GAUSS'S ${}_2F_1$ HYPERGEOMETRIC FUNCTION AND THE CONGRUENT NUMBER ELLIPTIC CURVE

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Abstract. Gauss's hypergeometric function gives a modular parameterization of period integrals of elliptic curves in Legendre normal form

$$E(\lambda): \quad y^2 = x(x-1)(x-\lambda).$$

We study a modular function which "measures" the variation of periods for the isomorphic curves $E(\lambda)$ and $E\left(\frac{\lambda}{\lambda-1}\right)$, and we show that it p-adically "interpolates" the cusp form for the "congruent number" curve $E(2)$, the case where these pairs collapse to a single curve.

1. Introduction and statement of results

For $\lambda \in \mathbb{C} \setminus \{0,1\}$, the Legendre normal form elliptic curve $E(\lambda)$ is given by

$$E(\lambda): \quad y^2 = x(x-1)(x-\lambda). \tag{1}$$

It is well known (for example, see [4]) that $E(\lambda)$ is isomorphic to the complex torus $\mathbb{C}/L_\lambda$, where $L_\lambda = \mathbb{Z}\omega_1(\lambda) + \mathbb{Z}\omega_2(\lambda)$, and the periods $\omega_1(\lambda)$ and $\omega_2(\lambda)$ are given by the integrals

$$\omega_1(\lambda) = \int_{-\infty}^{0} \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} \quad \text{and} \quad \omega_2(\lambda) = \int_{1}^{\infty} \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}.$$

These integrals can be expressed in terms of Gauss's hypergeometric function

$$ {}_2F_1(x) := {}_2F_1\left(\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ & 1 \end{matrix} ; x\right) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{2})_n}{(n!)^2} x^n, \tag{2}$$

where $(a)_n = a \cdot (a+1) \cdots (a+n-1)$. More precisely, for $\lambda \in \mathbb{C} \setminus \{0,1\}$ with $|\lambda|, |\lambda - 1| < 1$, we have

$$\omega_1(\lambda) = i\pi\, {}_2F_1(1-\lambda) \quad \text{and} \quad \omega_2(\lambda) = \pi\, {}_2F_1(\lambda). \tag{3}$$

The parameter λ is a "modular invariant". To make this precise, for z in $\mathbb{H}$, the upper half of the complex plane, we define the lattice $\Lambda_z := \mathbb{Z} + \mathbb{Z}z$,

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and let $\wp$ be the Weierstrass elliptic function associated to Λ_z. The function $\lambda(z)$ defined by

$$\lambda(z) := \frac{\wp(\frac{1}{2}) - \wp(\frac{z+1}{2})}{\wp(\frac{z}{2}) - \wp(\frac{z+1}{2})} = 16q^{\frac{1}{2}} \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1+q^{n-\frac{1}{2}}} \right)^8, \tag{4}$$

where $q := e^{2\pi i z}$, is a modular function on $\Gamma(2)$ that parameterizes the Legendre normal family above. In particular, we have that $\mathbb{C}/\Lambda_z \cong E(\lambda(z))$. Furthermore, for any lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\Im(\frac{\omega_1}{\omega_2}) > 0$, we have that $\mathbb{C}/\Lambda$ is isomorphic (over $\mathbb{C}$) to $E(\lambda)$ if and only if λ is in the orbit of $\lambda\left(\frac{\omega_1}{\omega_2}\right)$ under the action of the modular quotient $\mathrm{SL}_2(\mathbb{Z})/\Gamma(2) \cong \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$. This quotient is isomorphic to S_3, and the orbit of λ is

$$\left\{ \lambda,\ \frac{1}{\lambda},\ 1-\lambda, \frac{1}{1-\lambda},\ \frac{\lambda}{\lambda-1},\ \frac{\lambda-1}{\lambda} \right\}.$$

In view of this structure, it is natural to study expressions like

$$\omega_2(\lambda) - \omega_2\left(\frac{\lambda}{\lambda-1}\right) \tag{5}$$

which measures the difference between periods of the isomorphic elliptic curves $E(\lambda)$ and $E\left(\frac{\lambda}{\lambda-1}\right)$. Taking into account that $\lambda(z)$ has level 2, it is natural to consider the modular function

$$L(z) := \frac{{}_2F_1(\lambda(z)) - {}_2F_1\left(\frac{\lambda(z)}{\lambda(z)-1}\right)}{{}_2F_1(\lambda(2z)) - {}_2F_1\left(\frac{\lambda(2z)}{\lambda(2z)-1}\right)} = q^{-1} + 2q^3 - q^7 - 2q^{11} + \dots. \tag{6}$$

