CONGRUENCES FOR THE ANDREWS *spt*-FUNCTION

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ABSTRACT. Ramanujan-type congruences for the Andrews $spt(n)$ partition function have been found for prime moduli $5 \leq \ell \leq 37$ in work of Andrews [1] and Garvan [2]. We exhibit unexpectedly simple congruences for all $\ell \geq 5$. Confirming a conjecture of F. Garvan, we show that if $\ell \geq 5$ is prime and $\left(\frac{-\delta}{\ell}\right) = 1$, then

$$spt\left(\frac{\ell^2(\ell n + \delta) + 1}{24}\right) \equiv 0 \pmod{\ell}.$$

This gives $(\ell - 1)/2$ arithmetic progressions modulo ℓ^3 which support a mod ℓ congruence. This result follows from the surprising fact that the reduction of a certain mock theta function modulo ℓ, for every $\ell \geq 5$, is an eigenform of the Hecke operator $T(\ell^2)$.

1. Introduction and Statement of Results

Andrews recently [1] introduced the function $spt(n)$ which counts the number of smallest parts among the integer partitions of n. For $n = 4$ we have:

$$\underline{4}, \quad 3 + \underline{1}, \quad \underline{2} + \underline{2}, \quad 2 + \underline{1} + \underline{1}, \quad \underline{1} + \underline{1} + \underline{1} + \underline{1}.$$

The smallest parts are underlined, and so we have that $spt(4) = 10$. He [1] proved the following elegant Ramanujan-type congruences:

$$\begin{aligned} spt(5n + 4) &\equiv 0 \pmod{5}, \\ spt(7n + 5) &\equiv 0 \pmod{7}, \\ spt(13n + 6) &\equiv 0 \pmod{13}. \end{aligned}$$

Recently, Folsom and the author [3] (see also [4]) confirmed conjectures of Garvan and Sellers, and these results provide simple congruences modulo 2 and 3.

The situation is more complicated for primes $\ell \geq 5$. It is known that there are infinitely many congruences of the form

$$spt(an + b) \equiv 0 \pmod{\ell}.$$

This fact follows from work of Bringmann [5] (also see [6, 7]) on $N_2(n)$, the second rank moment, combined with earlier work of Ahlgren and the author on $p(n)$ [8, 9, 10]. However, explicit

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examples are only known for $\ell \leq 37$. For example, Garvan [2] has obtained:

$$\begin{aligned} spt(19^4 \cdot 11 \cdot n + 22006) &\equiv 0 \pmod{11}, \\ spt(7^4 \cdot 17 \cdot n + 243) &\equiv 0 \pmod{17}, \\ spt(5^4 \cdot 19 \cdot n + 99) &\equiv 0 \pmod{19}, \\ spt(13^4 \cdot 29 \cdot n + 18583) &\equiv 0 \pmod{29}. \end{aligned}$$

The moduli of the arithmetic progressions above involve (fourth) powers of special auxiliary primes, a feature shared by the congruences which arise from this theory. The congruences are constructed using these special primes, and these primes are guaranteed to exist by the theory of odd modular ℓ-adic Galois representations and the Chebotarev Density Theorem. To find a congruence, one is then required to search, prime by prime, for an auxiliary prime. This task is analogous to the simpler problem of finding the smallest prime $p \equiv 1 \pmod{\ell}$.

We establish new universal congruences for $spt(n)$ without relying on the existence of such primes. For aesthetics, we define $\widehat{s}(n)$ and $\widehat{p}(n)$ by:

$$\mathcal{S}(q) = \sum_{n=0}^{\infty} \widehat{s}(n) q^n := \sum_{n=1}^{\infty} spt(n) q^{24n-1}, \tag{1.1}$$

$$\mathcal{P}(q) = \sum_{n=-1}^{\infty} \widehat{p}(n) q^n := \sum_{n=0}^{\infty} p(n) q^{24n-1}. \tag{1.2}$$

We obtain the following congruences relating $\widehat{s}(n), \widehat{p}(n)$, and the Legendre symbol $\left(\frac{\bullet}{\ell}\right)$.

