The Last Words of a Genius

Ken Ono

This story begins with a cryptic letter written by a dying genius, the clues of which inspired scores of mathematicians to embark on an adventure which resembles an Indiana Jones movie. It is reminiscent of the quest for the Holy Grail, in which skillful knights confront great obstacles. But these knights are mathematicians, and the Grail is replaced by a mathematical "Rosetta Stone" that promises to reveal hidden truths in new worlds.

The Saga

Our drama begins on March 27, 1919, the date of Srinivasa Ramanujan's triumphant, but bittersweet, Indian homecoming. Five years earlier, accepting an invitation from the eminent British mathematician G. H. Hardy, the amateur Ramanujan had left for Cambridge University with the dream of making a name for himself in the world of mathematics. Now, stepping off the ship *Nagoya* in Bombay (now Mumbai), the two-time college dropout, who had intuited unimaginable formulas, returned as a world-renowned number theorist. He had achieved his goal. At the young age of thirty-one, Ramanujan had already made important contributions to a mindboggling array of subjects:[1] the distribution of prime numbers, hypergeometric series, elliptic functions, modular forms, probabilistic number theory, the theory of partitions, and q-series, among others. He had published over thirty papers, including seven with Hardy. In recognition of these accomplishments, Ramanujan was named a Fellow of Trinity College, and he was elected a Fellow of the Royal Society (F.R.S.), an honor shared by Sir Isaac Newton.

Sadly, the occasion of Ramanujan's homecoming was not one of celebration. He was a very sick man; he was much thinner than the rotund Ramanujan his Indian friends remembered. One of the main reasons for his declining health was malnutrition. He had been adhering to a strict vegetarian diet in a time and place with no adequate resources to support it. He also struggled with the severe change in climate. Accustomed to the temperate weather of south India, he did not have or did not wear appropriate clothing to protect him from the cool and damp Cambridge weather. These conditions took their toll, and he became gravely ill. He was diagnosed[2] with tuberculosis, and he returned to India seeking familiar surroundings, a forgiving climate, and a return to good health. Tragically, Ramanujan's health declined over the course of the following year, and he passed away on April 26, 1920, in Madras (now Chennai), with his wife Janaki by his side.

Ramanujan's Last Letter

Amazingly, in spite of his condition, Ramanujan spent his last year working on mathematics in isolation.

> *...through all the pain and fever,...Ramanujan, lying in bed, his head propped up on pillows, kept working. When he required it, Janaki would give him his slate; later she'd gather up the accumulated sheets of mathematics-covered paper ...and place them in the big leather box which he brought from England* (see p. 329 of [29]).

Janaki would later remember these last days (see p. 91 of [38]):

> *He was only skin and bones. He often complained of severe pain. In spite of it he was always busy*

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[1] *See* [4, 6, 7, 8, 25, 29, 35] *for more on Ramanujan and his achievements.*

[2] *The diagnosis of tuberculosis is now believed to be incorrect. D. A. B. Young examined the evidence pertaining to Ramanujan's illness, and he concluded that Ramanujan died of hepatic amoebiasis* [42].

doing his mathematics. …Four days before he died he was scribbling.

In a fateful last letter to Hardy, dated January 12, 1920, Ramanujan shared hints (see p. 220 of [7]) of his last theory.

I am extremely sorry for not writing you…I discovered very interesting functions recently which I call "Mock" theta functions.…they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples.

This mysterious letter set off a great adventure: the quest to realize the meaning behind these last words and to then unearth the implications of this understanding. These words exposed, in an unexplored territory of the world of mathematics, a padlocked wooden gate, beyond which was the promise of unknown mathematical treasures.

The Early Years

The letter, roughly four typewritten pages, consists of formulas for seventeen strange power series and a discussion of their asymptotics and behavior near the boundary of the unit disk. There are no proofs of any kind. Ramanujan also grouped these series based on their "order", a term he did not define. As a typical example, he offered

$$(1.1)\quad f(q) = \sum_{n=0}^{\infty} a_f(n)q^n := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2} = 1 + q - \cdots + 17503q^{99} + \ldots,$$

which he called a *third-order mock theta function.* He then miraculously claimed that

$$(1.2)\qquad a_f(n) \sim \frac{(-1)^{n-1}}{2\sqrt{n-\frac{1}{24}}} \cdot e^{\pi\sqrt{\frac{n}{6}-\frac{1}{144}}}.$$

Obviously Ramanujan knew much more than he revealed.

G. N. Watson was the first mathematician to take on the challenge. He worked for years, and on November 14, 1935, at a meeting of the London Mathematical Society, he celebrated his retirement as president of the Society with his now famous address [40]:

It is not unnatural that [one's] mode of approach to the preparation of his valedictory address should have taken the form of an investigation into the procedure of his similarly situated predecessors.…I was, however, deterred from this course…[Ramanujan's last] letter is the subject which I have chosen…; I doubt whether a more suitable title could be found for it than the title used by John H. Watson, M.D., for what he imagined to be his final memoir on Sherlock Holmes.

Watson chose the title *The Final Problem: An Account of the Mock Theta Functions.*

He proceeded to describe his findings, a medley of identities and formulas. Using "q-hypergeometric series", he reformulated Ramanujan's examples. For $f(q)$ he proved:

$$(1.3)\quad f(q) = \frac{2}{\prod_{n=1}^{\infty}(1-q^n)} \cdot \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{(3n^2+n)/2}}{1+q^n}.$$

He also proved identities relating the mock theta functions to *Mordell integrals*, such as

$$(1.4)\quad \int_0^{\infty} e^{-3\pi x^2} \cdot \frac{\sinh \pi x}{\sinh 3\pi x}\, dx = \frac{1}{e^{2\pi/3}\sqrt{3}} \times \sum_{n=0}^{\infty} \frac{e^{-2n(n+1)\pi}}{(1+e^{-\pi})^2(1+e^{-3\pi})^2\cdots(1+e^{-(2n+1)\pi})^2}.$$

He concluded by entrusting the quest to the next generation of mathematicians.

Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. …To his students such discoveries will be a source of delight and wonder until the time shall come when we too shall make our journey to that Garden of Proserpine [Persephone] …

Mathematicians continued the pursuit. In the late 1930s A. Selberg, as a high school student, published his first two mathematical papers on the subject. In the 1950s and 1960s, G. E. Andrews and L. Dragonette, employing Watson's results, finally confirmed Ramanujan's claimed asymptotic (1.2).[3] Many mathematicians, among them B. C. Berndt, Y.-S. Choi, B. Gordon, and R. McIntosh, progressed further along the lines set by Watson. After many technical calculations that run on for pages, these mathematicians mastered the asymptotics of Ramanujan's examples, amassed identities such as (1.3), and obtained analytic transformations relating these examples to integrals such as (1.4).