It turns out that $L(z)$ is a Hauptmodul for the genus zero congruence group $\Gamma_0(16)$. Here we study the p-adic properties of the Fourier expansion of $L(z)$ using the the theory of *harmonic Maass forms.* To make good use of this theory, we "normalize" $L(z)$ to obtain a weight 2 modular form whose poles are supported at the cusp infinity for a modular curve with positive genus. The first case where this occurs is $\Gamma_0(32)$, where the space of weight 2 cusp forms is generated by the unique normalized cusp form

$$g(z) := q\prod_{n=1}^{\infty}(1-q^{4n})^2(1-q^{8n})^2 = q - 2q^5 - 3q^9 + 6q^{13} + \dots. \tag{7}$$

Our normalization is

$$\mathfrak{F}(z) = \sum_{n=-1}^{\infty} C(n)q^n := -g(z)L(2z) = -q^{-1} + 2q^3 + q^7 - 2q^{11} + 5q^{15} + \dots. \tag{8}$$

Remark. It turns out that $\mathfrak{F}(z)$ satisfies the following identities:

$$\mathfrak{F}(z) = \frac{1}{2\pi i}\cdot\frac{d}{dz}L(z) - 4\frac{g(z)}{L(2z)} = L(z)\,{}_2F_1\left(\frac{\lambda(4z)}{\lambda(4z)-1}\right)\cdot {}_2F_1(\lambda(8z)).$$

The cusp form $g(z)$ plays a special role in the context of Legendre normal form elliptic curves. Under the Shimura-Taniyama correspondence, $g(z)$ is the cusp form which gives the Hasse-Weil L-function for $E(-1)$, the congruent number elliptic curve

$$E(-1): \quad y^2 = x^3 - x. \tag{9}$$

By the change of variable $x \mapsto x-1$, we have that $E(-1)$ is isomorphic to $E(2)$. Since $\lambda = \frac{\lambda}{\lambda-1}$ when $\lambda = 2$, we see that $g(z)$ is the cusp form corresponding to the "fixed point" of (5).

We show that $\mathfrak{F}(z)$ has some surprising p-adic properties which relate the Hauptmodul $L(z)$ to the cusp form $g(z)$. These properties are formulated using Atkin's U-operator

$$\sum a(n)q^n \mid U(m) := \sum a(mn)q^n. \tag{10}$$

Theorem 1.1. *If $p \equiv 3 \pmod 4$ is a prime for which $p \nmid C(p)$, then as a p-adic limit we have*

$$g(z) = \lim_{w\to\infty} \frac{\mathfrak{F}(z)|U(p^{2w+1})}{C(p^{2w+1})}.$$

Remark. The p-adic limit in Theorem 1.1 means that if we write $g(z) = \sum_{n=1}^{\infty} a_g(n)q^n$, then for all positive integers n the difference

$$\frac{C(np^{2w+1})}{C(p^{2w+1})} - a_g(n)$$

becomes uniformly divisible by arbitrarily large powers of p as $w \to +\infty$.

Remark. A short calculation in MAPLE shows that $p \nmid C(p)$ for every prime $p \equiv 3 \pmod 4$ less than 25,000. We speculate that there are no primes $p \equiv 3 \pmod 4$ for which $p \mid C(p)$.

Example. Here we illustrate the phenomenon in Theorem 1.1 for the primes $p = 3$ and 7. For convenience, we let

$$\mathfrak{F}_w(p; z) := \frac{\mathfrak{F}(z) \mid U(p^{2w+1})}{C(p^{2w+1})}. \tag{11}$$

If $p = 3$, then we have

$$\begin{aligned}
\mathfrak{F}_0(3;z) &= q + \tfrac{5}{2}q^5 + 6q^9 - 34q^{17} + \cdots \equiv g(z) && \pmod 3\\
\mathfrak{F}_1(3;z) &= q + \tfrac{5}{2}q^5 - \tfrac{519}{2}q^9 - \tfrac{39}{4}q^{13} - 1258q^{17} + \cdots \equiv g(z) && \pmod{3^2}\\
\mathfrak{F}_2(3;z) &= q - \tfrac{665}{346}q^5 + \tfrac{26923476}{173}q^9 + \cdots \equiv g(z) && \pmod{3^3}\\
\mathfrak{F}_3(3;z) &= q - \tfrac{150604045}{4487246}q^5 - \tfrac{340313285484369963465663}{8974492}q^9 + \cdots \equiv g(z) && \pmod{3^4}.
\end{aligned}$$

