DYSON'S RANKS AND MAASS FORMS

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For Jean-Pierre Serre in celebration of his 80th birthday.

1. Introduction and Statement of Results

"*The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi. This remains a challenge for the future.*"

Freeman Dyson, 1987
Ramanujan Centenary Conference

Dyson's quote (see page 20 of [16]) refers to 22 peculiar q-series, such as

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}, \tag{1.1}$$

which were defined by Ramanujan and Watson decades ago. In his last letter to Hardy dated January 1920 (see pages 127-131 of [27]), Ramanujan lists 17 such functions, and he gives 2 more in his "Lost Notebook" [27]. In his paper "The final problem: An account of the mock theta functions" [32], Watson defines 3 further functions.

Surprisingly, much remains unknown about these enigmatic series. Ramanujan's claims about their analytic properties remain open, and there is even debate concerning the rigorous definition of such a function. Despite these seemingly problematic issues, Ramanujan's mock theta functions indeed possess many striking properties, and they have been the subject of an astonishing number of important works (for example, see [5, 6, 7, 8, 12, 13, 14, 18, 19, 20, 23, 27, 28, 32, 33, 35, 36] to name a few). Watson predicted this high level of activity in his 1936 Presidential Address to the London Mathematical Society with his prophetic words (see page 80 of [32]):

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"Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. To his students such discoveries will be a source of delight and wonder until the time shall come when we too shall make our journey to that Garden of Proserpine (a.k.a. Persephone)..."

G. N. Watson, 1936.

In his 2002 Ph.D. thesis [36], written under the direction of Zagier, Zwegers made an important step in the direction of Dyson's "challenge for the future". He related many of Ramanujan's mock theta functions to real analytic vector valued modular forms. We make another step by establishing that Dyson's own rank generating function can be used to construct the desired "coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around old theta functions of Jacobi". We show that the specializations of his partition rank generating function $R(\zeta; q)$, where $\zeta \neq 1$ is a root of unity, are "holomorphic parts" of weak Maass forms. Moreover, we show that the "non-holomorphic parts" of these forms are period integrals of theta functions, thereby realizing Dyson's speculation that such a picture should involve theta functions. We shall use these results to systematically obtain Ramanujan-type congruences for Dyson's rank partition functions.

To describe the historical context of these results, we begin by recalling classical facts about partitions and modular forms which inspired Ramanujan to originally define the mock theta functions. A *partition* of a non-negative integer n is any non-increasing sequence of positive integers whose sum is n. As usual, let $p(n)$ denote the number of partitions of n. The partition function $p(n)$ has the well known infinite product generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}, \tag{1.2}$$

which coincides with $q^{\frac{1}{24}}/\eta(z)$, where

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \qquad\qquad (q := e^{2\pi i z})$$

is Dedekind's weight $1/2$ modular form. Modular forms have played a central role in the theory of partitions, largely due to the fact that many generating functions in the subject, such as (1.2), are related to infinite product modular forms such as Dedekind's eta-function and the Siegel-Klein forms.

On the other hand, many partition generating functions are "Eulerian" forms, also known as q-series, which do not naturally appear in modular form theory. However there are famous examples, such as the Rogers-Ramanujan identities

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\frac{1}{\prod_{n=1}^{\infty}(1-q^{5n-1})(1-q^{5n-4})},$$

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}=\frac{1}{\prod_{n=1}^{\infty}(1-q^{5n-2})(1-q^{5n-3})},$$

where Eulerian forms are essentially modular forms. As another example, we note that

$$\sum_{n=0}^{\infty}p(n)q^n=1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2}. \tag{1.3}$$

The mock theta functions stand out in this context. Although they are not modular, they possess striking properties which prompted Dyson to set forth his challenge of 1987. In this regard, the focus of our attention is a particularly exceptional family of such series, the specializations of Dyson's own rank generating function. In an effort to provide a combinatorial explanation of Ramanujan's congruences for $p(n)$, Dyson introduced [15] the so-called "rank" of a partition, a delightfully simple statistic. The rank of a partition is defined to be its largest part minus the number of its parts. More precisely, he conjectured that the partitions of $5n+4$ (resp. $7n+5$) form 5 (resp. 7) groups of equal size when sorted by their ranks modulo 5 (resp. 7)[1]. He further postulated the existence of another statistic, the so-called "crank"[2], which allegedly would explain all three Ramanujan congruences

$$\begin{aligned} p(5n+4)&\equiv 0 \pmod{5},\\ p(7n+5)&\equiv 0 \pmod{7},\\ p(11n+6)&\equiv 0 \pmod{11}.\end{aligned}$$

In 1954, Atkin and Swinnerton-Dyer proved [10] Dyson's rank conjectures.

If $N(m,n)$ denotes the number of partitions of n with rank m, then it is well known that

$$R(w;q):=1+\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}N(m,n)w^mq^n=1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(wq;q)_n(w^{-1}q;q)_n}, \tag{1.4}$$

[1] A short calculation reveals that this phenomenon cannot hold modulo 11.

[2] In 1988, Andrews and Garvan [9] found the crank, and they indeed confirmed Dyson's speculation that it explains the three Ramanujan congruences above. Recent work of Mahlburg [24] establishes that the Andrews-Dyson-Garvan crank plays an even more central role in the theory partition congruences. His work concerns partition congruences modulo arbitrary powers of all primes ≥ 5. Other work by Garvan, Kim and Stanton [17] gives a different "crank" for several other Ramanujan congruences.

where

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}),$$
$$(a;q)_\infty := \prod_{m=0}^{\infty}(1-aq^m).$$

Obviously, by letting $w=1$, we obtain (1.3).

Letting $w=-1$, we obtain the series

$$R(-1;q) = 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}.$$

This series is the mock theta function $f(q)$ given in (1.1). In earlier work [11], the present authors proved that $q^{-1}R(-1;q^{24})$ is the "holomorphic part" of a weak Maass form. This is a special case of our first result.

To make this precise, we begin by recalling the notion of a weak Maass form of half-integral weight $k\in\frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$. If $z=x+iy$ with $x,y\in\mathbb{R}$, then the weight k hyperbolic Laplacian is given by

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right). \tag{1.5}$$

If v is odd, then define ϵ_v by

$$\epsilon_v := \begin{cases}1 & \text{if } v\equiv 1 \pmod 4,\\ i & \text{if } v\equiv 3 \pmod 4.\end{cases} \tag{1.6}$$

A *weak Maass form of weight k on a subgroup* $\Gamma\subset\Gamma_0(4)$ is any smooth function $f:\mathbb{H}\to\mathbb{C}$ satisfying the following:

(1) For all $A=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$ and all $z\in\mathbb{H}$, we have[3]

$$f(Az)=\left(\frac{c}{d}\right)^{2k}\epsilon_d^{-2k}(cz+d)^k\ f(z).$$

(2) We have that $\Delta_k f=0$.

(3) The function $f(z)$ has at most linear exponential growth at all the cusps of Γ.

Suppose that $0<a<c$ are integers, and let $\zeta_c := e^{2\pi i/c}$. If $f_c := \frac{2c}{\gcd(c,6)}$, then define the theta function $\Theta\left(\frac{a}{c};\tau\right)$ by

$$\Theta\left(\frac{a}{c};\tau\right) := \sum_{m \pmod{f_c}}(-1)^m\sin\left(\frac{a\pi(6m+1)}{c}\right)\cdot\theta\left(6m+1,6f_c;\frac{\tau}{24}\right), \tag{1.7}$$

[3]This transformation law agrees with Shimura's notion of a half-integral weight modular form [30].

where

$$\theta(\alpha, \beta; \tau) := \sum_{n \equiv \alpha \pmod{\beta}} n e^{2\pi i \tau n^2}. \tag{1.8}$$

Throughout, let $\ell_c := \mathrm{lcm}(2c^2, 24)$, and let $\widetilde{\ell_c} := \ell_c/24$. It is well known that $\Theta\left(\frac{a}{c}; \ell_c\tau\right)$ is a cusp form of weight 3/2. Using this cuspidal theta function, we define the function $S_1\left(\frac{a}{c}; z\right)$ by the period integral

$$S_1\left(\frac{a}{c}; z\right) := \frac{-i\sin\left(\frac{\pi a}{c}\right)\ell_c^{\frac{1}{2}}}{\sqrt{3}} \int_{-\bar{z}}^{i\infty} \frac{\Theta\left(\frac{a}{c}; \ell_c\tau\right)}{\sqrt{-i(\tau+z)}}\, d\tau. \tag{1.9}$$

Using this notation, define $D\left(\frac{a}{c}; z\right)$ by

$$D\left(\frac{a}{c}; z\right) := -S_1\left(\frac{a}{c}; z\right) + q^{-\frac{\ell_c}{24}} R(\zeta_c^a; q^{\ell_c}). \tag{1.10}$$

Moreover, define the group Γ_c by

$$\Gamma_c := \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \ell_c^2 & 1 \end{pmatrix} \right\rangle. \tag{1.11}$$

Theorem 1.1. *If $0 < a < c$, then $D\left(\frac{a}{c}; z\right)$ is a weak Maass form of weight 1/2 on Γ_c.*

When $a/c = 1/2$, it turns out that $D\left(\frac{1}{2}; z\right)$ is a weak Maass form on $\Gamma_0(144)$ with Nebentypus character $\chi_{12}(\cdot) = \left(\frac{12}{\cdot}\right)$. This fact was established by the authors in [11], and it plays a central role in the proof of the Andrews-Dragonette Conjecture on the coefficients of $f(q)$. In view of this fact, it is natural to suspect that $D\left(\frac{a}{c}; z\right)$ is often a weak Maass form on a group larger than Γ_c. For odd c, we establish the following.