Theorem 1.1. *If $\ell \geq 5$ is prime, then*

$$\widehat{s}(\ell^2 n) \equiv \left(\frac{3}{\ell}\right)\left(1 - \left(\frac{-n}{\ell}\right)\right) \cdot \left(\widehat{s}(n) + \frac{n}{12} \cdot \widehat{p}(n)\right) \pmod{\ell}.$$

Remark. Theorem 1.1 may be reformulated in terms of the "mock theta function"

$$M(q) := \mathcal{S}(q) + \frac{1}{12} \cdot q \frac{d}{dq} \mathcal{P}(q) = -\frac{1}{12} \cdot q^{-1} + \frac{35}{12} \cdot q^{23} + \frac{65}{6} \cdot q^{47} + \dots. \tag{1.3}$$

We refer to $M(q)$ as a mock theta function because it is the holomorphic part of a harmonic Maass form. Although $M(q)$ is not an eigenform of any Hecke operators, Theorem 1.1 is equivalent to the assertion, for every prime $\ell \geq 5$, that

$$M(q)|T(\ell^2) \equiv \left(\frac{3}{\ell}\right) \cdot M(q) \pmod{\ell}.$$

Theorem 1.1 immediately gives the following corollary.

Corollary 1.2. *Suppose that $\ell \geq 5$ is prime. Then the following are true:*

(1) *If $\left(\frac{-n}{\ell}\right) = 1$, then*

$$\widehat{s}(\ell^2 n) \equiv 0 \pmod{\ell}.$$

(2) *We have that*

$$\widehat{s}(\ell^3 n) \equiv \left(\frac{3}{\ell}\right) \widehat{s}(\ell n) \pmod{\ell}.$$

Remark. Corollary 1.2 (1) gives distinct $0 < b_\ell(1) < \cdots < b_\ell\left(\frac{\ell-1}{2}\right) < \ell^3$ for which

$$spt\left(\ell^3 n + b_\ell(m)\right) \equiv 0 \pmod{\ell}.$$

Indeed, if $\left(\frac{-\delta}{\ell}\right) = 1$, then Corollary 1.2 (1) implies that

$$spt\left(\frac{\ell^2(\ell n + \delta) + 1}{24}\right) \equiv 0 \pmod{\ell}.$$

These congruences were conjectured by F. Garvan in July 2008 [11]. Garvan's Conjecture was inspired by work done by T. Garrett and her students in October 2007. For $\ell = 11$ the general result gives the five congruences:

$$spt(11^3 n + 479) \equiv spt(11^3 n + 842) \equiv spt(11^3 n + 1084) \equiv spt(11^3 n + 1205) \equiv spt(11^3 n + 1326) \equiv 0 \pmod{11}.$$

In Section 2 we prove Theorem 1.1 and Corollary 1.2 using work of Bringmann, and of Bruinier and the author. In Section 3 we conclude with several illuminating examples.

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2. Proofs

We assume that the reader is familiar with basic facts about modular forms and harmonic Maass forms (for background, see [12, 13, 14]). In [1], Andrews obtained the following generating function for $spt(n)$:

$$q^{\frac{1}{24}}\mathcal{S}(q^{\frac{1}{24}}) = \sum_{n=1}^{\infty} spt(n) q^n = \frac{1}{(q)_\infty} \cdot \sum_{n=1}^{\infty} \frac{q^n \cdot \prod_{m=1}^{n-1}(1-q^m)}{1-q^n} = q + 3q^2 + 5q^3 + \cdots, \tag{2.1}$$

where q is a formal parameter and $(q)_\infty = \prod_{n=1}^{\infty}(1-q^n)$. If we let $q := e^{2\pi i z}$, where z is in the upper-half of the complex plane, then we have the following important theorem[1] of Bringmann [5] which relates this generating function to a certain harmonic Maass form.

Theorem 2.1. *Define the function $\mathcal{M}(z)$ by*

$$\mathcal{M}(z) := \mathcal{S}(q) - \frac{\mathcal{D}(24z)}{12} - \frac{i}{4\pi\sqrt{2}} \cdot \int_{-\bar{z}}^{i\infty} \frac{\eta(24\tau)}{(-i(\tau+z))^{\frac{3}{2}}} \cdot d\tau,$$

where $\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is Dedekind's eta-function, and where

$$\mathcal{D}(24z) := \frac{1 - 24\sum_{n=1}^{\infty}\sum_{d|n} d q^{24n}}{\eta(24z)}.$$

Then $\mathcal{M}(z)$ is a weight 3/2 harmonic Maass form on $\Gamma_0(576)$ with Nebentypus $\chi_{12}(\bullet) := \left(\frac{12}{\bullet}\right)$.