If $p = 7$, then we have

$$\mathfrak{F}_0(7; z) = q + 40q^5 + 18q^9 + 104q^{13} + 51q^{17} + \cdots \equiv g(z) \pmod{7}$$

$$\mathfrak{F}_1(7; z) = q + \tfrac{19167440}{43}q^5 - \tfrac{93915}{43}q^9 + \tfrac{215354309456}{43}q^{13} + \cdots \equiv g(z) \pmod{7^2}.$$

Theorem 1.1 arises naturally in the theory of harmonic Maass forms. The proof depends on establishing a certain relationship between $\mathfrak{F}$ and g. This is achieved by viewing them as certain derivatives of the holomorphic and non-holomorphic parts of a harmonic weak Maass form that we explicitly construct as a Poincaré series. We then use recent work of Guerzhoy, Kent, and the second author [3] that explains how to relate such derivatives of a harmonic Maass form p-adically. (Cf. Section 2).

2. Proof of Theorem 1.1

Here we prove Theorem 1.1 after recalling crucial facts about harmonic Maass forms.

2.1. **Harmonic Maass forms and a certain Poincaré series.** We begin by recalling some basic facts about harmonic Maass forms (for example, see Sections 7 and 8 of [7]). Suppose that $k \geq 2$ is an even integer. The weight k hyperbolic Laplacian is defined by

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

A *harmonic weak Maass form of weight k on* $\Gamma_0(N)$ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying

(1) f is invariant under the usual $|_k\gamma$ slash operator for every $\gamma \in \Gamma_0(N)$.
(2) $\Delta_k f = 0$.
(3) There exists a polynomial

$$P_f = \sum_{n=0}^{n_f} c_f^+(-n)q^{-n} \in \mathbb{C}[q^{-1}]$$

such that $f(z) - P_f(z) = O(e^{-\varepsilon y})$ as $y \to \infty$ for some $\varepsilon > 0$. We require similar growth conditions at all other cusps of $\Gamma_0(N)$

The polynomial P_f, for a given cusp, is called the *principal part* of f at that cusp. The vector space of all forms satisfying these conditions is denoted by $H_k(N)$. Note that if $M_k^!(N)$ denotes the space of weakly holomorphic modular forms on $\Gamma_0(N)$ then $M_k^!(N) \subset H_k(N)$.

Any form $f \in H_{2-k}(N)$ has a natural decomposition as $f = f^+ + f^-$, where f^+ is holomorphic on $\mathbb{H}$ and f^- is a smooth non-holomorphic function on $\mathbb{H}$. Let D be the differential operator $\frac{1}{2\pi i}\frac{d}{dz}$ and let $\xi_r := 2iy^r\overline{\frac{\partial}{\partial \overline{z}}}$. Then we have that

$$D^{k-1}(f) = D^{k-1}(f^+) \in M_k^!(N) \quad \text{and} \quad \xi_{2-k}(f) = \xi_{2-k}(f^-) \in S_k(N), \tag{12}$$

where $S_k(N)$ is the space of weight k cusp forms on $\Gamma_0(N)$. In particular, there is a cusp form g_f of weight k attached to any Maass form f of weight $2-k$. Since $\xi_{2-k}(M_{2-k}^!(N)) = 0$, it follows that many harmonic Maass forms correspond to g_f. In [2], Bruinier, Rhoades, and the second author narrow down the correspondence by specifying certain additional restrictions on f. Specifically, they define a harmonic weak Maass form $f \in H_{2-k}(N)$ to be *good* for a normalized newform $g \in S_k(N)$, whose coefficients lie in a number field F_g, if the following conditions are satisfied:

(1) The principal part of f at the cusp ∞ belongs to $F_g[q^{-1}]$.
(2) The principal parts of f at other cusps (if any) are constant.
(3) $\xi_{2-k}(f) = \frac{g}{\|g\|^2}$, where $\|\cdot\|$ is the Petersson norm.

It is also shown in that paper that every newform has a corresponding good Maass form.

Theorem 1.1 depends on the interplay between the newform $g(z)$ in (7) and a certain harmonic Maass form which is intimately related to the Hauptmodul $L(z)$. These forms are constructed using Poincaré series.