Theorem 1.2. *If $0 < a < c$, where c is odd, then $D\left(\frac{a}{c}; z\right)$ is a weak Maass form of weight 1/2 on $\Gamma_1(144 f_c^2 \widetilde{\ell_c})$.*

Theorem 1.2 is implied by a general result about vector valued weight 1/2 weak Maass forms for the modular group $\mathrm{SL}_2(\mathbb{Z})$ (see Theorem 3.4), a result which is of independent interest.

Remark. We refer to $S_1\left(\frac{a}{c}; z\right)$ (resp. $q^{-\frac{\ell_c}{24}} R(\zeta_c^a; q^{\ell_c})$) as the *non-holomorphic* (resp. *holomorphic*) part of the Maass form $D\left(\frac{a}{c}; z\right)$. To justify this, one notes that $S_1\left(\frac{a}{c}; z\right)$ is non-holomorphic in z, and that

$$\frac{\partial}{\partial \bar{z}}\left(q^{-\frac{\ell_c}{24}} R(\zeta_c^a; q^{\ell_c})\right) = 0.$$

Here we have that $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. In particular, $q^{-\frac{\ell_c}{24}} R(\zeta_c^a; q^{\ell_c})$ is the part of the Fourier expansion of $D\left(\frac{a}{c}; z\right)$ which is given as a series expansion in $q = e^{2\pi i z}$ (see Proposition 4.1).

Theorems 1.1 and 1.2 provide a new perspective on the role that modular forms play in the theory of partitions. They imply that the generating functions for Dyson's rank partition functions are related to Maass forms and modular forms. If r and t are integers, then let $N(r,t;n)$ be the number of partitions of n whose rank is $r \pmod{t}$.

Theorem 1.3. *If $0 \leq r < t$ are integers, then*

$$\sum_{n=0}^{\infty}\left(N(r,t;n)-\frac{p(n)}{t}\right)q^{\ell_t n-\frac{\ell_t}{24}}$$

is the holomorphic part of a weak Maass form of weight 1/2 on Γ_t. Moreover, if t is odd, then it is on $\Gamma_1(144 f_t^2 \widetilde{\ell_t})$.

This result allows us to relate many "sieved" generating functions to *weakly holomorphic modular forms*, those forms whose poles (if there are any) are supported at cusps.

Theorem 1.4. *If $0 \leq r < t$ are integers, where t is odd, and $\mathcal{P} \nmid 6t$ is prime, then*

$$\sum_{\substack{n\geq 1\\ \left(\frac{24\ell_t n-\ell_t}{\mathcal{P}}\right)=-\left(\frac{-24\widetilde{\ell_t}}{\mathcal{P}}\right)}}\left(N(r,t;n)-\frac{p(n)}{t}\right)q^{\ell_t n-\frac{\ell_t}{24}}$$

is a weight 1/2 weakly holomorphic modular form on $\Gamma_1(144 f_t^2 \widetilde{\ell_t}\mathcal{P}^4)$.

These results are useful for studying Dyson's rank partition generating functions. Atkin and Swinnerton-Dyer [10] confirmed Dyson's conjecture that for every integer n and every r we have

$$N(r,5;5n+4)=\frac{p(5n+4)}{5}, \tag{1.12}$$

$$N(r,7;7n+5)=\frac{p(7n+5)}{7}, \tag{1.13}$$

thereby providing a combinatorial "explanation" of Ramanujan partition congruences with modulus 5 and 7. It is not difficult to use our results to give alternative proofs of these rank identities, as well as others of similar type.

Armed with Theorems 1.2, 1.4 and 3.4, one can obtain deeper results about ranks. They can be used to obtain asymptotic formulas for the $N(r,t;n)$ partition functions. Indeed, the present authors have already successfully employed the theory of weak Maass forms to solve the more difficult problem of obtaining exact formulas in the case of the functions $N(0,2;n)$ and $N(1,2;n)$ (see Theorem 1.1 of [11]). For odd t, one can use Theorem 3.4 and the "circle method" to obtain asymptotics. Since the details are messy and lengthy, for brevity we have chosen to address asymptotics in a later paper.

Here we turn to the question of congruences, the subject which originally motivated Dyson to define partition ranks. In this direction, we shall employ a method first used by the second author in [25] in his work on $p(n)$. We show that Dyson's rank partition functions satisfy congruences of Ramanujan type, a result which nicely complements the recent blockbuster paper [24] by Mahlburg on the Andrews-Garvan-Dyson crank.

Theorem 1.5. *Let t be a positive odd integer, and let $Q \nmid 6t$ be prime. If j is a positive integer, then there are infinitely many non-nested arithmetic progressions $An + B$ such that for every $0 \leq r < t$ we have*

$$N(r, t; An + B) \equiv 0 \pmod{Q^j}.$$

Three remarks.
1) The congruences in Theorem 1.5 may be viewed as a combinatorial decomposition of the partition function congruence

$$p(An + B) \equiv 0 \pmod{Q^j}.$$

2) By "non-nested", we mean that there are infinitely many arithmetic progressions $An + B$, with $0 \leq B < A$, with the property that there are no progressions which contain another progression.
3) Theorem 1.5 is in sharp contrast to Mahlburg's recent result [24] on the Andrews-Garvan-Dyson crank. For example, his results imply that congruences modulo Q^j exist for all the crank partition functions with modulus $t = Q$. On the other hand, Theorem 1.5 proves congruences for powers of those primes $Q \geq 5$ which do not divide the rank modulus t.

Conjecture. *Theorem 1.5 holds for those primes $Q \geq 5$ which divide t.*

To prove these theorems, we require a number of new results. First of all, the proof of Theorem 1.2 requires transformation laws for some new classes of mock theta functions. In Section 2, we derive these transformation formulas, and we recall recent work of Gordon and McIntosh [19]. In Section 3, we use the results of Section 2 to construct the vector valued Maass forms whose properties are the content of Theorem 3.4. We conclude Section 3 with proofs of Theorems 1.1, 1.2, and 1.3. In Section 4, we prove Theorem 1.4, and then give the proof of Theorem 1.5. The proof of Theorem 1.5 relies on Q-adic properties of weakly holomorphic half-integral weight modular forms, and Q-adic Galois representations associated to modular forms.

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2. Modular transformation formulas

Here we derive modular transformation formulas for $R(\zeta; q)$ and allied functions. In Section 2.1, we first recall transformation laws obtained recently by Gordon and McIntosh [19], and in Section 2.2 we derive transformation formulas for closely allied functions. In Section 2.3 we combine these results to produce an infinite family of vector valued modular forms under $\mathrm{SL}_2(\mathbb{Z})$.

2.1. **Transformation laws of Gordon and McIntosh.**

To state the transformation formulas of Gordon and McIntosh, we require the following series. If $0 < a < c$ are integers and $q := e^{2\pi i z}$, then we let

(2.1)

$$M\left(\frac{a}{c}; z\right) = M\left(\frac{a}{c}; q\right) := \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n+\frac{a}{c}}}{1 - q^{n+\frac{a}{c}}} \cdot q^{\frac{3}{2}n(n+1)},$$

$$M_1\left(\frac{a}{c}; z\right) = M_1\left(\frac{a}{c}; q\right) := \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} q^{n+\frac{a}{c}}}{1 + q^{n+\frac{a}{c}}} \cdot q^{\frac{3}{2}n(n+1)},$$

$$N\left(\frac{a}{c}; z\right) = N\left(\frac{a}{c}; q\right) := \frac{1}{(q;q)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n (1+q^n)\left(2 - 2\cos\left(\frac{2\pi a}{c}\right)\right)}{1 - 2q^n \cos\left(\frac{2\pi a}{c}\right) + q^{2n}} \cdot q^{\frac{n(3n+1)}{2}}\right),$$

$$N_1\left(\frac{a}{c}; z\right) = N_1\left(\frac{a}{c}; q\right) := \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (1 - q^{2n+1})}{1 - 2q^{n+\frac{1}{2}} \cos\left(\frac{2\pi a}{c}\right) + q^{2n+1}} \cdot q^{\frac{3n(n+1)}{2}}.$$

Two remarks.

1) Gordon and McIntosh show the following q-series identities

$$M\left(\frac{a}{c}; q\right) = \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(q^{\frac{a}{c}}; q)_n \cdot (q^{1-\frac{a}{c}}; q)_n}, \tag{2.2}$$

$$N\left(\frac{a}{c}; q\right) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{j=1}^{n}\left(1 - 2\cos\left(\frac{2\pi a}{c}\right) q^j + q^{2j}\right)}. \tag{2.3}$$

2) If $0 < a < c$ are integers, then (1.4) and (2.3) imply the important fact that

$$R(\zeta_c^a; q) = N\left(\frac{a}{c}; q\right). \tag{2.4}$$

To state their transformation laws, we require the following Mordell integrals

$$
(2.5) \qquad \begin{aligned}
J\left(\frac{a}{c};\alpha\right) &:= \int_0^\infty e^{-\frac{3}{2}\alpha x^2} \cdot \frac{\cosh\left(\left(\frac{3a}{c}-2\right)\alpha x\right)+\cosh\left(\left(\frac{3a}{c}-1\right)\alpha x\right)}{\cosh(3\alpha x/2)}\, dx,\\
J_1\left(\frac{a}{c};\alpha\right) &:= \int_0^\infty e^{-\frac{3}{2}\alpha x^2} \cdot \frac{\sinh\left(\left(\frac{3a}{c}-2\right)\alpha x\right)-\sinh\left(\left(\frac{3a}{c}-1\right)\alpha x\right)}{\sinh(3\alpha x/2)}\, dx.
\end{aligned}
$$

By modifying the seminal arguments of Watson [32], Gordon and McIntosh (see page 199 of [19]) proved the following theorem.

