MODULAR FORMS ARISING FROM $Q(n)$ AND DYSON'S RANK

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This paper is dedicated to Dennis Stanton, a combinatorist who really counts.

ABSTRACT. Let $R(w;q)$ be Dyson's generating function for partition ranks. For roots of unity $\zeta \neq 1$, it is known that $R(\zeta;q)$ and $R(\zeta;1/q)$ are given by harmonic Maass forms, Eichler integrals, and modular units. We show that modular forms arise from $G(w;q)$, the generating function for ranks of partitions into distinct parts, in a similar way. If $D(w;q) := (1+w)G(w;q) + (1-w)G(-w;q)$, then for roots of unity $\zeta \neq \pm 1$ we show that $q^{\frac{1}{12}} \cdot D(\zeta;q)D(\zeta^{-1};q)$ is a weight 1 modular form. Although $G(\zeta;1/q)$ is not well defined, we show that it gives rise to natural sequences of q-series whose limits involve infinite products (and modular forms when $\zeta = 1$). Our results follow from work of Fine on basic hypergeometric series.

1. Introduction and statement of results

There are many famous examples of q-series which coincide with modular forms (up to a power of q) when $q := e^{2\pi i\tau}$. For example, the celebrated Rogers-Ramanujan identities

$$R_1(q) := \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)},$$

$$R_2(q) := \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)},$$

provide series expansions of infinite products which are essentially weight 0 modular forms. As another example, the partition generating function satisfies

$$P(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2}.$$

Since $q^{-1}P(q^{24}) = 1/\eta(24\tau)$, where $\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is Dedekind's weight 1/2 modular form, this generating function is another instance of a modular form which is a basic hypergeometric type series.

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Dyson's rank generating function provides two infinite families of basic hypergeometric-type series which are related to automorphic forms. To make this precise, we first recall some notation. As usual, we let

$$(a;q)_n := \begin{cases} (1-a)(1-aq)\cdots(1-aq^{n-1}) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases} \tag{1.1}$$

and

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n). \tag{1.2}$$

Recall that the *rank* of a partition is its largest part minus its number of parts. For instance, the rank of the partition $(2,1,1,1)$ of 5 is $2-4=-2$. Let $N(m,n)$ denote the number of partitions of n with rank m. Then the generating function for $N(m,n)$ is

$$R(w;q) := 1 + \sum_{n=1}^{\infty} N(m,n) w^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq;q)_n (w^{-1}q;q)_n}.$$

The two families of functions $R(\zeta;q)$ and $R(\zeta;1/q)$, where ζ are roots of unity, are intimately related to automorphic forms. (We will use the rightmost expression as the fixed representation of R as a q-series throughout this paper.) We clearly have that

$$q^{-1}R(1;q^{24}) = \sum_{n=0}^{\infty} p(n) q^{24n-1} = \frac{1}{\eta(24\tau)}$$

is a weight -1/2 modular form. For $\zeta \neq 1$, a very different phenomenon holds. In this case Bringmann and the second author proved that $R(\zeta;q)$ is the "holomorphic part" of a weight 1/2 harmonic Maass form [4].

For $\widehat{R}(w;q) := R(w;1/q)$, another phenomenon occurs. A straightforward calculation shows that

$$\widehat{R}(w;q) = 1 + \sum_{n=1}^{\infty} \frac{q^n}{(wq;q)_n (w^{-1}q;q)_n}, \tag{1.3}$$

and then a classical identity of Ramanujan [1] asserts that

$$\widehat{R}(w;q) = (1-w)\sum_{n=0}^{\infty}(-1)^n w^{3n} q^{\frac{3n^2+n}{2}}(1-w^2q^{2n+1}) + \frac{w\sum_{n=0}^{\infty}(-1)^n w^{2n} q^{\frac{n^2+n}{2}}}{(wq;q)_\infty (w^{-1}q;q)_\infty}. \tag{1.4}$$

For generic roots of unity $w=\zeta$, the right hand side of (1.4) consists of two false theta series, which can be thought of as Eichler integrals (for example, see page 103 of [8]), and an infinite product which is essentially a modular form (see 1.7).