[1] Theorem 2.1 corrects a sign error in [5].

2.1. **Producing modular forms.** We use Theorem 2.1 to obtain modular forms from the harmonic Maass form $\mathcal{M}(z)$. By the q-series manipulations in [1] and (1.2), it is known that

$$q\frac{d}{dq}\mathcal{P}(q) = -\mathcal{D}(24z) = -\frac{1}{q} + 23q^{23} + 94q^{47} + 213q^{71} + 475q^{95} + 833q^{119} + \dots.$$

Therefore, (1.1) and (1.3) imply that $M(z) = M(q)$ is the holomorphic part of $\mathcal{M}(z)$, and so it is a mock theta function.

For each prime $\ell \geq 5$, we let $T(\ell^2)$ be the index ℓ^2 Hecke operator for weight 3/2 harmonic Maass forms with Nebentypus χ_{12}. On q-series, these operators are defined by

$$\left(\sum a(n)q^n\right) \mid T(\ell^2) := \sum \left(a(\ell^2 n) + \left(\frac{3}{\ell}\right)\left(\frac{-n}{\ell}\right)a(n) + \ell a(n/\ell^2)\right)q^n. \tag{2.2}$$

We define $M_\ell(z)$ by

$$M_\ell(z) = M_\ell(q) = \sum a_\ell(n)q^n := M(q) \mid T(\ell^2) - \left(\frac{3}{\ell}\right)(1+\ell)M(q). \tag{2.3}$$

The following theorem is crucial to the proof of Theorem 1.1.

Theorem 2.2. *Suppose that $\ell \geq 5$ is prime, and that*

$$F_\ell(z) := \eta(z)^{\ell^2} \cdot M_\ell(z/24).$$

Then $F_\ell(z)$ is a weight $(\ell^2+3)/2$ holomorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$.

Proof. The operator $\xi := 2iy^{\frac{3}{2}} \cdot \overline{\frac{\partial}{\partial \bar{z}}}$, where $y = \mathrm{Im}(z)$, has the property that $\xi(\mathcal{M}) = -\frac{1}{8\pi} \cdot \eta(24z)$. Since $\eta(24z)$ is an eigenform of the weight 1/2 Hecke operators, Lemma 7.4 of [15] implies that $M_\ell(z)$ is a weight 3/2 weakly holomorphic modular form on $\Gamma_0(576)$ with Nebentypus χ_{12}. Here we used the fact that the eigenvalue of $\eta(24z)$ for the index ℓ^2 weight 1/2 Hecke operator is $\chi_{12}(\ell)(1+\ell^{-1})$.

It is straightforward to check that $M_\ell(z)$ has coefficients in $\frac{1}{12}\mathbb{Z}$, and has the property that

$$M_\ell(z) = -\frac{\ell}{12} \cdot q^{-\ell^2} + \left(\frac{3}{\ell}\right) \cdot \frac{\ell}{12} \cdot q^{-1} + \sum_{\substack{n \geq 23 \\ n \equiv 23 \pmod{24}}} a_\ell(n)q^n. \tag{2.4}$$

Here we have used the fact that $\ell^2 \equiv 1 \pmod{24}$. Therefore, it follows that

$$F_\ell(24z) = \eta(24z)^{\ell^2} M_\ell(z) = -\frac{\ell}{12} + \dots$$

is a weight $(\ell^2+3)/2$ weakly holomorphic modular form on $\Gamma_0(576)$ with trivial Nebentypus whose nonzero coefficients are supported on exponents which are multiples of 24. In particular, we have that $F_\ell(z) = F_\ell(z+1)$. To prove that $F_\ell(z)$ is a weakly holomorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$, it suffices to prove that

$$F_\ell(-1/z) = z^{\frac{\ell^2+3}{2}} F_\ell(z).$$

To this end, let W be the Fricke involution (see Section 3.2 of [13]) which acts on weight $3/2$ modular forms on $\Gamma_0(576)$ by

$$(f \mid W)(z) := (\sqrt{24} \cdot \sqrt{-iz})^{-3} \cdot f\left(-\frac{1}{576z}\right).$$