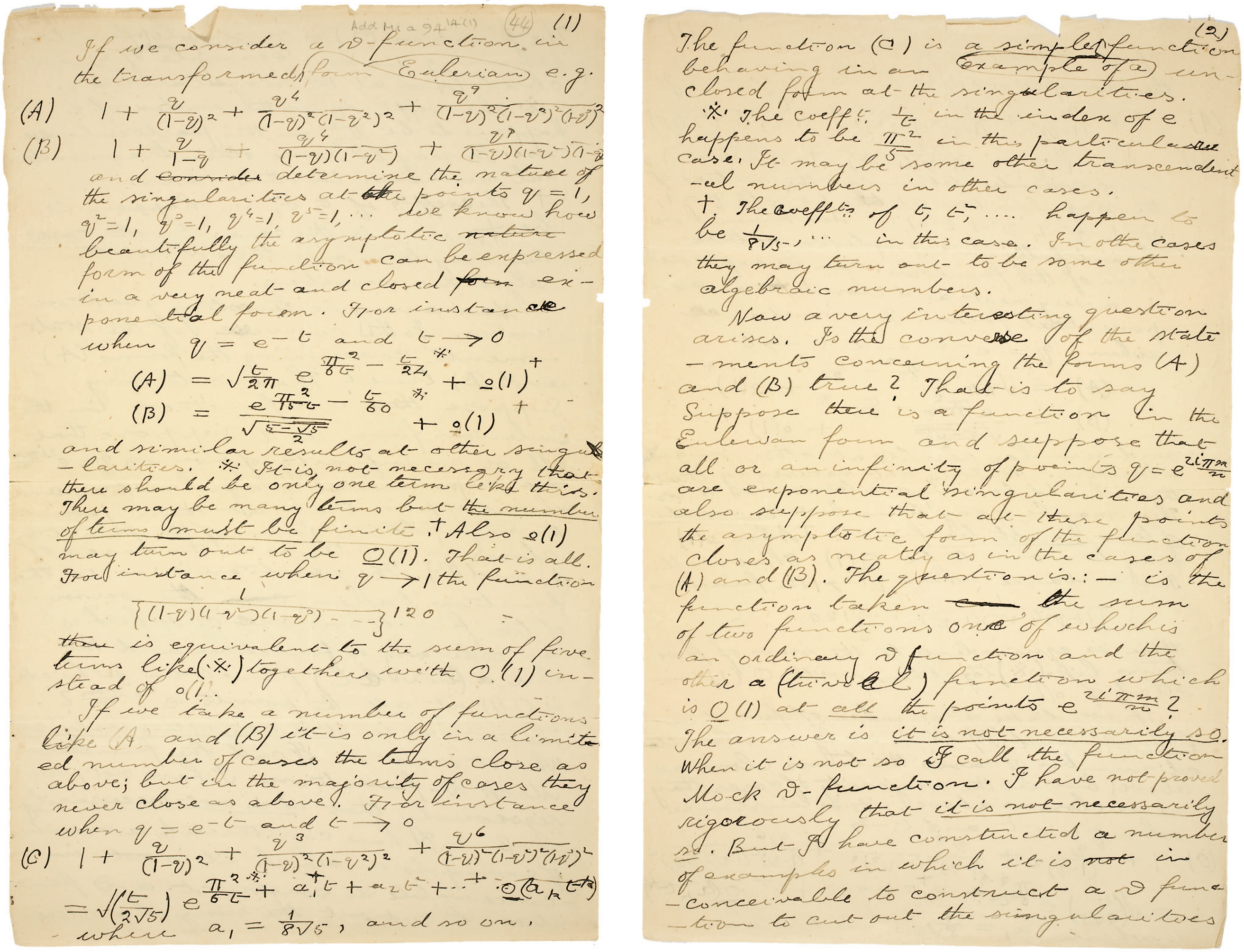
Mathematicians now had a grasp of the padlock which secured the wooden gate. But they still did not know the meaning behind Ramanujan's last words. Mathematicians had gathered a box full of formulas, but there would be little progress for the next ten years.

The "Lost Notebook"

In the spring of 1976, while searching through archived papers from Watson's estate in the Trinity College Library at Cambridge University, Andrews discovered the "Lost Notebook" [2]. The notebook, consisting of over 100 pages of Ramanujan's last works, was archived in a box among assorted papers collected from Watson's estate.

…the notebook and other material was discovered among Watson's papers by Dr. J. M. Whittaker,

[3] *K. Bringmann and the author have obtained an exact formula for $a_f(n)$ [12].*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If we consider a θ-function in the transformed Eulerian form e.g.

(A) $1 + \frac{q}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^2)^2} + \frac{q^9}{(1-q)^2(1-q^2)^2(1-q^3)^2}$

(B) $1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)}$

and determine the nature of the singularities at the points $q=1$, $q^2=1$, $q^3=1$, $q^4=1$, $q^5=1$, … we know how beautifully the asymptotic form of the function can be expressed in a very neat and closed exponential form. For instance when $q = e^{-t}$ and $t \to 0$

$$(A) = \sqrt{\frac{t}{2\pi}}\, e^{\frac{\pi^2}{6t} - \frac{t}{24}} + o(1)$$

$$(B) = \frac{e^{\frac{\pi^2}{15t} - \frac{t}{60}}}{\sqrt{\frac{5-\sqrt{5}}{2}}} + o(1)$$

and similar results at other singularities. * It is not necessary that there should be only one term like this. There may be many terms but the number of terms must be finite. † Also $o(1)$ may turn out to be $O(1)$. That is all. For instance when $q \to 1$ the function

$$\frac{1}{\{(1-q)(1-q^2)(1-q^3)\cdots\}^{120}}$$

is equivalent to the sum of five terms like (*) together with $O(1)$ instead of $o(1)$.

If we take a number of functions like (A) and (B) it is only in a limited number of cases the terms close as above; but in the majority of cases they never close as above. For instance when $q = e^{-t}$ and $t \to 0$

(C) $1 + \frac{q}{(1-q)^2} + \frac{q^3}{(1-q)^2(1-q^2)^2} + \frac{q^6}{(1-q)^2(1-q^2)^2(1-q^3)^2}$

$$= \sqrt{\left(\frac{t}{2\sqrt{5}}\right)}\, e^{\frac{\pi^2}{5t} + a_1 t + a_2 t^2 + \cdots + O(a_k t^k)}$$

where $a_1 = \frac{1}{8\sqrt{5}}$, and so on.

(2)

The function (C) is a simple example of a function behaving in an unclosed form at the singularities.