Here we address the analogous questions for the series

$$G(w;q) := \sum_{\substack{n\geq 0 \\ m\in\mathbb{Z}}} Q(m,n) w^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{\frac{n^2+n}{2}}}{(wq;q)_n}. \tag{1.5}$$

This is the generating function for $Q(m,n)$, the number of partitions of n into distinct parts with rank m. Using the fixed representation for G above, we similarly define the series $\widehat{G}(\zeta;q) := G(\zeta;1/q)$, and relate it to modular forms and other infinite products for roots of unity ζ.

The case of $G(\zeta;q)$ has been thoroughly investigated for $\zeta \in \{\pm 1, \pm i\}$. In particular, by considering the Sylvester's triangle of a partition one can show that

$$q^{\frac{1}{24}} G(1;q) = \frac{\eta(2\tau)}{\eta(\tau)}$$

is a weight 0 modular form. For $\zeta = \pm i$, we have the two false theta functions [9]

$$qG(\pm i; q^{24}) = \sum_{k=0}^{\infty} (\pm i)^k q^{(6k+1)^2} + \sum_{k=1}^{\infty} (\pm i)^{k-1} q^{(6k-1)^2}.$$

Although these two series are not modular, they are Eichler integrals of weight 3/2 modular forms. The case of $\zeta = -1$ is more exotic. If we define numbers $a(n)$ by $qG(-1;q^{24}) = \sum_{n=1}^{\infty} a(n) q^n$, then the function

$$\phi(\tau) := \sqrt{y} \sum_{n=1}^{\infty} a(n) e^{2\pi i n x} K_0(2\pi n y)$$

is a simple linear combination of twists of the Hecke-Maass cusp form constructed in [2, 5]. Here K_0 denotes the usual index zero K-Bessel function and $\tau = x + iy$ with $x, y \in \mathbb{R}$.

For roots of unity $\zeta \neq \pm 1$, it turns out that $G(\zeta;q)$, $G(-\zeta;q)$, $G(\zeta^{-1};q)$ and $G(-\zeta^{-1};q)$ together give a "factorization" of a certain weight 1 modular form. For convenience, we define the series

$$D(w;q) := (1+w)G(w;q) + (1-w)G(-w;q). \tag{1.6}$$

For roots of unity $\zeta \neq 1$, we require the weight 0 modular form (for example, see [7, 10])

$$\eta(\zeta;\tau) := q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1-\zeta q^n)(1-\zeta^{-1}q^n). \tag{1.7}$$

Theorem 1.1. *If $\zeta \neq \pm 1$ is a root of unity, then we have that $q^{\frac{1}{12}} \cdot D(\zeta;q) D(\zeta^{-1};q)$ is the weight 1 modular form*

$$q^{\frac{1}{12}} \cdot D(\zeta;q) D(\zeta^{-1};q) = 4 \cdot \frac{\eta(2\tau)^4}{\eta(\tau)^2 \eta(\zeta^2; 2\tau)}.$$

Example. Here we consider the case where $\zeta = i$. Since $D(i;q) = D(-i;q)$, Theorem 1.1 implies that $q^{\frac{1}{24}} D(i;q)$ is the weight 1/2 modular form

$$q^{\frac{1}{24}} D(i;q) = 2 \cdot \frac{\eta(2\tau)^3}{\eta(\tau)\eta(4\tau)} = 2q^{\frac{1}{24}} + 2q^{\frac{25}{24}} - 2q^{\frac{49}{24}} - \cdots.$$

Now we turn to the series $\widehat{G}(\zeta;q) := G(\zeta;1/q)$. A straightforward calculation (see Lemma 3.1) reveals the formal identity

$$\widehat{G}(w;q) = 1 + \sum_{n=1}^{\infty} \frac{(-w^{-1})^n}{(w^{-1}q;q)_n}. \tag{1.8}$$