Theorem 2.1. *Suppose that $0<a<c$ are integers, and that α and β have the property that $\alpha\beta=\pi^2$. If $q:=e^{-\alpha}$ and $q_1:=e^{-\beta}$, then we have*

$$
\begin{aligned}
q^{\frac{3a}{2c}\left(1-\frac{a}{c}\right)-\frac{1}{24}} \cdot M\left(\frac{a}{c};q\right) &= \sqrt{\frac{\pi}{2\alpha}}\csc\left(\frac{a\pi}{c}\right) q_1^{-\frac{1}{6}} \cdot N\left(\frac{a}{c};q_1^4\right) - \sqrt{\frac{3\alpha}{2\pi}} \cdot J\left(\frac{a}{c};\alpha\right),\\
q^{\frac{3a}{2c}\left(1-\frac{a}{c}\right)-\frac{1}{24}} \cdot M_1\left(\frac{a}{c};q\right) &= -\sqrt{\frac{2\pi}{\alpha}} q_1^{\frac{4}{3}} \cdot N_1\left(\frac{a}{c};q_1^2\right) - \sqrt{\frac{3\alpha}{2\pi}} \cdot J_1\left(\frac{a}{c};\alpha\right).
\end{aligned}
$$

2.2. **Modular transformation formulas for allied series.** Theorem 2.1 is not sufficient for fully understanding the modularity properties of the functions $N\left(\frac{a}{c};q\right) = R(\zeta_c^a;q)$ under the Möbius transformations arising from $\mathrm{SL}_2(\mathbb{Z})$. Indeed, under translations the functions M and M_1 transform to allied functions whose modularity properties must be deduced. To this end, it is necessary to define further series which will allow us to view the functions in the previous subsection as pieces of the components of a vector valued function whose transformations we shall determine under the generators of $\mathrm{SL}_2(\mathbb{Z})$. Suppose that c is a positive integer, and suppose that a and b are integers for which $0\le a,b<c$. Using this notation, define $M(a,b,c;z)$ by

$$
(2.6) \qquad M(a,b,c;z) = M(a,b,c;q) := \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n+\frac{a}{c}}}{1-\zeta_c^b q^{n+\frac{a}{c}}} \cdot q^{\frac{3}{2}n(n+1)}.
$$

In addition, if $\frac{b}{c}\notin\{0,\frac{1}{2},\frac{1}{6},\frac{5}{6}\}$, then define the integer $k(b,c)$ by

$$
(2.7) \qquad k(b,c) := \begin{cases} 0 & \text{if } 0<\frac{b}{c}<\frac{1}{6},\\ 1 & \text{if } \frac{1}{6}<\frac{b}{c}<\frac{1}{2},\\ 2 & \text{if } \frac{1}{2}<\frac{b}{c}<\frac{5}{6},\\ 3 & \text{if } \frac{5}{6}<\frac{b}{c}<1. \end{cases}
$$

Furthermore, throughout we let $e(\alpha) := e^{2\pi i\alpha}$. Using this notation, then define the series $N(a,b,c;z)$ by

$$N(a,b,c;z) = N(a,b,c;q) := \frac{1}{(q;q)_\infty}\left(\frac{ie\left(-\frac{a}{2c}\right)q^{\frac{b}{2c}}}{2\left(1-e\left(-\frac{a}{c}\right)q^{\frac{b}{c}}\right)} + \sum_{m=1}^{\infty} K(a,b,c,m;z)\cdot q^{\frac{m(3m+1)}{2}}\right), \tag{2.8}$$

where

$$K(a,b,c,m;z) := (-1)^m \frac{\sin\left(\frac{\pi a}{c} - \left(\frac{b}{c}+2k(b,c)m\right)\pi z\right) + \sin\left(\frac{\pi a}{c} - \left(\frac{b}{c}-2k(b,c)m\right)\pi z\right)q^m}{1-2\cos\left(\frac{2\pi a}{c}-\frac{2\pi bz}{c}\right)q^m + q^{2m}}. \tag{2.9}$$

Moreover, define the Mordell integral

$$J(a,b,c;\alpha) := \int_{-\infty}^{\infty} e^{-\frac{3}{2}\alpha x^2 + 3\alpha x\frac{a}{c}}\cdot \frac{\left(\zeta_c^b e^{-\alpha x} + \zeta_c^{2b}e^{-2\alpha x}\right)}{\cosh\left(3\alpha x/2 - 3\pi i\frac{b}{c}\right)}\, dx. \tag{2.10}$$

Using this notation, we obtain the following transformation laws.

Theorem 2.2. *Suppose that c is a positive integer, and that a and b are integers for which $0 \le a < c$, $0 < b < c$ and $\frac{b}{c} \notin \left\{\frac{1}{2}, \frac{1}{6}, \frac{5}{6}\right\}$. Furthermore, suppose that α and β have the property that $\alpha\beta = \pi^2$. If $q := e^{-\alpha}$ and $q_1 := e^{-\beta}$, then*

$$q^{\frac{3a}{2c}\left(1-\frac{a}{c}\right)-\frac{1}{24}}\cdot M(a,b,c;q) = \sqrt{\frac{8\pi}{\alpha}}e^{-2\pi i\frac{a}{c}k(b,c)+3\pi i\frac{b}{c}\left(\frac{2a}{c}-1\right)}\zeta_c^{-b}q_1^{\frac{4b}{c}k(b,c)-\frac{6b^2}{c^2}-\frac{1}{6}}\cdot N(a,b,c;q_1^4) - \sqrt{\frac{3\alpha}{8\pi}}\zeta_{2c}^{-5b}\cdot J(a,b,c;\alpha).$$

Three remarks.
1) Although b is non-zero in Theorem 2.2, note that $M(a,0,c;q) = M\left(\frac{a}{c};q\right)$. Therefore, Theorem 2.1, combined with Theorem 2.2, gives the appropriate transformation laws of every $M(a,b,c;q)$, where $0 < a < c$ and $0 \le b < c$, provided that $\frac{b}{c} \notin \left\{\frac{1}{2}, \frac{1}{6}, \frac{5}{6}\right\}$.
2) Observe that $M(1,1,2;q) = -M_1\left(\frac{1}{2};q\right)$. Therefore, the case where $2a = c$ and $\frac{b}{c} = \frac{1}{2}$ is also covered by Theorem 2.1.
3) One could also prove Theorem 2.2 by using arguments which are analogous to those developed by Zwegers in [36].

Proof of Theorem 2.2. To prove this theorem, we argue with contour integration in a manner which is very similar to earlier work of Watson [32]. We consider a contour integral which is basically the function $M(a,b,c;q)$. Define this integral I by

$$
\begin{aligned}
I := I_1 + I_2 := & \frac{1}{2\pi i}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{\pi}{\sin(\pi\tau)} \cdot \frac{e^{-\alpha\left(\tau+\frac{a}{c}\right)}}{1-\zeta_c^b e^{-\alpha\left(\tau+\frac{a}{c}\right)}} \cdot e^{-\frac{3}{2}\alpha\tau(\tau+1)} d\tau \\
& - \frac{1}{2\pi i}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{\pi}{\sin(\pi\tau)} \cdot \frac{e^{-\alpha\left(\tau+\frac{a}{c}\right)}}{1-\zeta_c^b e^{-\alpha\left(\tau+\frac{a}{c}\right)}} \cdot e^{-\frac{3}{2}\alpha\tau(\tau+1)} d\tau.
\end{aligned} \tag{2.11}
$$

Here $\varepsilon > 0$ is sufficiently small enough so that $1-\zeta_c^b e^{-\alpha\left(\tau+\frac{a}{c}\right)}$ is non-zero for $-\varepsilon \leq \mathrm{Im}(\tau) \leq \varepsilon$. This is indeed possible since $1-\zeta_c^b e^{-\alpha\left(\tau+\frac{a}{c}\right)} = 0$ if and only if

$$\tau = -\frac{a}{c} + \frac{2\pi i\left(\frac{b}{c}+n\right)}{\alpha} =: \tau_n$$

(here we need the condition that $b \neq 0$).