* The coeff. of $\frac{1}{t}$ in the index of e happens to be $\frac{\pi^2}{5}$ in this particular case. It may be some other transcendental numbers in other cases.

† The coeffts of $t, t^2, \ldots$ happen to be $\frac{1}{8\sqrt{5}}, \ldots$ in this case. In other cases they may turn out to be some other algebraic numbers.

Now a very interesting question arises. Is the converse of the statements concerning the forms (A) and (B) true? That is to say: Suppose there is a function in the Eulerian form and suppose that all or an infinity of points $q = e^{2i\pi m/n}$ are exponential singularities and also suppose that at these points the asymptotic form of the function closes as neatly as in the cases of (A) and (B). The question is: — is the function taken the sum of two functions one of which is an ordinary θ-function and the other a (trivial) function which is $O(1)$ at all the points $e^{2i\pi m/n}$? The answer is it is not necessarily so. When it is not so I call the function Mock θ-function. I have not proved rigorously that it is not necessarily so. But I have constructed a number of examples in which it is inconceivable to construct a θ-function to cut out the singularities

The five pages of Ramanujan's last letter to Hardy… (above and following pages)

who wrote the obituary of Professor Watson for the Royal Society. He passed the papers to Professor R. A. Rankin of Glasgow University, who, in December 1968, offered them to Trinity College so that they might join the other Ramanujan manuscripts…[2].

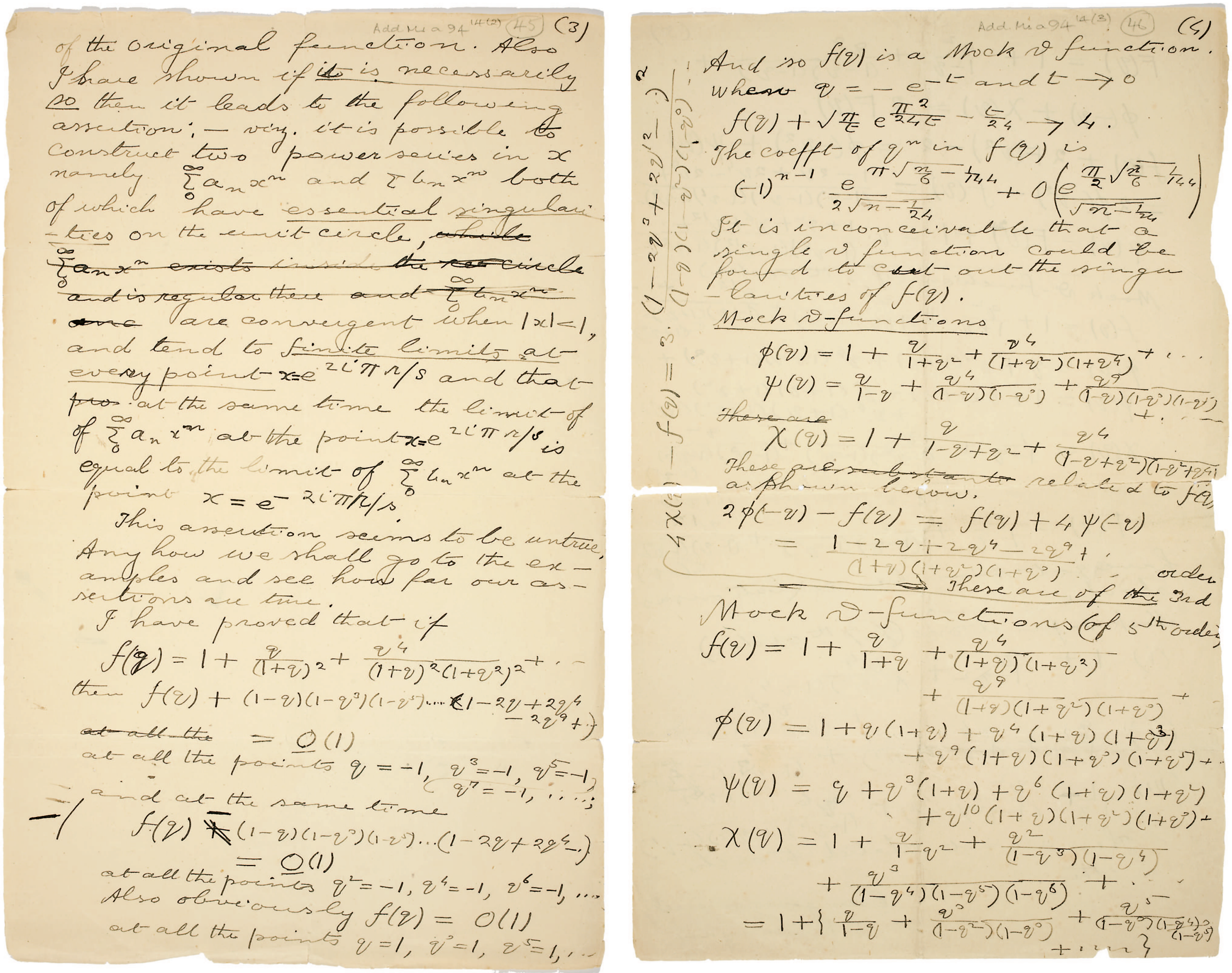
Janaki presumably sent Hardy the large leather box, the one filled with Ramanujan's last papers. Hardy passed it on to Watson in turn.

Although never truly lost, the sheaf of papers had survived the long journey from India only to then lie forgotten in the Trinity College Library. The journey was indeed extraordinary, for the manuscript almost met a catastrophic end. Whittaker, the son of Watson's famous coauthor E. T. Whittaker, recalled, in a letter to G. E. Andrews dated August 15, 1979, the scene of Watson's study at the time of his death in 1965 (see p. 304 of [7]):

…papers covered the floor of a fair sized room to the depth of about a foot, all jumbled together, and were to be incinerated in a few days. One could only make lucky dips [into the rubble] and, as Watson never threw away anything, the result might be a sheet of mathematics, but more probably a receipted bill or a draft of his income tax return for 1923. By an extraordinary stroke of luck one of my dips brought up the Ramanujan material.

The Lost Notebook allowed mathematicians to escape the seemingly eternal morass. In addition to listing some new mock theta functions, the scrawl contained many valuable clues: striking identities and relations, recorded without proofs of any kind. Thanks to these clues, mathematicians found many applications for the mock theta functions: L-functions in number theory, hypergeometric functions, partitions, Lie theory, modular forms, physics, and polymer chemistry, to name a few.