Unfortunately, this is not a well defined q-series. By that, we mean that if

$$\widehat{G}_m(w;q) := 1 + \sum_{n=1}^{m} \frac{(-w^{-1})^n}{(w^{-1}q;q)_n}, =: \sum_{n=0}^{\infty} c_m(n)q^n \tag{1.9}$$

then there are positive integers k such that $\lim_{m\to+\infty} c_m(k)$ is not well defined. However, it turns out that if $-\zeta^{-1} \neq 1$ is a root of unity of order m, then for each $0 \leq r < m$, we have that $\lim_{n\to\infty} c_{mn+r}(k)$ is well defined for each k, and so

$$\lim_{n\to+\infty} \widehat{G}_{mn+r}(\zeta;q) \tag{1.10}$$

is a well defined q-series. The next theorem describes these limits and shows that in the case $-\zeta^{-1} = -1$, it is the sum of a weight $-1/2$ modular form with a weight 0 modular form.

Theorem 1.2. *Suppose that $-\zeta^{-1} \neq 1$ is a primitive mth root of unity. If $0 \leq r < m$, then $\lim_{n\to\infty} \widehat{G}_{mn+r}(\zeta;q)$ is a well defined q-series and satisfies*

$$\lim_{n\to\infty} \widehat{G}_{mn+r}(\zeta;q) = \lim_{n\to\infty} \widehat{G}_{mn}(\zeta;q) + \frac{(-\zeta^{-1})^r - 1}{\zeta+1} \frac{1}{(\zeta^{-1}q;q)_\infty}.$$

Furthermore, if we define the sequence $b(n)$ by $\sum_{n=0}^{\infty}(-1)^n b(n)q^n = \prod_{n=1}^{\infty}(1+q^{2n-1})$, then

$$\lim_{n\to\infty} \widehat{G}_{2n}(1;q) = \frac{1}{2}\left(\sum_{n=0}^{\infty} b(n)q^n + \frac{1}{(q;q)_\infty}\right)$$

and

$$\lim_{n\to\infty} \widehat{G}_{2n+1}(1;q) = \frac{1}{2}\left(\sum_{n=0}^{\infty} b(n)q^n - \frac{1}{(q;q)_\infty}\right).$$

Remark. Notice that the sums

$$\sum_{n=0}^{\infty}(-1)^n b(n)q^{n-\frac{1}{24}} = q^{-\frac{1}{24}}\prod_{n=0}^{\infty}(1+q^{2n+1}) = \frac{\eta(2\tau)^2}{\eta(\tau)\eta(4\tau)},$$

and

$$\sum_{n=0}^{\infty} b(n)q^{n-\frac{1}{24}} = \frac{\eta(\tau)}{\eta(2\tau)}$$

are both modular forms of weight 0. Since $\eta(z)$ is a weight 1/2 modular form, it follows that $q^{-1/24}\lim_{n\to\infty}\widehat{G}_{2n}(1;q)$ and $q^{-1/24}\lim_{n\to\infty}\widehat{G}_{2n+1}(1;q)$ are both equal to a sum of a weight $-1/2$ modular form and a weight 0 modular form.

Example. To illustrate Theorem 1.2 in the case $\zeta = 1$, we list the first few series in the two convergent sequences:

$$\begin{aligned}
\widehat{G}_1(1;q) &= -q - q^2 - q^3 - q^4 - q^5 - q^6 - q^7 - q^8 - q^9 - \cdots \\
\widehat{G}_3(1;q) &= -q - q^2 - 2q^3 - 2q^4 - 3q^5 - 4q^6 - 5q^7 - 6q^8 - 8q^9 - \cdots \\
\widehat{G}_5(1;q) &= -q - q^2 - 2q^3 - 2q^4 - 4q^5 - 5q^6 - 7q^7 - 9q^8 - 13q^9 - \cdots \\
\widehat{G}_7(1;q) &= -q - q^2 - 2q^3 - 2q^4 - 4q^5 - 5q^6 - 8q^7 - 10q^8 - 15q^9 - \cdots \\
\widehat{G}_9(1;q) &= -q - q^2 - 2q^3 - 2q^4 - 4q^5 - 5q^6 - 8q^7 - 10q^8 - 16q^9 - \cdots ,
\end{aligned}$$