By construction, we have that the poles of the integrand only arise from the roots of $\sin(\pi\tau)$, and they are the points $\tau \in \mathbb{Z}$. The residue of the integrand in $\tau = n \in \mathbb{Z}$ equals

$$\frac{(-1)^n q^{n+\frac{a}{c}}}{1-\zeta_c^b \cdot q^{n+\frac{a}{c}}} \cdot q^{\frac{3n(n+1)}{2}}.$$

For $\mathrm{Re}(\tau) \to \infty$, the integrand is of rapid decay, and so the Residue Theorem implies that

$$I = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n+\frac{a}{c}}}{1-\zeta_c^b \cdot q^{n+\frac{a}{c}}} \cdot q^{\frac{3n(n+1)}{2}} = (q;q)_\infty \cdot M(a,b,c;q). \tag{2.12}$$

We now compute the integrals I_1 and I_2. We first consider I_2. Using (2.11) and the identity

$$\frac{1}{\sin(\pi\tau)} = -2i\sum_{n=0}^{\infty} e^{(2n+1)\pi i\tau},$$

which holds for $\tau \in \mathbb{H}$, we find that

$$I_2 = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} 2\pi i \frac{e^{(2n+1)\pi i\tau-\alpha\left(\tau+\frac{a}{c}\right)-\frac{3}{2}\alpha\tau(\tau+1)}}{1-\zeta_c^b \cdot e^{-\alpha\left(\tau+\frac{a}{c}\right)}}\, d\tau =: \frac{1}{2\pi i}\sum_{n=0}^{\infty} J_n.$$

We now reformulate I_2 in a useful way by shifting the paths of integration through the points ω_n, the saddle points of

$$\exp\left((2n+1)\pi i\tau - \frac{3}{2}\alpha\tau^2\right).$$

These are the points given by

$$\omega_n = \frac{(2n+1)\pi i}{3\alpha}.$$

By the Residue Theorem, we have

$$J_n := \int_{-\infty+i\epsilon}^{\infty+i\epsilon} f = \int_{-\infty+\omega_n}^{\infty+\omega_n} f + \sum_{\tau_{m,n}} \mathrm{Res}[f(\tau_{m,n})],$$

where $\tau_{m,n}$ are those poles τ_m of the integrand f lying between the original contour and the new contour (i.e. those τ_m for which $m \geq 0$ and $\mathrm{Im}(\tau_m) < \mathrm{Im}(\omega_n)$). These turn out to be those points τ_m for which $m \geq 0$ and that satisfy

$$\frac{(2n+1)}{3} > 2\left(\frac{b}{c} + m\right). \tag{2.13}$$

(That no poles lie on the path of integration follows from the condition that $\frac{b}{c} \notin \left\{\frac{1}{2}, \frac{1}{6}, \frac{5}{6}\right\}$). Using definition (2.7), we have that (2.13) is equivalent to

$$n \geq 3m + k(b,c).$$

At the points τ_m, the integrand has the residue

$$\begin{aligned}
\lambda_{n,m} \quad &:= \quad \frac{2\pi i}{\alpha} \cdot e^{(2n+1)\pi i \tau_m - \alpha\left(\tau_m + \frac{a}{c}\right) - \frac{3}{2}\alpha\tau_m(\tau_m+1)} \\
&= \quad \frac{2\pi i}{\alpha} \cdot \zeta_c^{-b} \cdot e^{-(2n+1)\pi i \frac{a}{c} + 3\pi i\left(m+\frac{b}{c}\right)\left(2\frac{a}{c}-1\right)} \cdot q^{-\frac{3a}{2c}\left(1-\frac{a}{c}\right)} \cdot q_1^{2(2n+1)\left(\frac{b}{c}+m\right) - 6\left(\frac{b}{c}+m\right)^2}.
\end{aligned}$$

Hence the Residue Theorem, combined with a reordering of summation, implies that

$$I_2 = \sum_{m \geq 0} \sum_{n \geq 3m+k(b,c)} \lambda_{n,m} + \sum_{n \geq 0} J_n', \tag{2.14}$$

where

$$J_n' := \int_{-\infty+\omega_n}^{\infty+\omega_n} \frac{e^{(2n+1)\pi i\tau - \alpha\left(\tau+\frac{a}{c}\right) - \frac{3}{2}\alpha\tau(\tau+1)}}{1 - \zeta_c^b \cdot e^{-\alpha\left(\tau+\frac{a}{c}\right)}}\, d\tau.$$

Using the fact that

$$\lambda_{n+1,m} = e^{-2\pi i\frac{a}{c}} \cdot q_1^{4\left(m+\frac{b}{c}\right)} \cdot \lambda_{n,m},$$

we find that

(2.15)
$$\sum_{m\geq 0}\sum_{n\geq 3m+k(b,c)}\lambda_{n,m}=\sum_{m\geq 0}\frac{\lambda_{3m+k(b,c),m}}{1-e^{-2\pi i\frac{a}{c}}\cdot q_1^{4\left(m+\frac{b}{c}\right)}}$$
$$=\frac{2\pi i}{\alpha}\cdot e^{-(2k(b,c)+1)\frac{a}{c}\pi i-3\pi i\left(1-2\frac{a}{c}\right)\frac{b}{c}}\cdot\zeta_c^{-b}\cdot q^{-\frac{3a}{2c}\left(1-\frac{a}{c}\right)}\cdot q_1^{2(2k(b,c)+1)\frac{b}{c}-\frac{6b^2}{c^2}}$$
$$\times\sum_{m=0}^{\infty}\frac{(-1)^m q_1^{6m^2+2(2k(b,c)+1)m}}{1-e^{-2\pi i\frac{a}{c}}\cdot q_1^{4\left(m+\frac{b}{c}\right)}}.$$

Now we compute the integral I_1 by arguing as above using the identity
$$\frac{1}{\sin(\pi\tau)}=2i\sum_{n=0}^{\infty}e^{-(2n+1)\pi i\tau},$$
which holds for $-\tau\in\mathbb{H}$. Again by the Residue Theorem, we find that
$$I_1=\sum_{m\geq 1}\sum_{n\geq 3m-k(b,c)}\mu_{n,m}+\sum_{n\geq 0}K_n',\tag{2.16}$$
where
$$K_n':=\int_{-\infty+\widetilde{\omega}_n}^{\infty+\widetilde{\omega}_n}\frac{e^{(2n+1)\pi iz-\alpha\left(z+\frac{a}{c}\right)-\frac{3}{2}\alpha z(z+1)}}{1-\zeta_c^b\cdot e^{-\alpha\left(z+\frac{a}{c}\right)}}\,dz.$$
Here the points $\widetilde{\omega}_n$ are given by
$$\widetilde{\omega}_n:=-\frac{(2n+1)\pi i}{3\alpha},$$
and
$$\begin{aligned}\mu_{n,m}&:=\frac{2\pi i}{\alpha}\cdot e^{-(2n+1)\pi i\tau_{-m}-\alpha\left(\tau_{-m}+\frac{a}{c}\right)-\frac{3}{2}\alpha\tau_{-m}(\tau_{-m}+1)}\\&=\frac{2\pi i}{\alpha}\cdot\zeta_c^{-b}\cdot e^{(2n+1)\pi i\frac{a}{c}+3\pi i\left(-m+\frac{b}{c}\right)\left(2\frac{a}{c}-1\right)}\cdot q^{-\frac{3a}{2c}\left(1-\frac{a}{c}\right)}\cdot q_1^{2(2n+1)\left(-\frac{b}{c}+m\right)-6\left(\frac{b}{c}-m\right)^2}.\end{aligned}$$
As in the case of I_2, we obtain
$$\sum_{m\geq 1}\sum_{n\geq 3m-k(b,c)}\mu_{n,m}=\frac{2\pi i}{\alpha}e^{(-2k(b,c)+1)\frac{a}{c}\pi i-3\pi i\left(1-2\frac{a}{c}\right)\frac{b}{c}}\cdot\zeta_c^{-b}\cdot q^{-\frac{3a}{2c}\left(1-\frac{a}{c}\right)}\cdot q_1^{2(2k(b,c)-1)\frac{b}{c}-\frac{6b^2}{c^2}}$$
$$\times\sum_{m=1}^{\infty}\frac{(-1)^m q_1^{6m^2+2(-2k(b,c)+1)m}}{1-e^{2\pi i\frac{a}{c}}\cdot q_1^{4\left(m-\frac{b}{c}\right)}}.$$

This fact, combined with (2.14), (2.15), and (2.16), implies that

$$I_1+I_2=\frac{4\pi}{\alpha}e^{-2k(b,c)\frac{a}{c}\pi i+3\pi i\left(-1+2\frac{a}{c}\right)\frac{b}{c}}\cdot\zeta_c^{-b}\cdot q^{-\frac{3a}{2c}\left(1-\frac{a}{c}\right)}\cdot q_1^{4k(b,c)\frac{b}{c}-\frac{6b^2}{c^2}}$$
$$\times\left(\frac{ie^{-\pi i\frac{a}{c}}\cdot q_1^{\frac{2b}{c}}}{2\cdot\left(1-e^{-2\pi i\frac{a}{c}}\cdot q_1^{\frac{4b}{c}}\right)}+\sum_{m=1}^{\infty}\widetilde{K}(a,b,c,m;q_1)\cdot q_1^{6m^2+2m}\right)+\sum_{n\geq0}\left(J_n'+K_n'\right),$$

where

$$\widetilde{K}(a,b,c,m;q_1)$$
$$:=(-1)^m\frac{\sin\left(\frac{\pi a}{c}-i\beta\left(\frac{2b}{c}+4k(b,c)m\right)\right)+\sin\left(\frac{\pi a}{c}-i\beta\left(\frac{2b}{c}-4k(b,c)m\right)\right)q_1^{4m}}{1-2\cos\left(2\pi\frac{a}{c}-4i\frac{b}{c}\beta\right)\cdot q_1^{4m}+q_1^{8m}}.$$

By (2.8), we then find that

(2.17)
$$I_1+I_2=\frac{4\pi}{\alpha}e^{-2k(b,c)\frac{a}{c}\pi i+3\pi i\left(-1+2\frac{a}{c}\right)\frac{b}{c}}\zeta_c^{-b}q^{-\frac{3a}{2c}\left(1-\frac{a}{c}\right)}q_1^{4k(b,c)\frac{b}{c}-\frac{6b^2}{c^2}}\left(q_1^4;q_1^4\right)_\infty N\left(a,b,c;q_1^4\right)$$
$$+\sum_{n\geq0}\left(J_n'+K_n'\right).$$