The Lost Notebook notably surrendered new sorts of identities that, as we shall see, go on to play a crucial role in the quest. Andrews proved identities [3] relating mock theta functions to



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of the original function. Also I have shown if it is necessarily so then it leads to the following assertion:— viz. it is possible to construct two power series in x namely $\sum a_n x^n$ and $\sum b_n x^n$ both of which have essential singularities on the unit circle, are convergent when $|x|<1$, and tend to finite limits at every point $x = e^{2i\pi r/s}$ and that at the same time the limit of $\sum_0^\infty a_n x^n$ at the point $x = e^{2i\pi r/s}$ is equal to the limit of $\sum_0^\infty b_n x^n$ at the point $x = e^{-2i\pi r/s}$.

This assertion seems to be untrue. Any how we shall go to the examples and see how far our assertions are true.

I have proved that if

$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots$

then $f(q) + (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-2q^9+\cdots) = O(1)$ at all the points $q=-1$, $q^3=-1$, $q^5=-1$, $q^7=-1$, …; and at the same time

$f(q) - (1-q)(1-q^2)(1-q^3)\cdots(1-2q+2q^4-\cdots) = O(1)$

at all the points $q^2=-1$, $q^4=-1$, $q^6=-1$, … Also obviously $f(q) = O(1)$ at all the points $q=1$, $q^3=1$, $q^5=1$, …

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And so $f(q)$ is a Mock ϑ function. When $q = -e^{-t}$ and $t \to 0$

$f(q) + \sqrt{\frac{\pi}{t}}\, e^{\frac{\pi^2}{24t} - \frac{t}{24}} \to 4$.

The coeff of q^n in $f(q)$ is

$(-1)^{n-1}\frac{e^{\pi\sqrt{\frac{n}{6}-\frac{1}{144}}}}{2\sqrt{n-\frac{1}{24}}} + O\left(\frac{e^{\frac{\pi}{2}\sqrt{\frac{n}{6}-\frac{1}{144}}}}{\sqrt{n-\frac{1}{24}}}\right)$

It is inconceivable that a single ϑ function could be found to cut out the singularities of $f(q)$.

Mock ϑ-functions

$\phi(q) = 1 + \frac{q}{1+q^2} + \frac{q^4}{(1+q^2)(1+q^4)} + \cdots$

$\psi(q) = \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^9}{(1-q)(1-q^3)(1-q^5)} + \cdots$

$\chi(q) = 1 + \frac{q}{1-q+q^2} + \frac{q^4}{(1-q+q^2)(1-q^2+q^4)} + \cdots$

These are related to $f(q)$ as shown below.

$2\phi(-q) - f(q) = f(q) + 4\psi(-q) = \frac{1-2q+2q^4-2q^9+\cdots}{(1+q)(1+q^2)(1+q^3)\cdots}$

These are of the 3rd order

$4\chi(q) - f(q) = 3\frac{(1-2q^3+2q^{12}-\cdots)}{(1-q)(1-q^2)(1-q^3)\cdots}$

Mock ϑ-functions (of 5th order)

$f(q) = 1 + \frac{q}{1+q} + \frac{q^4}{(1+q)(1+q^2)} + \frac{q^9}{(1+q)(1+q^2)(1+q^3)} + \cdots$

$\phi(q) = 1 + q(1+q) + q^4(1+q)(1+q^3) + q^9(1+q)(1+q^3)(1+q^5) + \cdots$

$\psi(q) = q + q^3(1+q) + q^6(1+q)(1+q^2) + q^{10}(1+q)(1+q^2)(1+q^3) + \cdots$

$\chi(q) = 1 + \frac{q}{1-q^2} + \frac{q^2}{(1-q^3)(1-q^4)} + \frac{q^3}{(1-q^4)(1-q^5)(1-q^6)} + \cdots$

$= 1 + \left\{\frac{q}{1-q} + \frac{q^3}{(1-q^2)(1-q^3)} + \frac{q^5}{(1-q^3)(1-q^4)(1-q^5)} + \cdots\right\}$

indefinite binary quadratic forms. For example, he proved:

(1.5)

$$f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)\cdots(1+q^n)} = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \cdot \left(\sum_{\substack{n+j\ge 0\\ n-j\ge 0}} - \sum_{\substack{n+j<0\\ n-j<0}}\right)(-1)^j q^{\frac{5}{2}n^2+\frac{1}{2}n-j^2}.$$

D. Hickerson confirmed [26] identities in which sums of mock theta functions are infinite products. For example, for $f_0(q)$ and the mock theta function $\Phi(q)$, he showed that

$$f_0(q) + 2\Phi(q^2) = \prod_{n=1}^{\infty} \frac{(1-q^{5n})(1-q^{10n-5})}{(1-q^{5n-4})(1-q^{5n-1})}.$$

As indefinite binary quadratic forms and infinite products appear in modular form theory, these identities finally provided evidence linking mock theta functions to modular forms, the "ordinary theta functions" of Ramanujan's last letter.

A *modular form* is a holomorphic function on the upper half complex plane $\mathbb{H}$ which is tamed by Möbius transformations $\gamma\tau := \frac{a\tau+b}{c\tau+d}$ for $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Loosely speaking,[4] a *weight k modular form on a subgroup* $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ that satisfies

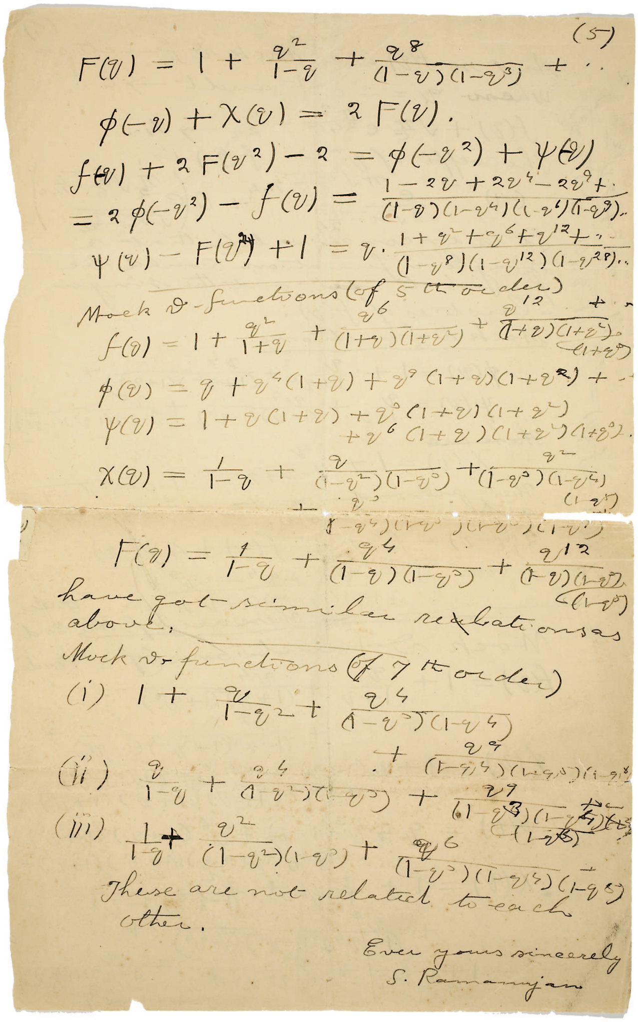
(1.6) $$f(\gamma\tau) = (c\tau+d)^k f(\tau)$$

for all $\gamma \in \Gamma$ and is meromorphic "at the cusps".