and

$$\begin{aligned}
\widehat{G}_2(1;q) &= 1 + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + 4q^9 + \cdots \\
\widehat{G}_4(1;q) &= 1 + q^2 + q^3 + 3q^4 + 3q^5 + 5q^6 + 6q^7 + 9q^8 + 10q^9 + \cdots \\
\widehat{G}_6(1;q) &= 1 + q^2 + q^3 + 3q^4 + 3q^5 + 6q^6 + 7q^7 + 11q^8 + 13q^9 + \cdots \\
\widehat{G}_8(1;q) &= 1 + q^2 + q^3 + 3q^4 + 3q^5 + 6q^6 + 7q^7 + 12q^8 + 14q^9 + \cdots \\
\widehat{G}_{10}(1;q) &= 1 + q^2 + q^3 + 3q^4 + 3q^5 + 6q^6 + 7q^7 + 12q^8 + 14q^9 + \cdots .
\end{aligned}$$

We conclude the introduction with the following open problem.

Problem. Assume the notation and hypotheses in Theorem 1.1. For integers $m > 2$, determine whether the q-series

$$\lim_{n \to +\infty} \widehat{G}_{mn}(\zeta;q)$$

appears naturally in the theory of automorphic forms.

In Section 2 we recall some classical identities of Fine, and we prove Theorem 1.1. In Section 3 we prove Theorem 1.2.

2. Identities of Fine and the proof of Theorem 1.1

First we recall classical work of Fine on q-hypergeometric series (a.k.a. *basic* hypergeometric series).

2.1. **Fine's basic hypergeometric series.** Theorem 1.1 follows from the combinatorial properties of Fine's basic hypergeometric function

$$F(a,b,t) := 1 + \sum_{n=1}^{\infty} \frac{(aq;q)_n}{(bq;q)_n} t^n. \tag{2.1}$$

The next lemma (see (6.1), (6.2) and (7.3) of [6]) gives important closed formulas for certain specializations of this series.

Lemma 2.1. *Assuming that the following series are well defined:*

$$F(a,0,t) = \frac{1}{1-t}\sum_{n=0}^{\infty}\frac{(-at)^n q^{\frac{n^2+n}{2}}}{(tq;q)_n}, \tag{2.2}$$

$$F(a,1,t) = \frac{(atq;q)_\infty}{(t;q)_\infty}, \tag{2.3}$$

$$\sum_{n=0}^{\infty}\frac{(t;q)_n q^n}{(bq;q)_n(q;q)_n} = \frac{(t;q)_\infty}{(bq;q)_\infty(q;q)_\infty}\cdot F(b/t,0,t). \tag{2.4}$$

The next identity (see (7.1) of [6]) is the key observation which leads to Theorem 1.1.

Lemma 2.2. *Assume that the series below are well defined*

$$\begin{aligned}
S &= F(a,0,t),\\
H &= \frac{(b/a;q)_\infty}{(bq;q)_\infty(b/at;q)_\infty},\\
G &= \frac{(b/at)}{1-(b/at)}\sum_{n=0}^{\infty}\frac{(b/a;q)_n q^n}{(bq;q)_n(bq/at;q)_n},\\
F &= F(a,b,t).
\end{aligned}$$

Then we have that

$$F + G = HS.$$

2.2. **Proof of Theorem 1.1.** In Lemma 2.2, let $t=\zeta$ and let $a=-1/\zeta$. By (2.2) we have that

$$S = \frac{1}{1-\zeta}\sum_{n=0}^{\infty}\frac{q^{\frac{n^2+n}{2}}}{(\zeta q;q)_n} = \frac{1}{1-\zeta}\cdot G(\zeta;q).$$