Hence the proof of the theorem essentially boils down to the computation of

$$\sum_{n\geq0}\left(J_n'+K_n'\right).$$

We first compute the J_n' integrals. For this we need the identity

$$\frac{t}{1-t}=\frac{t^{-\frac{1}{2}}+t^{\frac{1}{2}}+t^{\frac{3}{2}}}{t^{-\frac{3}{2}}-t^{\frac{3}{2}}},$$

which we apply when $t=\zeta_c^b\cdot e^{-\alpha\left(\tau+\frac{a}{c}\right)}$. This identity implies that the integrand in J_n' equals

$$\zeta_{2c}^{-5b}\cdot e^{(2n+1)\pi i\tau+\frac{3}{2}\alpha\frac{a}{c}-\frac{3}{2}\alpha\tau^2}\frac{\left(\zeta_c^b\cdot e^{-\alpha\left(\tau+\frac{a}{c}\right)}+\zeta_c^{2b}\cdot e^{-2\alpha\left(\tau+\frac{a}{c}\right)}+\zeta_c^{3b}\cdot e^{-3\alpha\left(\tau+\frac{a}{c}\right)}\right)}{\left(\zeta_{2c}^{-3b}\cdot e^{\frac{3}{2}\alpha\left(\tau+\frac{a}{c}\right)}-\zeta_{2c}^{3b}\cdot e^{-\frac{3}{2}\alpha\left(\tau+\frac{a}{c}\right)}\right)}.$$

In the integrand we now put $\tau=-\frac{a}{c}+p+x$, where

$$p:=\frac{(2n+1)\pi i}{3\alpha},$$

and where x is a real variable running from $-\infty$ to ∞. This easily gives

$$J_n' = \frac{(-1)^{n+1}i}{2} \cdot \zeta_{2c}^{-5b} \cdot q_1^{\frac{(2n+1)^2}{6}} \cdot q^{-\frac{3a}{2c}\left(1-\frac{a}{c}\right)}$$
$$\int_{\mathbb{R}} \frac{\left(\zeta_c^b \cdot e^{-\frac{(2n+1)\pi i}{3}} e^{-\alpha x} + \zeta_c^{2b} \cdot e^{-\frac{2(2n+1)\pi i}{3}} e^{-2\alpha x} - \zeta_c^{3b} e^{-3\alpha x}\right)}{\cosh\left(\frac{3}{2}\alpha x - 3\pi i \frac{b}{c}\right)} \cdot e^{-\frac{3}{2}\alpha x^2 + 3\alpha \frac{a}{c} x}\, dx.$$

In the same way, we obtain

$$K_n' = \frac{(-1)^n i}{2} \cdot \zeta_{2c}^{-5b} q_1^{\frac{(2n+1)^2}{6}} q^{-\frac{3a}{2c}\left(1-\frac{a}{c}\right)}$$
$$\int_{\mathbb{R}} \frac{\left(\zeta_c^b \cdot e^{\frac{(2n+1)\pi i}{3}} e^{-\alpha x} + \zeta_c^{2b} \cdot e^{\frac{2(2n+1)\pi i}{3}} e^{-2\alpha x} - \zeta_c^{3b} e^{-3\alpha x}\right)}{\cosh\left(\frac{3}{2}\alpha x - 3\pi i \frac{b}{c}\right)} \cdot e^{-\frac{3}{2}\alpha x^2 + 3\alpha \frac{a}{c} x}\, dx$$

since we have that

$$\sin\left(\frac{(2n+1)\pi}{3}\right) = \sin\left(\frac{2(2n+1)\pi}{3}\right),$$

for every integer n we obtain the expression

$$\begin{aligned} J_n' + K_n' &= (-1)^{n+1} \cdot \zeta_{2c}^{-5b} \cdot q_1^{\frac{(2n+1)^2}{6}} \cdot q^{-\frac{3a}{2c}\left(1-\frac{a}{c}\right)} \cdot \sin\left(\frac{(2n+1)\pi}{3}\right) \\ &\qquad \int_{\mathbb{R}} \frac{\left(\zeta_c^b \cdot e^{-\alpha x} + \zeta_c^{2b} \cdot e^{-2\alpha x}\right)}{\cosh\left(\frac{3}{2}\alpha x - 3\pi i \frac{b}{c}\right)} e^{-\frac{3}{2}\alpha x^2 + 3\alpha \frac{a}{c} x}\, dx \\ &= (-1)^{n+1} \cdot \zeta_{2c}^{-5b} \cdot q_1^{\frac{(2n+1)^2}{6}} \cdot q^{-\frac{3a}{2c}\left(1-\frac{a}{c}\right)} \cdot \sin\left(\frac{(2n+1)\pi}{3}\right) J(a,b,c;\alpha). \end{aligned}$$

Now by Euler's identity (see page 464 of [34])

$$2\sum_{n=0}^{\infty} (-1)^n \sin\left(\frac{(2n+1)\pi}{3}\right) q_1^{\frac{(2n+1)^2}{6}} = \sqrt{3} \cdot q_1^{\frac{1}{6}} \cdot \left(q_1^4; q_1^4\right)_\infty,$$

we find that

$$\sum_{n=0}^{\infty} \left(J_n' + K_n'\right) = -\frac{\sqrt{3}}{2} \cdot \zeta_{2c}^{-5b} \cdot q_1^{\frac{1}{6}} \cdot q^{-\frac{3}{2}\frac{a}{c}\left(1-\frac{a}{c}\right)} \left(q_1^4; q_1^4\right)_\infty J(a,b,c;\alpha).$$

This fact, combined with (2.11), (2.12), and (2.17) then gives

$$\begin{aligned}(q;q)_\infty M(a,b,c;q)\\ &= \frac{4\pi}{\alpha}\cdot e^{-2k(b,c)\frac{a}{c}\pi i+3\pi i\left(-1+2\frac{a}{c}\right)\frac{b}{c}}\cdot\zeta_c^{-b}\cdot q^{-\frac{3a}{2c}\left(1-\frac{a}{c}\right)}\cdot q_1^{4k(b,c)\frac{b}{c}-\frac{6b^2}{c^2}}\cdot\left(q_1^4;q_1^4\right)_\infty\cdot N\left(a,b,c;q_1^4\right)\\ &\qquad -\frac{\sqrt{3}}{2}\cdot\zeta_{2c}^{-5b}\cdot q_1^{\frac{1}{6}}\cdot q^{-\frac{3}{2}\frac{a}{c}\left(1-\frac{a}{c}\right)}\cdot\left(q_1^4;q_1^4\right)_\infty J(a,b,c;\alpha).\end{aligned}$$

By the transformation law for Dedekind's eta-function, it is straightforward to deduce that

$$(q;q)_\infty = \sqrt{\frac{2\pi}{\alpha}}\cdot q^{-\frac{1}{24}}\cdot q_1^{\frac{1}{6}}\left(q_1^4;q_1^4\right)_\infty,$$

from which the statement of the theorem follows easily. □

2.3. **An infinite family of vector valued Maass forms.** It turns out that the transformations in Theorems 2.1 and 2.2 allow us to produce an infinite family of vector valued weight 1/2 weak Maass forms, one for every positive odd integer c. To this end, it suffices to determine the images of the components of these forms under the generators of $\mathrm{SL}_2(\mathbb{Z})$:

$$z\mapsto z+1 \quad\text{and}\quad z\mapsto -\frac{1}{z}.$$

If c is a positive odd integer, then for every pair of integers $0\le a,b<c$ define the functions

$$\mathcal{N}\left(\frac{a}{c};q\right) = \mathcal{N}\left(\frac{a}{c};z\right) := \csc\left(\frac{a\pi}{c}\right)\cdot q^{-\frac{1}{24}}\cdot N\left(\frac{a}{c};q\right), \tag{2.18}$$

$$\mathcal{M}\left(\frac{a}{c};q\right) = \mathcal{M}\left(\frac{a}{c};z\right) := 2q^{\frac{3a}{2c}\cdot\left(1-\frac{a}{c}\right)-\frac{1}{24}}\cdot M\left(\frac{a}{c};q\right), \tag{2.19}$$

$$\mathcal{M}(a,b,c;q) = \mathcal{M}(a,b,c;z) := 2q^{\frac{3a}{2c}\cdot\left(1-\frac{a}{c}\right)-\frac{1}{24}}\cdot M(a,b,c;q), \tag{2.20}$$

$$\begin{aligned}\mathcal{N}(a,b,c;q) &= \mathcal{N}(a,b,c;z)\\ &:= 4e^{-2\pi i\frac{a}{c}k(b,c)+3\pi i\frac{b}{c}\left(\frac{2a}{c}-1\right)}\cdot\zeta_c^{-b}\cdot q^{\frac{b}{c}k(b,c)-\frac{3b^2}{2c^2}-\frac{1}{24}}\cdot N(a,b,c;q).\end{aligned} \tag{2.21}$$

Remark. Notice that a must be non-zero for the function $\mathcal{N}\left(\frac{a}{c};q\right)$.

