtains weakly holomorphic forms of half-integer weight which are congruent to cusp forms modulo Q. By the Shimura correspondence, we obtain even integer weight cusp forms, which by Lemma 3.30 of [23], are annihilated modulo Q by infinitely many Hecke operators T(p). Since the Shimura correspondence is Hecke equivariant, it follows that infinitely many half-integral weight Hecke operators $T(p^2)$ annihilate these cusp forms modulo Q. The proof follows from the formula for the action of these operators.

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