Now by letting $b=1$ we have

$$\begin{aligned}
H &= \frac{(-\zeta;q)_\infty}{(q;q)_\infty(-1;q)_\infty} = \frac{1+\zeta}{2}\cdot\frac{(-\zeta q;q)_\infty}{(q;q)_\infty(-q;q)_\infty},\\
G &= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-\zeta;q)_n q^n}{(q;q)_n(-q;q)_n},\\
F &= \sum_{n=0}^{\infty}\frac{(-\zeta^{-1}q;q)_n\zeta^n}{(q;q)_n} = F(-\zeta^{-1},1,\zeta).
\end{aligned}$$

By (2.3), we then have that

$$F = \frac{1}{1-\zeta}\cdot\frac{(-q;q)_\infty}{(\zeta q;q)_\infty}.$$

Combining these observations with Lemma 2.2, we can solve for $G(\zeta;q)$ to obtain

$$G(\zeta;q)=\frac{2}{1+\zeta}\cdot\frac{(q;q)_\infty(-q;q)_\infty^2}{(\zeta q;q)_\infty(-\zeta q;q)_\infty}-\frac{1-\zeta}{1+\zeta}\cdot\frac{(q;q)_\infty(-q;q)_\infty}{(-\zeta q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-\zeta;q)_nq^n}{(q;q)_n(-q;q)_n}. \tag{2.5}$$

By (2.4), where $t=-\zeta$ and $b=-1$, followed by (2.2), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-\zeta;q)_nq^n}{(q;q)_n(-q;q)_n}&=\frac{(-\zeta;q)_\infty}{(-q;q)_\infty(q;q)_\infty}\cdot F(\zeta^{-1},0,-\zeta)\\&=\frac{(-\zeta q;q)_\infty}{(q;q)_\infty(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{\frac{n^2+n}{2}}}{(-\zeta q;q)_n}\\&=\frac{(-\zeta q;q)_\infty}{(q;q)_\infty(-q;q)_\infty}\cdot G(-\zeta;q).\end{aligned}$$

Plugging this into (2.5), one easily finds that

$$D(\zeta;q)=(1+\zeta)G(\zeta;q)+(1-\zeta)G(-\zeta;q)=2\cdot\frac{(-q;q)_\infty^2(q;q)_\infty}{(\zeta^2q^2;q^2)_\infty}.$$

Using (1.7) and the definition of $\eta(\tau)$, the proof follows easily by multiplying out $D(\zeta;q)D(\zeta^{-1};q)$.

3. Proof of Theorem 1.2

To prove Theorem 1.2, we first prove the following two lemmas.

Lemma 3.1. *We have that*

$$\widehat{G}(w;q)=1+\sum_{n=1}^{\infty}\frac{(-w^{-1})^n}{(w^{-1}q;q)_n}. \tag{3.1}$$

Proof. We have

$$\begin{aligned}\widehat{G}(w;q)&=\widehat{G}(w;1/q)\\&=1+\sum_{n=1}^{\infty}\frac{1}{q^{n(n+1)/2}}\frac{1}{(1-w/q)(1-w/q^2)\cdots(1-w/q^n)}\\&=1+\sum_{n=1}^{\infty}\frac{1}{q^{n(n+1)/2}}\frac{-w^{-1}q}{(1-w^{-1}q)}\frac{-w^{-1}q^2}{(1-w^{-1}q^2)}\cdots\frac{-w^{-1}q^n}{(1-w^{-1}q^n)}\\&=1+\sum_{n=1}^{\infty}\frac{(-w^{-1})^n}{(w^{-1}q;q)_n}\end{aligned}$$

as desired. ∎

Lemma 3.2. *Let $-\zeta^{-1}$ be an mth root of unity, and let k be a positive integer. For any positive integer $n > k$ and any positive integer r, the coefficient of q^k in $\widehat{G}_{mn+r}(\zeta; q)$ is the same as the coefficient of q^k in*

$$\widehat{G}_{mn}(\zeta; q) + \frac{(-\zeta^{-1})^r - 1}{\zeta + 1} \frac{1}{(\zeta^{-1}q; q)_\infty}.$$