Theorem 2.3. *Suppose that c is a positive odd integer, and that a and b are integers for which $0\le a<c$ and $0<b<c$.*

(1) *For all $z \in \mathbb{H}$ we have*

$$\begin{aligned}
\mathcal{N}\left(\frac{a}{c}; z+1\right) &= \zeta_{24}^{-1} \cdot \mathcal{N}\left(\frac{a}{c}; z\right), \\
\mathcal{N}(a,b,c; z+1) &= \zeta_{2c^2}^{3b^2} \cdot \zeta_{24}^{-1} \cdot \mathcal{N}(a-b,b,c;z), \\
\mathcal{M}\left(\frac{a}{c}; z+1\right) &= \zeta_{2c}^{5a} \cdot \zeta_{2c^2}^{-3a^2} \cdot \zeta_{24}^{-1} \cdot \mathcal{M}(a,a,c;z), \\
\mathcal{M}(a,b,c; z+1) &= \zeta_{2c}^{5a} \cdot \zeta_{2c^2}^{-3a^2} \cdot \zeta_{24}^{-1} \cdot \mathcal{M}(a,a+b,c;z),
\end{aligned}$$

where a is required to be non-zero in the first and third formula.

(2) *For all $z \in \mathbb{H}$ we have*

$$\begin{aligned}
\frac{1}{\sqrt{-iz}} \cdot \mathcal{N}\left(\frac{a}{c}; -\frac{1}{z}\right) &= \mathcal{M}\left(\frac{a}{c}; z\right) + 2\sqrt{3}\sqrt{-iz} \cdot J\left(\frac{a}{c}; -2\pi i z\right), \\
\frac{1}{\sqrt{-iz}} \cdot \mathcal{N}\left(a,b,c; -\frac{1}{z}\right) &= \mathcal{M}(a,b,c;z) + \zeta_{2c}^{-5b}\sqrt{3}\sqrt{-iz} \cdot J(a,b,c;-2\pi i z), \\
\frac{1}{\sqrt{-iz}} \cdot \mathcal{M}\left(\frac{a}{c}; -\frac{1}{z}\right) &= \mathcal{N}\left(\frac{a}{c}; z\right) - \frac{2\sqrt{3}i}{z} \cdot J\left(\frac{a}{c}; \frac{2\pi i}{z}\right), \\
\frac{1}{\sqrt{-iz}} \cdot \mathcal{M}\left(a,b,c; -\frac{1}{z}\right) &= \mathcal{N}(a,b,c;z) - \zeta_{2c}^{-5b}\frac{\sqrt{3}i}{z} \cdot J\left(a,b,c; \frac{2\pi i}{z}\right),
\end{aligned}$$

where a is required to be non-zero in the first and third formula.

Remark. Strictly speaking, the functions in Theorem 2.3 do not always have the property that their defining parameters lie in the interval $[0, c)$. For example, this occurs whenever $a - b$ (resp. $a + b$) is not in the interval $[0, c)$. In such cases, one defines the corresponding functions in the obvious way, and then observes that the resulting functions equal, up to a precise root of unity, the corresponding functions where $a - b$ (resp. $a + b$) are replaced by their reduced residue classes modulo c. Lastly, the reader should recall the first remark after Theorem 2.2.

Proof of Theorem 2.3. The first claim follows from the definitions of the series. The second claim follows from Theorems 2.1 and 2.2 by letting $\alpha = -2\pi i z$ and $\frac{2\pi i}{z}$. □

3. Weak Maass forms

Here we prove Theorems 1.1 and 3.4 using the results from the previous section. In Section 3.1, we explicitly construct the non-holomorphic and holomorphic parts of the functions $D\left(\frac{a}{c}; z\right)$, we derive their images under the generators of Γ_c, and we prove Theorem 1.1.

3.1. **The Non-holomorphic and holomorphic parts of $D\left(\frac{a}{c};z\right)$.** Using Theorem 2.1, here we construct a weak Maass form of weight 1/2 using $N\left(\frac{a}{c};q\right)$. The arguments we employ are analogous to those employed by Zwegers in his work on Ramanujan's mock theta functions (for example, see Section 3 of [35], or [36]).

We begin with the transformation formulas for the relevant series. As in the introduction, suppose that $0<a<c$ are integers. Define the vector valued function $F\left(\frac{a}{c};z\right)$ by

$$\begin{aligned} F\left(\frac{a}{c};z\right) &:= \left(F_1\left(\frac{a}{c};z\right), F_2\left(\frac{a}{c};z\right)\right)^T \\ &= \left(\sin\left(\frac{\pi a}{c}\right)\mathcal{N}\left(\frac{a}{c};\ell_c z\right), \sin\left(\frac{\pi a}{c}\right)\mathcal{M}\left(\frac{a}{c};\ell_c z\right)\right)^T, \end{aligned} \tag{3.1}$$

where $\ell_c := \mathrm{lcm}(2c^2, 24)$, as defined in the introduction. Similarly, define the vector valued (non-holomorphic) function $G\left(\frac{a}{c};z\right)$ by

$$\begin{aligned} G\left(\frac{a}{c};z\right) &= \left(G_1\left(\frac{a}{c};z\right), G_2\left(\frac{a}{c};z\right)\right)^T \\ &:= \left(2\sqrt{3}\sin\left(\frac{\pi a}{c}\right)\sqrt{-i\ell_c z}\cdot J\left(\frac{a}{c};-2\pi i\ell_c z\right), \frac{2\sqrt{3}\sin\left(\frac{\pi a}{c}\right)}{i\ell_c z}\cdot J\left(\frac{a}{c};\frac{2\pi i}{\ell_c z}\right)\right)^T. \end{aligned} \tag{3.2}$$

The transformations in Theorem 2.1 imply that these two vector valued functions are intertwined by the generators of Γ_c.

Lemma 3.1. *Assume the notation and hypotheses above. For $z\in\mathbb{H}$, we have*

$$\begin{aligned} F\left(\frac{a}{c};z+1\right) &= F\left(\frac{a}{c};z\right), \\ \frac{1}{\sqrt{-i\ell_c z}}\cdot F\left(\frac{a}{c};-\frac{1}{{\ell_c}^2 z}\right) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot F\left(\frac{a}{c};z\right) + G\left(\frac{a}{c};z\right). \end{aligned}$$

Proof. The first transformation law follows from the simple fact that both components of $F\left(\frac{a}{c};z\right)$ are given as series in q with integer exponents. The second transformation follows from Theorem 2.3. ☐

The Mordell vector $G\left(\frac{a}{c};z\right)$ appearing in Lemma 3.1 may be interpreted in terms of period integrals of the theta function $\Theta\left(\frac{a}{c};\tau\right)$. The next lemma makes this precise.

Lemma 3.2. *Assume the notation and hypotheses above. For $z\in\mathbb{H}$, we have*

$$G\left(\frac{a}{c};z\right) = \frac{i\ell_c^{\frac{1}{2}}\sin\left(\frac{\pi a}{c}\right)}{\sqrt{3}}\int_0^{i\infty}\frac{\left((-i\ell_c\tau)^{-\frac{3}{2}}\Theta\left(\frac{a}{c};-\frac{1}{\ell_c\tau}\right), \Theta\left(\frac{a}{c};\ell_c\tau\right),\right)^T}{\sqrt{-i(\tau+z)}}\,d\tau.$$

Proof. For brevity, we only prove the asserted formula for the first component of $G\left(\frac{a}{c};z\right)$. The proof of the second component follows in the same way.

By analytic continuation, we may assume that $z = it$ with $t > 0$. By a change of variables, using (2.5), we find that

$$J\left(\frac{a}{c};\frac{2\pi}{\ell_c t}\right) = \ell_c t \cdot \int_0^\infty e^{-3\ell_c \pi t x^2} \cdot \frac{\cosh\left(\left(\frac{3a}{c}-2\right)2\pi x\right)+\cosh\left(\left(\frac{3a}{c}-1\right)2\pi x\right)}{\cosh(3\pi x)}\,dx.$$

Using the Mittag-Leffler theory of partial fraction decompositions (see e.g. [34] pages 134-136), a direct calculation shows that

$$\frac{\cosh\left(\left(\frac{3a}{c}-2\right)2\pi x\right)+\cosh\left(\left(\frac{3a}{c}-1\right)2\pi x\right)}{\cosh(3\pi x)}$$
$$= \frac{-i}{\sqrt{3}\pi}\sum_{n\in\mathbb{Z}}\frac{(-1)^n\sin\left(\frac{\pi a(6n+1)}{c}\right)}{x-i\left(n+\frac{1}{6}\right)} - \frac{i}{\sqrt{3}\pi}\sum_{n\in\mathbb{Z}}\frac{(-1)^n\sin\left(\frac{\pi a(6n+1)}{c}\right)}{-x-i\left(n+\frac{1}{6}\right)}.$$

By introducing the extra term $\frac{1}{i\left(n+\frac{1}{6}\right)}$, we just have to consider

$$\int_{-\infty}^\infty e^{-3\pi\ell_c t x^2}\sum_{n\in\mathbb{Z}}(-1)^n\sin\left(\frac{\pi a(6n+1)}{c}\right)\left(\frac{1}{x-i\left(n+\frac{1}{6}\right)}+\frac{1}{i\left(n+\frac{1}{6}\right)}\right)\,dx.$$

Since this expression is absolutely convergent, thanks to Lebesgue's Theorem of dominated convergence, we may interchange summation and integration to obtain

$$J\left(\frac{a}{c};\frac{2\pi}{\ell_c t}\right) = \frac{-\ell_c i t}{\sqrt{3}\pi}\sum_{n\in\mathbb{Z}}(-1)^n\sin\left(\frac{\pi a(6n+1)}{c}\right)\int_{-\infty}^\infty\frac{e^{-3\pi\ell_c t x^2}}{x-i\left(n+\frac{1}{6}\right)}\,dx.$$