If f has Nebentypus χ, and if $\ell \nmid 576$ is prime, then it is well known that

$$f \mid W \mid T(\ell^2) = \overline{\chi(\ell^2)} \cdot f \mid T(\ell^2) \mid W.$$

If we let $A_\ell(z) := F_\ell(24z)$, then this commutation relation implies that

$$\begin{aligned} A_\ell\left(-\frac{1}{576z}\right) &= (\sqrt{24} \cdot \sqrt{-iz})^3 \cdot \eta\left(-\frac{1}{24z}\right)^{\ell^2} \cdot \left(M \mid T(\ell^2) \mid W - \left(\frac{3}{\ell}\right)(1+\ell)M \mid W\right) \\ &= (\sqrt{24} \cdot \sqrt{-iz})^3 \cdot \eta\left(-\frac{1}{24z}\right)^{\ell^2} \cdot \left(M \mid W \mid T(\ell^2) - \left(\frac{3}{\ell}\right)(1+\ell)M \mid W\right). \end{aligned}$$

Using the fact that

$$\eta(-1/z) = \sqrt{-iz} \cdot \eta(z),$$

we then find that

$$A_\ell\left(-\frac{1}{576z}\right) = (\sqrt{24} \cdot \sqrt{-iz})^{\ell^2+3} \eta(24z)^{\ell^2} \cdot \left(M \mid W \mid T(\ell^2) - \left(\frac{3}{\ell}\right)(1+\ell)M \mid W\right).$$

Bringmann proves that $\mathcal{M}(z)$ is an eigenform of W with multiplier arising from Dedekind's eta-function (see Section 4 of [5]). A reformulation of her result shows that

$$\mathcal{M}\left(-\frac{1}{576z}\right) = -(-24iz)^{\frac{3}{2}} \cdot \mathcal{M}(z).$$

Combining these facts, we have that

$$A_\ell\left(-\frac{1}{576z}\right) = (24z)^{\frac{\ell^2+3}{2}} \cdot A_\ell(z).$$

Letting $z \to z/24$ gives

$$F_\ell(-1/z) = A_\ell\left(-\frac{1}{24z}\right) = z^{\frac{\ell^2+3}{2}} \cdot A_\ell(z/24) = z^{\frac{\ell^2+3}{2}} \cdot F_\ell(z).$$

Therefore, $F_\ell(z)$ is a weight $(\ell^2+3)/2$ weakly holomorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$. Since it is holomorphic at infinity, it is a holomorphic modular form, and this completes the proof. □

2.2. **Proof of Theorem 1.1 and Corollary 1.2.** We now prove Theorem 1.1.

Proof of Theorem 1.1. By (2.4), we have that

$$F_\ell(24z) = \eta(24z)^{\ell^2} \cdot M_\ell(z) = \left(q^{\ell^2} - \dots\right) \cdot \left(-\frac{\ell}{12}q^{-\ell^2} - \frac{\ell}{12}q^{-1} + \sum_{\substack{n \geq 23 \\ n \equiv 23 \pmod{24}}} a_\ell(n) q^n\right).$$

Since the coefficients of $M_\ell(z)$ are ℓ-integral, $F_\ell(24z) \pmod{\ell}$ is well defined. Moreover, it follows that $\mathrm{ord}_\ell(F_\ell(24z)) \geq \ell^2 + 23$. Here ord_ℓ denotes the smallest exponent whose coefficient

is non-zero modulo ℓ. Therefore, we have that $\mathrm{ord}_\ell(F_\ell(z)) \geq (\ell^2+23)/24$. However, $F_\ell(z)$ is a weight $(\ell^2+3)/2$ holomorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$, and it is well known that every f in this space with ℓ-integral coefficients has either $\mathrm{ord}_\ell(f) \leq (\ell^2+3)/24$ or $\mathrm{ord}_\ell(f) = +\infty$. This follows from the existence of "diagonal bases" for spaces of modular forms on $\mathrm{SL}_2(\mathbb{Z})$. Therefore we have that $\mathrm{ord}_\ell(F_\ell(z)) = +\infty$, which in turn implies that $M_\ell(z) \equiv 0 \pmod{\ell}$. The theorem now follows from (1.3), (2.2) and (2.3). □

Proof of Corollary 1.2. Claim (1) follows since the right hand side is $0 \pmod{\ell}$ in Theorem 1.1. Claim (2) follows by replacing n by $n\ell$ in Theorem 1.1 since $\left(\frac{-n\ell}{\ell}\right) = 0$. □

3. Examples

Here we give examples which illustrate the results and modular forms in this paper.