Proof. For $n > k$, the coefficient of q^k in $\frac{1}{(\zeta^{-1}q;q)_n}$ is the same as that in $\frac{1}{(\zeta^{-1}q;q)_\infty}$. Thus for $n > k$, the coefficient of q^k in

$$\widehat{G}_{mn+r}(\zeta; q) = 1 + \sum_{t=1}^{mn+r} \frac{(-\zeta^{-1})^t}{(\zeta^{-1}q; q)_t} = \widehat{G}_{mn} + \sum_{t=mn+1}^{mn+r} \frac{(-\zeta^{-1})^t}{(\zeta^{-1}q; q)_t}$$

is the same as the coefficient of q^k in

$$\begin{aligned} \widehat{G}_{mn}(\zeta; q) + \sum_{t=mn+1}^{mn+r} \frac{(-\zeta^{-1})^t}{(\zeta^{-1}q; q)_\infty} &= \widehat{G}_{mn}(\zeta; q) + \frac{1}{(\zeta^{-1}q; q)_\infty} \sum_{t=1}^{r} (-\zeta^{-1})^t \\ &= \widehat{G}_{mn}(\zeta; q) + \frac{(-\zeta^{-1})^r - 1}{\zeta + 1} \frac{1}{(\zeta^{-1}q; q)_\infty} \end{aligned}$$

and the claim follows. □

3.1. **Proof of Theorem 1.2.** Setting $r = m$ in Lemma 3.2, it follows that for $n > k$, the coefficient of q^k in $\widehat{G}_{m(n+1)}(\zeta; q)$ is the same as that of $\widehat{G}_{mn}(\zeta; q)$. Thus, using Lemma 3.2 repeatedly, we see that the coefficient of q^k in $\widehat{G}_{m(n+a)+r}(\zeta; q)$ is the same as that of $\widehat{G}_{mn+r}(\zeta; q)$ for any positive integers a and r. Therefore, the coefficient of each power of q is eventually constant in the series $\widehat{G}_{mn+r}(\zeta; q)$, and so the functions $\lim_{n\to\infty} \widehat{G}_{mn+r}(\zeta; q)$ are well-defined q-series.

Taking the limit as $n \to \infty$ on both sides of Lemma 3.2, it follows that

$$\lim_{n\to\infty} \widehat{G}_{mn+r}(\zeta; q) = \lim_{n\to\infty} \widehat{G}_{mn}(\zeta; q) + \frac{(-\zeta^{-1})^r - 1}{\zeta + 1} \frac{1}{(\zeta^{-1}q; q)_\infty}. \tag{3.2}$$

We now examine the case $-\zeta^{-1} = -1$. Notice that by (3.2), we have

$$\lim_{n\to\infty} \widehat{G}_{2n+1}(1; q) - \lim_{n\to\infty} \widehat{G}_{2n}(1; q) = \frac{-1}{(q; q)_\infty}. \tag{3.3}$$

Now, define $p_k(n)$ to be the number of partitions of n whose largest part is at most k, and define $p(n, k)$ to be the number of partitions of n whose largest part is equal to k. It is clear by inspection that the coefficient of q^t in $\frac{1}{(q;q)_k}$ is $p_k(t)$ for any positive integers t and k.

Let $c(t)$ be the coefficient of q^t in $\lim_{n\to\infty} \widehat{G}_{2n+1}(1; q) + \lim_{n\to\infty} \widehat{G}_{2n}(1; q)$. Suppose t is even. Then $c(t)$ equal to the coefficient of q^t in

$$\widehat{G}_{t+1}(1; q) + \widehat{G}_t(1; q) = 2 + 2 \sum_{k=1}^{t} \frac{(-1)^k}{(q; q)_k} + \frac{-1}{(q; q)_{t+1}},$$

so

$$c(t) = 2(p_0(t) - p_1(t) + \cdots - p_{t-1}(t) + p_t(t)) - p_{t+1}(t)$$

since t is even by assumption. Let $\mathrm{Even}(t)$ denote the number of partitions of t whose largest part is even, and let $\mathrm{Odd}(t)$ denote the number of partitions of t whose largest part is odd. Notice that $\mathrm{Even}(t) + \mathrm{Odd}(t) = p(t)$, $p_t(t) = p_{t+1}(t) = p(t)$, and $p_k(t) = p(t,k) + p(t,k-1) + \cdots + p(t,1)$. Therefore,