Despite this evidence, the essence of Ramanujan's theory continued to elude mathematicians. The problem was that the Lost Notebook is merely a bundle of pages that "…contains over six hundred mathematical formulae listed one after the other without proof[5]…there are only a few words scattered here and there…" (p. 89 and p. 96 of [2]). Instead of furnishing the missing key, the Notebook

[4] *If k is not an integer, then (1.6) must be suitably modified.*

[5] *Almost all of the results on q-series in the Lost Notebook have now been proved [4].*



(5)

$F(q) = 1 + \frac{q^2}{1-q} + \frac{q^8}{(1-q)(1-q^3)} + \cdots$

$\phi(-q) + \chi(q) = 2F(q).$

$f(q) + 2F(q^2) - 2 = \phi(-q^2) + \psi(q)$

$= 2\phi(-q^2) - f(q) = \frac{1 - 2q + 2q^4 - 2q^9 + \cdots}{(1-q)(1-q^4)(1-q^6)(1-q^9)\cdots}$

$\psi(q) - F(q^4) + 1 = q \cdot \frac{1 + q^2 + q^6 + q^{12} + \cdots}{(1-q^8)(1-q^{12})(1-q^{28})\cdots}$

Mock ϑ-functions (of 5th order)

$f(q) = 1 + \frac{q^2}{1+q} + \frac{q^6}{(1+q)(1+q^2)} + \frac{q^{12}}{(1+q)(1+q^2)(1+q^3)} + \cdots$

$\phi(q) = q + q^4(1+q) + q^9(1+q)(1+q^2) + \cdots$

$\psi(q) = 1 + q(1+q) + q^3(1+q)(1+q^2) + q^6(1+q)(1+q^2)(1+q^3) \cdots$

$\chi(q) = \frac{1}{1-q} + \frac{q}{(1-q^2)(1-q^3)} + \frac{q^2}{(1-q^3)(1-q^4)(1-q^5)} + \frac{q^3}{(1-q^4)(1-q^5)(1-q^6)(1-q^7)}$

$F(q) = \frac{1}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^{12}}{(1-q)(1-q^3)(1-q^5)}$

have got similar relations as above.

Mock ϑ-functions (of 7th order)

(i) $1 + \frac{q}{1-q^2} + \frac{q^4}{(1-q^3)(1-q^4)} + \frac{q^9}{(1-q^4)(1-q^5)(1-q^6)}$

(ii) $\frac{q}{1-q} + \frac{q^4}{(1-q^2)(1-q^3)} + \frac{q^9}{(1-q^3)(1-q^4)(1-q^5)}$

(iii) $\frac{1}{1-q} + \frac{q^2}{(1-q^2)(1-q^3)} + \frac{q^6}{(1-q^3)(1-q^4)(1-q^5)}$

These are not related to each other.

Ever yours sincerely
S. Ramanujan

provided a hammer in the form of countless identities. So armed, mathematicians burst through the wooden gate, only to find a long dusty hallway lined with locked iron doors. When the dust settled, they could read the signs on the doors, and with this knowledge they finally understood the widespread scope of the mock theta functions. At the Ramanujan Centenary Conference in 1987, F. Dyson eloquently summed up the dilemma [21]:

The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi. This remains a challenge for the future. My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include mock theta-functions…

Zwegers's Thesis

By the late 1990s little progress had been made. Then in 2002, in a brilliant Ph.D. thesis [45] written under D. Zagier, S. Zwegers made sense of the mock theta functions. By understanding the meaning behind identities such as (1.3–1.5) and by notably making use of earlier work of Lerch and Mordell, he found the answer: *real analytic modular forms*. In the solution, one must first "complete" the mock theta functions by adding a nonholomorphic function, a so-called "period integral".

For the mock theta functions $f(q)$ (see (1.1)) and

(1.7)

$$\omega(q) = \sum_{n=0}^{\infty} a_\omega(n) q^n := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2},$$

Zwegers [44] defined the *vector-valued mock theta function* (here $q := e^{2\pi i\tau}$)

$$F(\tau) = (F_0(\tau),\ F_1(\tau),\ F_2(\tau))^T := (q^{-\frac{1}{24}} f(q),\ 2q^{\frac{1}{3}}\omega(q^{\frac{1}{2}}),\ 2q^{\frac{1}{3}}\omega(-q^{\frac{1}{2}}))^T.$$

Then using theta functions $g_0(z), g_1(z)$, and $g_2(z)$, where

$$g_0(z) := \sum_{n=-\infty}^{\infty} (-1)^n \left(n + \frac{1}{3}\right) e^{3\pi i\left(n+\frac{1}{3}\right)^2 z}$$

($g_1(z)$ and $g_2(z)$ are similar), he defined the vector-valued nonholomorphic function

(1.8)

$$G(\tau) = (G_0(\tau),\ G_1(\tau),\ G_2(\tau))^T := 2i\sqrt{3}\int_{-\overline{\tau}}^{i\infty} \frac{(g_1(z),\ g_0(z),\ -g_2(z))^T}{\sqrt{-i(\tau+z)}}\, dz.$$

He completed $F(\tau)$ to obtain $H(\tau) := F(\tau) - G(\tau)$, and he proved that [44]

$$H(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(\tau)$$

and

$$H(-1/\tau) = \sqrt{-i\tau} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(\tau),$$

where $\zeta_n := e^{2\pi i/n}$. As $SL_2(\mathbb{Z}) = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$, this gives a vector version of (1.6), and so the *vector-valued mock theta function* $F(\tau)$ is the *holomorphic part* of the vector-valued real analytic modular form $H(\tau)$.