3.1. **Explicit formulas for $M_5(z)$ and $M_7(z)$.** Here we compute the level 1 modular forms $F_5(z)$ and $F_7(z)$ in terms of $\Delta(z) := \eta(z)^{24}$, and the usual Eisenstein series

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sum_{d|n} d^3q^n \quad \text{and} \quad E_6(z) = 1 - 504\sum_{n=1}^{\infty}\sum_{d|n} d^5q^n.$$

For $\ell = 5$, we find that

$$M_5(z) = -\frac{5}{12}\cdot q^{-25} - \frac{5}{12}\cdot q^{-1} + \frac{492205}{6}\cdot q^{23} + \dots,$$
$$F_5(24z) = \eta(24z)^{25}\cdot M_5(z) = -\frac{5}{12} + 10q^{24} + 81930q^{48} + 15943240q^{72} + \dots.$$

Theorem 2.2 implies that $F_5(z)$ is a weight 14 holomorphic modular form, and we find that

$$F_5(z) = -\frac{5}{12}\cdot E_4(z)^2E_6(z) = -\frac{5}{12} + 10q + 81930q^2 + \dots.$$

Therefore, we have that

$$M_5(z) = -\frac{5}{12}\cdot\frac{E_4(24z)^2E_6(24z)}{\eta(24z)^{25}}.$$

For $\ell = 7$, we find that $F_7(z)$ is the weight 26 modular form

$$F_7(z) = -\frac{7}{12}\cdot E_4(z)^5E_6(z) + \frac{5215}{12}\cdot\Delta(z)E_4(z)^2E_6(z),$$

which in turn implies that

$$M_7(z) = -\frac{1}{12}\cdot\left(\frac{7E_4(24z)^5E_6(24z) - 5215\Delta(24z)E_4(24z)^2E_6(24z)}{\eta(24z)^{49}}\right).$$

3.2. **Example of Corollary 1.2 (1).** If $\ell \geq 5$ is prime, then let $O_\ell(q)$ be the series

$$O_\ell(q) := \sum_{\left(\frac{-n}{\ell}\right)=1} \widehat{s}(\ell^2 n) q^n.$$

By Corollary 1.2 (1), we have that $O_\ell(q) \equiv 0 \pmod{\ell}$. If $\ell = 11$, then we indeed see that

$$O_{11}(q)$$
$$= 1234141921846851217211 0q^{95} + 819052154915850436964574391585q^{167} + \cdots \equiv 0 \pmod{11}.$$

3.3. **Example of Corollary 1.2 (2).** If $\ell \geq 5$ is prime, then let

$$T_\ell^{(1)}(q) := \sum_{n=1}^{\infty} \widehat{s}(\ell n) q^n,$$

$$T_\ell^{(3)}(q) := \sum_{n=1}^{\infty} \widehat{s}(\ell^3 n) q^n.$$

Corollary 1.2 (2) then asserts that

$$T_\ell^{(3)}(q) \equiv \left(\frac{3}{\ell}\right) T_\ell^{(1)}(q) \pmod{\ell}.$$

For $\ell = 11$, we find that

$$\left(\frac{3}{11}\right) \cdot T_{11}^{(1)}(q) = 26q^{13} + 1048q^{37} + 16562q^{61} + \cdots \equiv 4q^{13} + 3q^{37} + 7q^{61} + \ldots \pmod{11}$$

and that

$$\begin{aligned} T_{11}^{(3)}(q) &= 342156714900173087653891 1832q^{13} \\ &\quad + 721427557133531761496593371848380785660101905536q^{37} \\ &\quad + 120494776849783345014198876429157577016120072623960718684904344q^{61} + \ldots \\ &\equiv 4q^{13} + 3q^{37} + 7q^{61} + \ldots \pmod{11}. \end{aligned}$$

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