$$\begin{aligned} c(t) &= -2((p_1(t) - p_0(t)) + (p_3(t) - p_2(t)) + \cdots + (p_{t-1}(t) - p_{t-2}(t))) + p(t) \\ &= -2(p(t,1) + p(t,3) + p(t,5) + \cdots + p(t,t-1)) + p(t) \\ &= -2\,\mathrm{Odd}(t) + (\mathrm{Even}(t) + \mathrm{Odd}(t)) \\ &= \mathrm{Even}(t) - \mathrm{Odd}(t). \end{aligned}$$

Now suppose t is an odd integer. In this case,

$$\begin{aligned} c(t) &= 2(p_0(t) - p_1(t) + \cdots + p_{t-1}(t) - p_t(t)) + p_{t+1}(t) \\ &= 2((p_2(t) - p_1(t)) + \cdots + (p_{t-1}(t) - p_{t-2}(t))) - p(t) \\ &= 2(p(t,2) + p(t,4) + \cdots + p(t,t-1)) - p(t) \\ &= 2\,\mathrm{Even}(t) - (\mathrm{Even}(t) + \mathrm{Odd}(t)) \\ &= \mathrm{Even}(t) - \mathrm{Odd}(t). \end{aligned}$$

Thus $c(t) = \mathrm{Even}(t) - \mathrm{Odd}(t)$ for any positive integer t.

Recall that $b(n)$ is defined by $\sum_{n=0}^{\infty}(-1)^n b(n) q^n = \prod_{n=1}^{\infty}(1+q^{2n-1})$, so $b(n)$ is equal to $(-1)^n$ times the number of partitions of n into distinct odd parts. A bijection of Sylvester (c.f. [3], p. 28) shows that $b(n)$ is equal to the difference between the number of partitions of n having an even number of parts and the number of partitions of n having an odd number of parts. By taking the conjugates of the partitions of n (interchanging the rows and columns of their Young diagrams), it follows that $b(n) = \mathrm{Even}(n) - \mathrm{Odd}(n) = c(n)$ for all n. Therefore,

$$\lim_{n\to\infty} \widehat{G}_{2n+1}(1;q) + \lim_{n\to\infty} \widehat{G}_{2n}(1;q) = \sum_{n=0}^{\infty} b(n) q^n. \tag{3.4}$$

Combining (3.3) and (3.4), the conclusion follows.

Remark. We have presented the above proof of Theorem 1.2 in order to explore the combinatorial properties of the related partition functions. Here we give an alternate proof (due to the referee) that uses only manipulation of q-series. For any positive integer n, we have

$$\begin{aligned}
\widehat{G}_n(1;q)+\widehat{G}_{n-1}(1;q) &= \sum_{k=0}^{n}\frac{(-1)^k}{(q;q)_k}+\sum_{k=0}^{n-1}\frac{(-1)^k}{(q;q)_k}\\
&= \sum_{k=0}^{n}\frac{(-1)^k}{(q;q)_k}-\sum_{k=1}^{n}\frac{(-1)^k}{(q;q)_{k-1}}\\
&= 1+\sum_{k=1}^{n}(-1)^k\left\{\frac{1}{(q;q)_k}-\frac{1}{(q;q)_{k-1}}\right\}\\
&= 1+\sum_{k=1}^{n}\frac{(-1)^kq^k}{(q;q)_k}\\
&= \sum_{k=0}^{n}\frac{(-1)^kq^k}{(q;q)_k}
\end{aligned}$$

Taking the limit as $n\to\infty$ yields

$$\lim_{n\to\infty}\widehat{G}_{2n+1}(1;q)+\lim_{n\to\infty}\widehat{G}_{2n}(1;q)=\sum_{k=0}^{\infty}\frac{(-1)^kq^k}{(q;q)_k}.$$

Setting $a=0$ and $t=-q$ in the identity 2.3, we find that this sum simplifies to $\frac{1}{(-q;q)_\infty}$, as desired.

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