Generalizing identities such as (1.3), in which mock theta functions are related to "Appell-Lerch" series, Zwegers also produced infinite families of mock theta functions that eclipse Ramanujan's list. For $\tau \in \mathbb{H}$ and $u, v \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$, he defined the function

$$\mu(u, v; \tau) := \frac{z^{1/2}}{\vartheta(v;\tau)} \cdot \sum_{n\in\mathbb{Z}} \frac{(-w)^n q^{n(n+1)/2}}{1 - zq^n}, \tag{1.9}$$

where $z := e^{2\pi i u}$, $w := e^{2\pi i v}$, $q := e^{2\pi i \tau}$ and $\vartheta(v;\tau) := \sum_{\nu \in \mathbb{Z}+\frac{1}{2}} e^{\pi i \nu} w^{\nu} q^{\nu^2/2}$. Using a function $R(u-v;\tau)$ which resembles the components of (1.8), he then defined

$$\hat{\mu}(u,v;\tau) := \mu(u,v;\tau) + \frac{i}{2} R(u-v;\tau). \tag{1.10}$$

He proved that $\hat{\mu}(u,v;\tau)$ is a nonholomorphic *Jacobi form*, a function whose specializations at "torsion points" give weight 1/2 real analytic modular forms. This function satisfies transformations that imply (1.6) for these specializations; for example, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, then there is an explicit root of unity $\chi(\gamma)$ for which

$$\hat{\mu}\left(\frac{u}{c\tau+d}, \frac{v}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = \chi(\gamma)^{-3}(c\tau+d)^{\frac{1}{2}} e^{-\pi i c(u-v)^2/(c\tau+d)} \cdot \hat{\mu}(u,v;\tau).$$

Exploring New Worlds

Armed with Zwegers's landmark thesis, mathematicians have begun to explore [36, 43] the worlds behind the iron doors. Here we sample some of the discoveries that this author has obtained with his collaborators.

Harmonic Maass Forms

The mock theta functions turn out to be holomorphic parts of distinguished real analytic modular forms, the *harmonic Maass forms*, which were recently introduced by J. H. Bruinier and J. Funke [16].

Loosely speaking, a *weight k harmonic Maass form* is a smooth function $M : \mathbb{H} \to \mathbb{C}$ satisfying (1.6) and $\Delta_k(M) = 0$, which also has (at most)[6] linear exponential growth at cusps. Here the *hyperbolic Laplacian* Δ_k, where $\tau = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$, is given by

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The Fourier expansions of these forms have been the object of our explorations. In terms of the incomplete gamma function $\Gamma(\alpha; x) := \int_x^\infty e^{-t} t^{\alpha-1}\, dt$, every weight $2-k$ harmonic Maass form $M(\tau)$, where $k > 1$, has an expansion of the form

$$M(\tau) = \sum_{n \gg -\infty} c_M^+(n) q^n + \sum_{n<0} c_M^-(n) \Gamma(k-1, 4\pi |n| y) q^n. \tag{2.11}$$

Obviously, $M(\tau)$ naturally decomposes into two pieces, a *holomorphic part*

$$M^+(\tau) := \sum_{n \gg -\infty} c_M^+(n) q^n$$

and its complement $M^-(\tau)$, the *nonholomorphic part*. The mock theta functions and Zwegers's μ-function give holomorphic parts of weight 1/2 harmonic Maass forms.

Harmonic Maass forms are generalizations of modular forms; a modular form is a harmonic Maass form $M(\tau)$ where $M^-(\tau) = 0$. Because modular forms appear prominently in mathematics, one then expects the mock theta functions and harmonic Maass forms to have far-reaching implications. Our first forays in the long dusty hallway have been profitable, and we have obtained results [12, 13, 14, 15, 17, 32, 36, 37] on a wide array of subjects: partitions and q-series, Moonshine, Donaldson invariants, Borcherds products, and elliptic curves, among others. We now describe some of these results.

Partitions

A *partition* of a nonnegative integer n is any nonincreasing sequence of positive integers that sum to n. If $p(n)$ denotes the number of partitions of n, then Ramanujan famously proved that

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

In an effort to provide a combinatorial explanation of these congruences, Dyson defined the *rank* of a partition to be its largest part minus the number of its parts. For example, the table below includes the ranks of the partitions of 4.

Partition	Rank	Rank mod 5
4	$4-1=3$	3
$3+1$	$3-2=1$	1
$2+2$	$2-2=0$	0
$2+1+1$	$2-3=-1$	4
$1+1+1+1$	$1-4=-3$	2

Based on numerics, Dyson [20] made the following conjecture whose truth provides a combinatorial explanation of Ramanujan's congruences modulo 5 and 7.

Conjecture (Dyson)**.** The partitions of $5n+4$ (resp. $7n+5$) form 5 (resp. 7) groups of equal size when sorted by their ranks modulo 5 (resp. 7).

In 1954 A. O. L. Atkin and H. P. F. Swinnerton-Dyer proved [5] Dyson's conjecture.[7]

There is now a robust theory of partition congruences modulo every integer M coprime to 6 [1, 34], and typical congruences look more like

$$p(48037937n + 1122838) \equiv 0 \pmod{17}.$$

One naturally asks: what role, if any, do Dyson's original guesses play within this theory?

[6] *In this paper we use a slightly stronger condition; we assume the existence of "principal parts" at cusps.*

[7] *A short calculation reveals that the obvious generalization of the conjecture cannot hold for 11.*

K. Bringmann and the author [13] investigated this question, and in their work they related $N(r,t;n)$, the number of partitions of n with rank congruent to $r \pmod{t}$, to harmonic Maass forms. They essentially proved that

$$\sum_{n=0}^{\infty}\left(N(r,t;n)-\frac{p(n)}{t}\right)q^n$$

is the holomorphic part of a weight 1/2 harmonic Maass form. This result, combined with Shimura's theory of half-integral weight modular forms and the Deligne-Serre theory of Galois representations, implies that ranks "explain" infinite classes of congruences.

Theorem 2.1 (Th. 1.5 of [13])**.** *If $Q \geq 5$ is prime and $j \geq 1$, then there are positive integers t and arithmetic progressions $An+B$ such that*

$$N(r,t;An+B) \equiv 0 \pmod{Q^j}$$

for every $0 \leq r < t$. In particular, we have that $p(An+B) \equiv 0 \pmod{Q^j}$.

Moonshine

In the late 1970s J. McKay and J. Thompson [39] observed that the first few coefficients of the classical elliptic modular function

$$\begin{aligned} j(z) - 744 &\\ &= q^{-1} + 196884q + 21493760q^2 \\ &\quad + 864299970q^3 + \cdots \end{aligned}$$

are certain linear combinations of the dimensions of the irreducible representations of the *Monster* group. For example, the degrees of the four "smallest" irreducible representations are: 1, 196883, 21296876, and 842609326, and the first few coefficients are:

$$\begin{aligned} 1 &= 1 \\ 196884 &= 196883 + 1 \\ 21493760 &= 21296876 + 196883 + 1 \\ 864299970 &= 842609326 + 21296876 + 2\cdot 196883 + 2\cdot 1. \end{aligned}$$

J. Conway and S. Norton [18] expanded on these observations, and they formulated a series of deep conjectures, the so-called *Monstrous Moonshine Conjectures*. These conjectures have now been settled, and thanks to the work of many authors, most notably R. E. Borcherds [9], there is a beautiful theory, involving string theory, vertex operator algebras, and generalized Kac-Moody superalgebras, in which connections between objects like the j-function and the Monster are revealed.