We first recall the definition of (holomorphic) Poincaré series. Let $\Gamma_0(N)_\infty$ denote the stabilizer of ∞ in $\Gamma_0(N)$ and set $e(z) := e^{2\pi i z}$. For integers m, $k > 2$ and positive N, the classical holomorphic Poincaré series is defined by

$$P(m,k,N;z) := \sum_{\gamma \in \Gamma_0(N)_\infty \backslash \Gamma_0(N)} e(tz)|_k\gamma = q^m + \sum_{n=1}^{\infty} a(m,k,N;n)q^n.$$

We extend the definition to the case $k = 2$ using "Hecke's trick". For a positive integer m, we have that $P(m,k,N;z) \in S_k(N)$ and $P(-m,k,N;z) \in M_k^!(N)$. The Poincaré series $P(-m,k,N;z)$ is holomorphic at all cusps except ∞ where the principal part is q^{-m}.

The coefficients of these functions are infinite sums of Kloosterman sums multiplied with the I_n and J_n Bessel functions. The modulus c Kloosterman sum $K_c(a,b)$ is

$$K_c(a,b) := \sum_{v \in (\mathbb{Z}/c\mathbb{Z})^\times} e\left(\frac{av + bv^{-1}}{c}\right).$$

It is well known (for example, see [5] or Proposition 6.1 of [2]) that for positive integers m we have

$$a(m,k,N;n) = 2\pi(-1)^{\frac{k}{2}}\left(\frac{n}{m}\right)^{\frac{k-1}{2}} \cdot \sum_{c=1}^{\infty} \frac{K_{Nc}(m,n)}{Nc} \cdot J_{k-1}\left(\frac{4\pi\sqrt{mn}}{Nc}\right),$$

and

$$a(-m,k,N;n) = 2\pi(-1)^{\frac{k}{2}}\left(\frac{n}{m}\right)^{\frac{k-1}{2}} \cdot \sum_{c=1}^{\infty} \frac{K_{Nc}(-m,n)}{Nc} \cdot I_{k-1}\left(\frac{4\pi\sqrt{mn}}{Nc}\right).$$

Furthermore, for positive m, the Petersson norm of the cusp form $P(m,k,N;z)$ is given by

$$\|P(m,k,N;z)\|^2 = \frac{(k-2)!}{(4\pi m)^{k-1}}(1+a(m,k,N;m)). \tag{13}$$

These Poincaré series are related to the Maass-Poincaré series which we now briefly recall. Let $M_{\nu,\mu}(z)$ be the usual Whittaker function given by

$$M_{\nu,\mu}(z) = e^{-\frac{z}{2}} z^{\mu+\frac{1}{2}} {}_1F_1(\mu-\nu+\frac{1}{2}; 1+2\mu; z),$$

where ${}_1F_1(a,b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{z^n}{n!}$. For $y>0$ set

$$\mathcal{M}^*_{-m,k}(x+iy) := e(-mx)(4\pi my)^{-\frac{k}{2}} M_{-\frac{k}{2},\frac{1-k}{2}}(4\pi my).$$

Then, for $k>2$ the Poincaré series

$$Q(-m,k,N;z) := \sum_{\gamma\in\Gamma_0(N)_\infty\backslash\Gamma_0(N)} \mathcal{M}^*_{-m,k}(z)|_k\gamma$$

is in $H_{2-k}(N)$ (for example, see [2]). This series converges normally for for $k>2$, and we can extend its definition to the case $k=2$ using analytic continuation to get a form in $H_0(N)$. These different Poincaré series are connected via the differential operators D and ξ_{2-k} as follows (see §6.2 of [2]):

$$D^{k-1}(Q(-m,k,N;z)) = -m^{k-1}P(-m,k,N;z), \tag{14}$$

and

$$\xi_{2-k}(Q(-m,k,N;z)) = \frac{(4\pi m)^{k-1}}{(k-2)!}\cdot P(m,k,N;z). \tag{15}$$

The following lemma relates $\mathfrak{F}$ and g using these Poincaré series.

Lemma 2.1. *The following are true:*

(1) We have that

$$g(z) = \frac{P(1,2,32;z)}{(1+a(1,2,32;1))} \quad \text{and} \quad \mathfrak{F}(z) = -P(-1,2,32;z).$$

(2) We have that $Q(-1,2,32;z)$ is good for g.

(3) We have that $D(Q(-1,2,32;z)) = \mathfrak{F}(z)$.