For all $s\in\mathbb{R}\setminus\{0\}$, we have the identity

$$\int_{-\infty}^\infty\frac{e^{-\pi t x^2}}{x-is}\,dx = \pi i s\int_0^\infty\frac{e^{-\pi u s^2}}{\sqrt{u+t}}\,du$$

(this follows since both sides are solutions of $\left(-\frac{\partial}{\partial t}+\pi s^2\right)f(t) = \frac{\pi i s}{\sqrt{t}}$ and have the same limit 0 as $t\mapsto\infty$ and hence are equal). Hence we may conclude that

$$J\left(\frac{a}{c};\frac{2\pi}{\ell_c t}\right) = \frac{\ell_c t}{6\sqrt{3}}\sum_{n\in\mathbb{Z}}(-1)^n(6n+1)\sin\left(\frac{\pi a(6n+1)}{c}\right)\int_0^\infty\frac{e^{-\pi(n+1/6)^2 u}}{\sqrt{u+3\ell_c t}}\,du.$$

Substituting $u = -3\ell_c i\tau$, and interchanging summation and integration (which is allowed by Lebesgue's Theorem of dominated convergence) gives

$$J\left(\frac{a}{c};\frac{2\pi}{\ell_c t}\right) = \frac{-it\ell_c^{\frac{3}{2}}}{6}\int_0^{i\infty}\frac{\sum_{n\in\mathbb{Z}}(-1)^n(6n+1)\sin\left(\frac{\pi a(6n+1)}{c}\right)e^{3\pi i\ell_c\tau\left(n+\frac{1}{6}\right)^2}}{\sqrt{-i(\tau+it)}}\,d\tau.$$

Now the claim follows since one can easily see that the sum over n coincides with definition (1.7). ☐

To prove Theorem 1.1, we must determine the necessary modular transformation properties of the vector

$$S\left(\frac{a}{c};z\right)=\left(S_1\left(\frac{a}{c};z\right),S_2\left(\frac{a}{c};z\right)\right)$$
$$:=\frac{-i\sin\left(\frac{\pi a}{c}\right)\ell_c^{\frac{1}{2}}}{\sqrt{3}}\int_{-\bar{z}}^{i\infty}\frac{\left(\Theta\left(\frac{a}{c};\ell_c\tau\right),(-i\ell_c\tau)^{-\frac{3}{2}}\Theta\left(\frac{a}{c};-\frac{1}{\ell_c\tau}\right)\right)^T}{\sqrt{-i(\tau+z)}}\,d\tau. \tag{3.3}$$

Since $\Theta\left(\frac{a}{c};\ell_c z\right)$ is a cusp form, the integral above is absolutely convergent. The next lemma shows that $S\left(\frac{a}{c};z\right)$ satisfies the same transformations as $F\left(\frac{a}{c};z\right)$.

Lemma 3.3. *Assume the notation and hypotheses above. For $z\in\mathbb{H}$, we have*

$$\begin{aligned}S\left(\frac{a}{c};z+1\right)&=S\left(\frac{a}{c};z\right),\\ \frac{1}{\sqrt{-i\ell_c z}}\cdot S\left(\frac{a}{c};-\frac{1}{\ell_c{}^2z}\right)&=\begin{pmatrix}0&1\\1&0\end{pmatrix}\cdot S\left(\frac{a}{c};z\right)+G\left(\frac{a}{c};z\right).\end{aligned}$$

Proof. Using the Fourier expansion of $\Theta\left(\frac{a}{c};z\right)$, one easily sees that

$$S_1\left(\frac{a}{c};z+1\right)=S_1\left(\frac{a}{c};z\right).$$

Using classical facts about theta functions (for example, see equations (2.4) and (2.5) of [30]), we also have that

$$S_2\left(\frac{a}{c};z+1\right)=S_2\left(\frac{a}{c};z\right).$$

Hence, it suffices to prove the second transformation law. We directly compute

$$\frac{1}{\sqrt{-i\ell_c z}}\cdot S\left(\frac{a}{c};-\frac{1}{\ell_c{}^2z}\right)$$
$$=\frac{i\sin\left(\frac{\pi a}{c}\right)\ell_c^{\frac{1}{2}}}{\sqrt{3}\sqrt{-i\ell_c z}}\int_{\frac{1}{\ell_c{}^2\bar{z}}}^{i\infty}\frac{\left(\Theta\left(\frac{a}{c};\ell_c\tau\right),(-i\ell_c\tau)^{-\frac{3}{2}}\Theta\left(\frac{a}{c};-\frac{1}{\ell_c\tau}\right)\right)^T}{\sqrt{-i\left(\tau-\frac{1}{\ell_c{}^2z}\right)}}\,d\tau.$$

By making the change of variable $\tau \mapsto -\frac{1}{{\ell_c}^2\tau}$, we obtain

$$\frac{1}{\sqrt{-i\ell_c z}} \cdot S\left(\frac{a}{c}; -\frac{1}{{\ell_c}^2 z}\right) = \frac{i\sin\left(\frac{\pi a}{c}\right){\ell_c}^{\frac{1}{2}}}{\sqrt{3}} \int_0^{-\bar{z}} \frac{\left((-i\ell_c\tau)^{-\frac{3}{2}}\Theta\left(\frac{a}{c}; -\frac{1}{\ell_c\tau}\right), \Theta\left(\frac{a}{c}, \ell_c\tau\right)\right)^T}{\sqrt{-i\left(\tau+z\right)}}\, d\tau.$$

Consequently, we obtain the desired conclusion

$$\frac{1}{\sqrt{-i\ell_c z}} \cdot S\left(\frac{a}{c}; -\frac{1}{{\ell_c}^2 z}\right) - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot S\left(\frac{a}{c}; z\right) = \frac{i\sin\left(\frac{\pi a}{c}\right){\ell_c}^{\frac{1}{2}}}{\sqrt{3}} \int_0^{i\infty} \frac{\left((-i\ell_c\tau)^{-\frac{3}{2}}\Theta\left(\frac{a}{c}; -\frac{1}{\ell_c\tau}\right), \Theta\left(\frac{a}{c}; \ell_c\tau\right)\right)^T}{\sqrt{-i\left(\tau+z\right)}}\, d\tau = G\left(\frac{a}{c}; z\right).$$

□

Proof of Theorem 1.1. Using (2.1), (2.4), (2.18), and (3.1), we find that we have already determined the transformation laws satisfied by $D\left(\frac{a}{c}; z\right)$. Since we have

$$\begin{pmatrix} 1 & 0 \\ {\ell_c}^2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -{\ell_c}^2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{{\ell_c}^2} \\ 1 & 0 \end{pmatrix},$$

where the first and third matrices on the right provide the same Möbius transformation on $\mathbb{H}$, the transformation laws for $D\left(\frac{a}{c}; z\right)$ follow from Lemma 3.1 and Lemma 3.3.

Now we show that $D\left(\frac{a}{c}; z\right)$ is annihilated by

$$\Delta_{\frac{1}{2}} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{iy}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = -4y^{\frac{3}{2}}\frac{\partial}{\partial z}\sqrt{y}\frac{\partial}{\partial \bar{z}}.$$

Since $q^{-\frac{\ell_c}{24}}R(\zeta_b^a; q^{\ell_c})$ is a holomorphic function in z, we get

$$\frac{\partial}{\partial \bar{z}}\left(D\left(\frac{a}{c}; z\right)\right) = -\frac{\partial}{\partial \bar{z}}\left(S_1\left(\frac{a}{c}; z\right)\right) = \frac{\sin\left(\frac{\pi a}{c}\right)}{\sqrt{6y}} \cdot \Theta\left(\frac{a}{c}; -\ell_c\bar{z}\right).$$

Hence, we find that $\sqrt{y}\frac{\partial}{\partial \bar{z}}\left(D\left(\frac{a}{c}; z\right)\right)$ is anti-holomorphic, and so

$$\frac{\partial}{\partial z}\sqrt{y}\frac{\partial}{\partial \bar{z}}\left(D\left(\frac{a}{c}; z\right)\right) = 0.$$

To complete the proof, it suffices to show that $D\left(\frac{a}{c}; z\right)$ has at most linear exponential growth at cusps. The period integral $S_1\left(\frac{a}{c}; z\right)$ is convergent since $\Theta\left(\frac{a}{c}; \ell_c\tau\right)$ is a weight 3/2 cusp form (for example, see Section 2 of [30]). This fact, combined with the transformation laws in Theorems 1.1 and 1.2, allow us to conclude that $D\left(\frac{a}{c}; z\right)$ has at most linear exponential growth at cusps. □

3.2. **Vector valued weak Maass forms of weight 1/2.** Theorem 1.1 is a hint of a more general modular transformation law which holds for larger groups than Γ_c. Using Theorem 2.3, here we produce an infinite family of vector valued weak Maass forms for $\mathrm{SL}_2(\mathbb{Z})$.