In the 1980s, in the spirit of Moonshine, V. G. Kac and D. H. Peterson [27] established the modularity of similar characters that arise in the study of infinite-dimensional affine Lie algebras. As a generalization of this work, ten years ago Kac and M. Wakimoto [28] computed characters of the affine Lie superalgebras $g\ell(m,1)^\wedge$ and $s\ell(m,1)^\wedge$. These characters are not modular, and Kac asked whether they might be related to harmonic Maass forms. Bringmann and the author have confirmed [14] this speculation; these characters are pieces of nonholomorphic modular functions. We consider the character for the $s\ell(m,1)^\wedge$ modules $L(\Lambda_{(s)})$, where $L(\Lambda_{(s)})$ is the irreducible $s\ell(m,1)^\wedge$ module with highest weight $\Lambda_{(s)}$. If $m \geq 2$ and $s \in \mathbb{Z}$, then the work of Kac and Wakimoto implies that

Ramanujan passport photo.

(2.12)

$$tr_{L(\Lambda_{(s)})}q^{L_0} = 2q^{\frac{m-2-12s}{24}}\cdot\frac{\eta(2\tau)^2}{\eta(\tau)^{m+2}} \cdot \sum_{k=(k_1,k_2,\ldots,k_{m-1})\in\mathbb{Z}^{m-1}} \frac{q^{\frac{1}{2}\sum_{i=1}^{m-1}k_i(k_i+1)}}{1+q^{|k|-s}},$$

where $|k| := \sum_{i=1}^{m-1}k_i$ and $\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is Dedekind's eta function. Using the function R in (1.10), Bringmann and the author defined the function

(2.13)

$$\mathcal{T}_{m,s}(\tau) := tr_{L(\Lambda_{(s)})}q^{L_0} - 2^{m-1}q^{\frac{m-2}{24}}\frac{\eta(2\tau)^{2m}}{\eta(\tau)^{2m+1}}R(-s\tau;(m-1)\tau),$$

and they showed that

$$\frac{\eta(\tau)^{2m+1}}{\eta(2\tau)^{2m}}\cdot tr_{L(\Lambda_{(s)})}q^{L_0}$$

is (up to a power of q) a mock theta function. As a consequence, they proved:

Theorem 2.2 (Th. 1.1 of [14])**.** *If $m \geq 2$ and $s \in \mathbb{Z}$, then $\mathcal{T}_{m,s}(\tau)$ is (up to a power of q) a nonholomorphic modular function.*

Donaldson Invariants

In recent work with A. Malmendier [32], it is shown that the mock theta function

$$M(q) := q^{-\frac{1}{8}} \times \sum_{n=0}^{\infty}\frac{(-1)^{n+1}q^{(n+1)^2}(1-q)(1-q^3)\cdots(1-q^{2n-1})}{(1+q)^2(1+q^3)^2\cdots(1+q^{2n+1})^2}$$

is a "topological invariant" for $\mathbb{C}P^2$. This claim pertains to the differentiable topology of 4-manifolds.

In the early 1980s S. Donaldson proved (for example, see [19]) that the diffeomorphism class of a compact, simply connected, differentiable 4-manifold X is not necessarily determined by its intersection form. In his work, he famously defined the *Donaldson invariants*, diffeomorphism invariants of X obtained as graded homogeneous polynomials on the homology ring with integer coefficients.

There are two families of invariants corresponding to the $SU(2)$ and the $SO(3)$ gauge theories. The author and Malmendier considered the $SO(3)$ case for the simplest manifold, the complex projective plane $\mathbb{C}P^2$ with the Fubini-Study metric. The invariants are difficult to work out even in this case; indeed, they were not computed until the work of L. Göttsche [22] in 1996, assuming the Kotschick-Morgan Conjecture. Göttsche, H. Nakajima, and K. Yoshioka have recently confirmed [23] this provisional description of the Donaldson "zeta-function"

$$Z(p,S) = \sum_{m,n} \phi_{mn} \cdot \frac{p^m S^n}{m!n!}. \tag{2.15}$$

In the mid-1990s G. Moore and E. Witten conjectured [33] "u-plane integral" formulas for this zeta function. Their work relies on the following identifications:

$$\begin{array}{ccc} u\text{-plane integrals} & \longleftrightarrow & \text{Donaldson invariants for } \mathbb{C}P^2 \\ \updownarrow & & \\ \text{elliptic surface} & & \end{array}$$

Here the rational elliptic surface is the universal curve for the modular group $\Gamma_0(4)$, which can be identified with $\mathbb{C}P^1$ minus 3 points with singular fibers. In addition, the rational elliptic surface is endowed with an analytical marking such that the generic fibers correspond to elliptic curves that are parameterized by $\mathbb{C}$-lattices $\langle \omega, \tau\omega \rangle$ in the usual way. Then the u-plane zeta function is given as a "regularized" integral

(2.16)
$$Z_{UP}(p,S) := -\frac{8}{\sqrt{2\pi}} \cdot \int_{UP}^{reg} \frac{du \wedge d\overline{u}}{\sqrt{\operatorname{Im}\tau}} \cdot \frac{d\overline{\tau}}{d\overline{u}} \cdot \frac{\Delta^{\frac{1}{8}}}{\omega^{\frac{1}{2}}} \cdot e^{2up+S^2\hat{T}} \cdot \overline{\eta(\tau)^3}.$$

Here Δ is the discriminant of the corresponding elliptic curves, and $\hat{T}$ is defined by the renormalization flow on the elliptic surface.

The Moore-Witten Conjecture in this case is that $Z(p,S) = Z_{UP}(p,S)$. Using (2.16), the author and Malmendier reformulated this conjecture in terms of harmonic Maass forms arising from $M(q)$, and they then used Zwegers's μ-function to prove the following theorem.