Proof. Since g and $P(1,2,32;z)$ are both nonzero cusp forms in the one dimensional space $S_2(32)$, the first equality follows easily. For the second equality, note that $\mathfrak{F}$ and $-P(-1,2,32;z)$ have the same principal part at ∞ and no constant term, hence their difference must be in $S_2(32)$, hence a multiple of g. Further, since $K_{32c}(-1,1)=0$ for all $c\geq 1$, we see that the coefficient of q in both $\mathfrak{F}$ and $-P(-1,2,32;z)$ is zero, and it follows that they must be equal. The proof of the "goodness" of Q follows at from the properties of Q listed above and from (13) and (15). Claim (3) now follows from (14). □

2.2. **Proof of Theorem 1.1.** Theorem 1.1 is a consequence of the following theorem which was recently proved by Guerzhoy, Kent, and the second author.

Theorem 2.2. [Theorem 1.2 (2) of [3]] *Let $g\in S_k(N)$ be a normalized CM newform. Suppose that $f\in H_{2-k}(N)$ is good for g and set*

$$F := D^{k-1}f = \sum_{n\gg-\infty} c(n)q^n.$$

If p is an inert prime in the CM field of g such that $p^{k-1}\nmid c(p)$, and if

$$\lim_{w\to\infty} p^{-w(k-1)}F|U(p^{2w+1})\neq 0. \tag{16}$$

Then as a p-adic limit we have

$$g=\lim_{w\to\infty}\frac{F|U(p^{2w+1})}{c(p^{2w+1})}.$$

We require a lemma regarding the existence of certain modular functions with integral coefficients that are holomorphic away from the cusp infinity.

Lemma 2.3. *Let $\mathbb{Z}((q))$ denote the ring of Laurent series in q over $\mathbb{Z}$.*

(1) For each positive integer $n\not\equiv 1 \pmod 4$ there exists a modular function

$$\phi_n = q^{-n}+O(q)\in M_0^!(32)\cap\mathbb{Z}((q))$$

such that ϕ_n is holomorphic at all cusps except infinity.

(2) For each $n\geq 5$ with $n\equiv 1 \pmod 4$ there exists a modular function

$$\phi_n = q^{-n}+a_{-1}q^{-1}+O(q)\in M_0^!(32)\cap\mathbb{Z}((q))$$

such that ϕ_n is holomorphic at all cusps except infinity.

In both cases, the coefficients of $\phi_n(z)$ vanish for all indices not congruent to $-n \pmod 4$.

Proof. This follows by induction. Specifically, let $L(z)$ be as in 6 and set

$$\phi_2(z) := L(2z) = q^{-2} + 2q^6 - q^{14} + \dots$$
$$\phi_3(z) := L(z)L(2z) = q^{-3} + 2q + q^5 + 2q^9 + \dots.$$

Both ϕ_2 and ϕ_3 are modular functions of level 32 with integer coefficients. It is clear one can inductively construct polynomials

$$\Psi_n(x,y) = \sum t_n(i,j)x^iy^j \in \mathbb{Z}[x,y]$$

such that $\Psi_n(\phi_2(z), \phi_3(z))$ satisfies the conditions on the principal parts in Lemma 2.3. For example

$$\phi_7(z) = \phi_3(7)\phi_2(z)^2 - 2\phi_3(7) = q^{-7} + q + 8q^5 + 2q^9 + \dots.$$

Furthermore, if n is even (resp. $n \equiv 3 \pmod 4$, resp. $n \equiv 1 \pmod 4$) then one sees that $\Psi_n(x,y) = \Psi_n(x,1)$ (i.e it is purely a polynomial in x) (resp. $\Psi_n(x,y)$ equals y multiplied by a polynomial in x^2, resp. $\Psi_n(x,y)$ equals xy multiplied by a polynomial in x^2). This remark establishes the last assertion. □

This sequence of modular functions turns out to be closely related to $\mathfrak{F}$ as follows.

Corollary 2.4. *If $n \geq 2$ and $\phi_n(z) = \sum_{l=-n}^{\infty} A_n(l)q^l$, then $C(n) = -A_n(1)$.*

Proof. Since $C(n) = 0$ whenever $n \not\equiv 3 \pmod 4$, then by part (3) of the above lemma, the corollary follows trivially for such n. For $n \equiv 3 \pmod 4$, the meromorphic differential $\mathfrak{F}(z)\phi_n(z)dz$ is holomorphic everywhere except at the cusp infinity. Recall that the sum of residues of a meromorphic differential is zero. Furthermore, the residue at ∞ of the differential $h(z)dz$ (for any weight 2 form h) is a multiple of the constant term in its q-expansion. Since $\mathfrak{F}(z) = DQ(-1, 2, 32; z)$ we see that $\mathfrak{F}$ has no constant term at any cusp, and hence $\mathfrak{F}\phi_n$ vanishes at all cusps except ∞. It follows that the residue at the cusp ∞ must be zero, and the result follows since the constant term of the q-expansion of $\mathfrak{F}(z)\phi_n(z)$ is $C(n) + A_n(1)$. □