Theorem 2.3 (Th. 1.1 of [32]). *The Moore-Witten Conjecture for the $SO(3)$-gauge theory on $\mathbb{C}P^2$ is true. In particular, we have that $Z(p,S) = Z_{UP}(p,S)$.*

Borcherds Products

Recently R. E. Borcherds provided [10, 11] a striking description for the exponents in the infinite product expansion of many modular forms with a *Heegner divisor*. He proved that the exponents in these expansions are coefficients of weight 1/2 modular forms. As an example, the classical Eisenstein series $E_4(\tau)$ factorizes as

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sum_{d|n} d^3 q^n = (1-q)^{-240}(1-q^2)^{26760} \cdots = \prod_{n=1}^{\infty} (1-q^n)^{c(n)},$$

where the $c(n)$ are the coefficients $b(n^2)$ of a weight 1/2 meromorphic modular form

$$G(\tau) = \sum_{n \geq -3} b(n)q^n = q^{-3} + 4 - 240q + 26760q^4 + \cdots - 4096240q^9 + \cdots.$$

Bruinier and the author [17] have generalized this phenomenon to allow for exponents that are coefficients of weight 1/2 harmonic Maass forms. For brevity, we give examples of *generalized Borcherds products* that arise from the mock theta functions $f(q)$ and $\omega(q)$. To this end, let $1 < D \equiv 23 \pmod{24}$ be square-free, and for $0 \leq j \leq 11$ let

(2.17)
$$H_j(\tau) = \sum_{n \gg -\infty} C(j;n)q^n := \begin{cases} 0 & \text{if } j = 0,3,6,9, \\ \left(\frac{j}{3}\right) q^{-1} f(q^{24}) & \text{if } j = 1,5,7,11, \\ \left(\frac{j}{3}\right) 2q^8 \left(\omega(q^{12}) - \omega(-q^{12})\right) & \text{if } j = 2,10, \\ (-1)^{\frac{j}{4}} 2q^8 \left(\omega(q^{12}) + \omega(-q^{12})\right) & \text{if } j = 4,8. \end{cases}$$

If $e(\alpha) := e^{2\pi i\alpha}$ and $\left(\frac{-D}{b}\right)$ is the Jacobi-Kronecker quadratic residue symbol, then define

$$P_D(X) := \prod_{b \mod D} (1 - e(-b/D)X)^{\left(\frac{-D}{b}\right)}. \tag{2.18}$$

Using this rational function, we then define the *generalized Borcherds product* $\Psi_D(\tau)$ by

$$\Psi_D(\tau) := \prod_{m=1}^{\infty} P_D(q^m)^{C(\overline{m};Dm^2)}. \tag{2.19}$$

The exponents come from (2.17), and $\overline{m}$ is the residue class of m modulo 12.

Theorem 2.4 (§8.2 of [17]). *The function $\Psi_D(\tau)$ is a weight 0 meromorphic modular form on $\Gamma_0(6)$ with a discriminant $-D$ Heegner divisor (see §5 of* [17] *for the explicit divisor).*

This theorem has an interesting consequence for the parity of the partition function. Very little is known about this parity; indeed, it was not even known that $p(n)$ takes infinitely many even and odd values until 1959 [30]. Using Theorem 2.4 and the fact that

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2} \equiv \sum_{n=0}^{\infty} p(n)q^n \pmod 4,$$

the author proved the following result for the partition numbers evaluated at the values of certain quadratic polynomials.

Theorem 2.5 (Corollary 1.4 of [37]). *If $\ell \equiv 23 \pmod{24}$ is prime, then there are infinitely many m coprime to 6 for which $p\left(\frac{\ell m^2+1}{24}\right)$ is even. Moreover, the first such m is bounded by $12h(-\ell)+2$, where $h(-\ell)$ is the class number of $\mathbb{Q}(\sqrt{-\ell})$.*

Elliptic Curve L-Functions

If $E/\mathbb{Q}$ is an elliptic curve

$$E/\mathbb{Q}: \ y^2 = x^3 + Ax + B,$$

then $E(\mathbb{Q})$, its $\mathbb{Q}$-rational points, forms a finitely generated abelian group. The Birch and Swinnerton-Dyer Conjecture, one of the Clay millennium prize problems, predicts that

$$\mathrm{ord}_{s=1}(L(E,s)) = \text{Rank of } E(\mathbb{Q}),$$

where $L(E,s)$ is the Hasse-Weil L-function for E. There is no known procedure for computing $\mathrm{ord}_{s=1}(L(E,s))$. Determining when $\mathrm{ord}_{s=1}(L(E,s)) \le 1$ for elliptic curves in a family of quadratic twists already requires the deep theorems of Kohnen [31] and Waldspurger [41], and of Gross and Zagier [24]. These results, however, involve very disparate criteria for deducing the analytic behavior at $s=1$.

Using generalized Borcherds products [17], Bruinier and the author have produced a single device that encompasses these criteria. We present a special case of these results. Suppose that E has prime conductor, and suppose further that the sign of the functional equation of $L(E,s)$ is -1. If Δ is a fundamental discriminant of a quadratic field, then let $E(\Delta)$ be the *quadratic twist* elliptic curve

$$E(\Delta): \ \Delta y^2 = x^3 + Ax + B.$$

Using harmonic Maass forms and their generalized Borcherds products, the author and Bruinier show that the coefficients of certain harmonic Maass forms encode the vanishing of central derivatives (resp. values) of the L-functions for the elliptic curves $E(\Delta)$.

Theorem 2.6 (Th. 1.1 of [17]). *Assuming the hypotheses above, there is a weight 1/2 harmonic Maass form*

$$M_E(\tau) = \sum_{n \gg -\infty} c_M^+(n)q^n + \sum_{n<0} c_M^-(n)\Gamma(1/2; 4\pi|n|y)q^n,$$

and a nonzero constant $\alpha(E)$ that satisfies:

(1) *If $\Delta < 0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right) = 1$, then*

$$L(E(\Delta),1) = \alpha(E)\cdot\sqrt{|\Delta|}\cdot c_M^-(\Delta)^2.$$

(2) *If $\Delta > 0$ is a fundamental discriminant for which $\left(\frac{\Delta}{p}\right) = 1$, then $L'(E(\Delta),1) = 0$ if and only if $c_M^+(\Delta)$ is algebraic.*

The Path Ahead

As we have seen, Ramanujan's deathbed letter set into motion an implausible adventure, one whose first act is now over. It was about the theory of harmonic Maass forms and its implications for many subjects: partitions and q-series, Moonshine, Donaldson invariants, mathematical physics, Borcherds products, and L-functions of elliptic curves, to name a few. This theory has provided satisfying answers to the first challenges: to understand the meaning behind Ramanujan's last words and to realize the expectation that this understanding would reveal and open new doors in the interconnected world of mathematics.

Every step along the way has evoked wonder—the enigmatic letter, the Lost Notebook, and the work of many minds. If the past is the road map to the future, then the yet unwritten acts promise forays, by intrepid mathematicians of today and tomorrow, into new worlds presently populated with seemingly unattainable mathematical truths.

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