Proof of Theorem 1.1. By Theorem 2.2, Lemma 2.1, and the fact that those primes inert in $\mathbb{Q}(i)$, the CM field for g, are the primes $p \equiv 3 \pmod 4$, it suffices to prove equation (16) under the assumption that $p \nmid C(p)$.

Recall that the weight k m-th Hecke operator $T(m)$ (see [6, 7]) acts on $M_k^!(N)$ by

$$f|T(m)(z) = f|U(p)(z) + p^{k-1}f(pz). \qquad (17)$$

It is obvious from the definition that integrality of the coefficients is preserved for forms of positive weight. In particular, for

$$\mathfrak{F} = -q^{-1} + 2q^3 + q^7 - 2q^{11} + \dots,$$

we get

$$\mathfrak{F} \mid_2 T(p) = -pq^{-p} + C(p)q + O(q^2),$$

and $\mathfrak{F} \mid_2 T(p)$ is holomorphic at all cusps except ∞. For $p \equiv 3 \pmod 4$ Lemma 2.3 and Corollary 2.4 give that

$$\mathfrak{F} \mid_2 T(p)(z) = \phi_p'(z) = \sum_{n=-p}^{\infty} a_{\phi_p'}(n)q^n = \sum_{n=-p}^{\infty} nA_p(n)q^n. \tag{18}$$

From (17) we get

$$\mathfrak{F}|U(p) = \phi_p'(z) - p\mathfrak{F}(pz).$$

Acting by $U(p^2)$ gives

$$\mathfrak{F}|U(p^3) = \phi_p'|U(p^2) - p\mathfrak{F}(z)|U(p),$$

and it follows by induction that

$$p^{-w}\mathfrak{F}|U(p^{2w+1}) = \sum_{l=1}^{w} p^{-l}\phi_p'|U(p^{2l}) - \mathfrak{F}|U(p). \tag{19}$$

If

$$\lim_{w\to\infty} p^{-w}\mathfrak{F}|U(p^{2w+1}) = 0,$$

then

$$\mathfrak{F}|U(p) = \sum_{l=1}^{\infty} p^{-l}\phi_p'|U(p^{2l}).$$

(The convergence here is p-adic). Focusing on the coefficient of q gives

$$C(p) = \sum_{l=1}^{\infty} p^{-l}a_{\phi_p'}(p^{2l}) = \sum_{l=1}^{\infty} p^{-l}p^{2l}(A_p(p^{2l})).$$

Hence

$$C(p) = p\sum_{l=1}^{\infty} p^{l-1}(A_p(p^{2l})),$$

which contradicts the hypothesis that $p \nmid C(p)$. Thus hypothesis (16) is satisfied thereby proving the theorem. □

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References

[1] J. H. Bruinier, K. Ono, *Heegner divisors, L-functions and harmonic weak Maass forms*, Annals of Mathematics, to appear

[2] J. H. Bruinier, K. Ono, and R. C. Rhoades, *Differential operators for harmonic weak Maass forms and the vanishing of Hecke eigenvalues,* Math. Ann. 342 (2008), 673-693

[3] P. Guerzhoy, Z. Kent, and K. Ono, *p-adic coupling of mock modular forms and shadows*, Proc. Natl. Acad. Sci., USA, accepted for publication.

[4] D. Husemöller, *Elliptic Curves*, Springer Verlag, Graduate Texts in Mathematics, 111 (2004)

[5] H. Iwaniec, *Topics in the classical theory of automorphic forms*, Grad Studies in Math. 17, Amer. Math. Soc., Providence, R.I., 1997

[6] K. Ono, *Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-Series*, Amer. Math. Soc., (2004).

[7] K. Ono, *Unearthing the visions of a master: harmonic Maass forms and number theory*, Harvard-MIT Current Developments in Mathematics 2008, International Press.

[8] B. Schoeneberg, *Elliptic modular functions*, Springer-Verlag, Grundlehren der Mathematischen Wissenschaften, 203 (1974).

[9] G. Shimura, *Introduction to the arithmetic theory of automorphic functions* (Princeton University Press, 1971).

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