Suppose that c is a positive odd integer. For integers $0 \leq a < c$ and $0 < b < c$, define the functions

$$T_1\left(\frac{a}{c};z\right) := -\frac{i}{\sqrt{3}}\int_{-\bar{z}}^{i\infty} \frac{\Theta\left(\frac{a}{c};\tau\right)}{\sqrt{-i(\tau+z)}}\,d\tau, \tag{3.4}$$

$$T_2\left(\frac{a}{c};z\right) := -\frac{i}{\sqrt{3}}\int_{-\bar{z}}^{i\infty} \frac{(-i\tau)^{-\frac{3}{2}}\Theta\left(\frac{a}{c};-\frac{1}{\tau}\right)}{\sqrt{-i(\tau+z)}}\,d\tau, \tag{3.5}$$

$$T_1\left(a,b,c;z\right) := -\frac{\zeta_{2c}^{-5b}}{2\sqrt{3}}\int_{-\bar{z}}^{i\infty} \frac{\Theta\left(a,b,c;\tau\right)}{\sqrt{-i(\tau+z)}}\,d\tau, \tag{3.6}$$

$$T_2\left(a,b,c;z\right) := -\frac{\zeta_{2c}^{-5b}}{2\sqrt{3}}\int_{-\bar{z}}^{i\infty} \frac{(-i\tau)^{-\frac{3}{2}}\Theta\left(a,b,c;-\frac{1}{\tau}\right)}{\sqrt{-i(\tau+z)}}\,d\tau. \tag{3.7}$$

If we let $t_c := \mathrm{lcm}(c,6)$, then define $\Theta(a,b,c;\tau)$ by

$$\Theta(a,b,c;\tau) := \sum_{m \pmod{t_c}} (-1)^m \sin\left(\frac{\pi}{3}(2m+1)\right) e^{2\pi i m\frac{a}{c}} \cdot \theta\left(2cm+6b+c, 2ct_c; \frac{\tau}{24c^2}\right). \tag{3.8}$$

Recall that the theta functions $\theta(\alpha,\beta;\tau)$ are defined by (1.8). Using this notation, define the following functions

$$\mathcal{G}_1\left(\frac{a}{c};z\right) := \mathcal{N}\left(\frac{a}{c};z\right) - T_1\left(\frac{a}{c};z\right), \tag{3.9}$$

$$\mathcal{G}_2\left(\frac{a}{c};z\right) := \mathcal{M}\left(\frac{a}{c};z\right) - T_2\left(\frac{a}{c};z\right), \tag{3.10}$$

$$\mathcal{G}_1\left(a,b,c;z\right) := \mathcal{N}\left(a,b,c;z\right) - T_1\left(a,b,c;z\right), \tag{3.11}$$

$$\mathcal{G}_2\left(a,b,c;z\right) := \mathcal{M}\left(a,b,c;z\right) - T_2\left(a,b,c;z\right). \tag{3.12}$$

These functions constitute a vector valued weak Maass form of weight 1/2. Here we recall this notion more precisely. A *vector valued weak Maass form of weight k for* $\mathrm{SL}_2(\mathbb{Z})$ is any finite set of smooth functions, say $v_1(z),\dots,v_m(z) : \mathbb{H} \to \mathbb{C}$, which satisfy the following:

(1) If $1 \leq n_1 \leq m$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, then there is a root of unity $\epsilon(A,n_1)$ and an index $1 \leq n_2 \leq m$ for which
$$v_{n_1}(Az) = \epsilon(A,n_1)(cz+d)^k v_{n_2}(z)$$
for all $z \in \mathbb{H}$.

(2) For each $1 \leq n \leq m$ we have that $\Delta_k v_n = 0$.

If c is a positive odd integer, then let V_c be the "vector" of functions defined by

$$\begin{aligned} V_c := &\left\{ \mathcal{G}_1\left(\frac{a}{c}; z\right), \mathcal{G}_2\left(\frac{a}{c}; z\right) \ : \ \text{with } 0 < a < c \right\} \\ &\bigcup \left\{ \mathcal{G}_1(a,b,c;z), \mathcal{G}_2(a,b,c;z) \ : \ (a,b) \text{ with } 0 \leq a < c \text{ and } 0 < b < c \right\}. \end{aligned}$$

Theorem 3.4. *Assume the notation above. If c is a positive odd integer, then V_c is a vector valued weak Maass form of weight $1/2$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$.*

Sketch of the proof. The proof of Theorem 3.4 follows along the lines of the proof of Theorem 1.1. Therefore, for brevity here we simply provide a sketch of the proof and make key observations.

As in the proof of Lemma 3.2, one first shows that

$$\begin{aligned} \frac{2\sqrt{3}}{iz} \cdot J\left(\frac{a}{c}; \frac{2\pi i}{z}\right) &= \frac{i}{\sqrt{3}} \int_0^{i\infty} \frac{\Theta\left(\frac{a}{c}; \tau\right)}{\sqrt{-i(\tau+z)}}\, d\tau, \\ 2\sqrt{3}\sqrt{-iz} \cdot J\left(\frac{a}{c}; -2\pi i z\right) &= \frac{i}{\sqrt{3}} \int_0^{i\infty} \frac{(-i\tau)^{-\frac{3}{2}} \Theta\left(\frac{a}{c}; -\frac{1}{\tau}\right)}{\sqrt{-i(\tau+z)}}\, d\tau, \\ \frac{\zeta_{2c}^{-5b}\sqrt{3}}{iz} \cdot J\left(a,b,c; \frac{2\pi i}{z}\right) &= \frac{\zeta_{2c}^{-5b}}{6c} \int_0^{i\infty} \frac{\Theta\left(a,b,c;\tau\right)}{\sqrt{-i(\tau+z)}}\, d\tau, \\ \zeta_{2c}^{-5b}\sqrt{3}\sqrt{-iz} \cdot J\left(a,b,c; -2\pi i z\right) &= \frac{\zeta_{2c}^{-5b}}{6c} \int_0^{i\infty} \frac{(-i\tau)^{-\frac{3}{2}} \Theta\left(a,b,c; -\frac{1}{\tau}\right)}{\sqrt{-i(\tau+z)}}\, d\tau. \end{aligned}$$

Arguing as in the proof of Lemma 3.3, one then establishes that the functions T_i satisfy the same transformation laws under the generators of $\mathrm{SL}_2(\mathbb{Z})$ as the corresponding functions $\mathcal{N}$ and $\mathcal{M}$ appearing in (3.9)-(3.12). That the functions $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy suitable transformation laws under $\mathrm{SL}_2(\mathbb{Z})$ follows easily from the "closure" of the formulas in Theorem 2.3.

To complete the proof, it suffices to show that each component is annihilated by the weight $1/2$ hyperbolic Laplacian $\Delta_{\frac{1}{2}}$, and satisfies the required growth conditions at the cusps. These facts follow *mutatis mutandis* as in the proof of Theorem 1.1. □

Sketch of the Proof of Theorem 1.2. By Theorem 3.4, the transformation laws of the components of the given vector valued weak Maass forms are completely determined under all of $\mathrm{SL}_2(\mathbb{Z})$. Observe that $D\left(\frac{a}{c}; z\right)$ is the image of $\mathcal{G}_1\left(\frac{a}{c}; z\right)$ by letting $z \to \ell_c z$. Therefore, the modular transformation properties of $D\left(\frac{a}{c}; z\right)$ are inherited by the modularity properties of $\Theta\left(\frac{a}{c}; \ell_c \tau\right)$ when applied to the definition of $S_1\left(\frac{a}{c}; z\right)$. By Proposition 2.1 of [30], it is known that $\Theta\left(\frac{a}{c}; \ell_c \tau\right)$ is on $\Gamma_1(144 f_c^2 \widetilde{\ell_c})$, and the result follows. □

Remark. The phenomenon above where the modularity properties of a theta function imply the modular transformation laws of a Maass form was first observed by Hirzebruch and Zagier [21]. In their work, the period integral of the classical Jacobi theta function $\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2\tau}$ is the non-holomorphic part of their $\Gamma_0(4)$ weight $3/2$ Maass form $\mathcal{F}(z)$. The modularity in Theorem 1.2 follows *mutatis mutandis* (see page 92 of [21]).

3.3. **Proof of Theorem 1.3.** Now we use Theorems 1.1 and 1.2 to prove Theorem 1.3. If $0 \le r < t$ are integers, then we begin by claiming that

$$\sum_{n=0}^{\infty} N(r,t;n)q^n = \frac{1}{t}\sum_{n=0}^{\infty} p(n)q^n + \frac{1}{t}\sum_{j=1}^{t-1} \zeta_t^{-rj}\cdot R(\zeta_t^j;q). \tag{3.13}$$

There is just one partition of 0, the empty partition. We define its rank to be 0. Since we have

$$\sum_{n=0}^{\infty} p(n)q^n = R(1;q),$$

it follows that the right hand side of (3.13) is

$$\frac{1}{t}\sum_{j=0}^{t-1} \zeta_t^{-rj}\cdot R(\zeta_t^j;q).$$

Therefore the nth coefficient of this series, say $a(n)$, is given by

$$a(n) = \frac{1}{t}\sum_{j=0}^{t-1}\zeta_t^{-rj}\sum_{m=-\infty}^{\infty}\zeta_t^{mj}N(m,n) = \frac{1}{t}\sum_{m=-\infty}^{\infty}N(m,n)\sum_{j=0}^{t-1}\zeta_t^{(m-r)j}.$$

Equation (3.13) follows since the inner sum is t if $m \equiv r \pmod{t}$, and is 0 otherwise.

By Theorems 1.1, 1.2, and (3.13), we obtain

$$\sum_{n=0}^{\infty}\left(N(r,t;n) - \frac{p(n)}{t}\right)q^{\ell_t n - \frac{\ell_t}{24}} = \frac{1}{t}\sum_{j=1}^{t-1}\zeta_t^{-rj}S_1\left(\frac{j}{t};z\right) + \frac{1}{t}\sum_{j=1}^{t-1}\zeta_t^{-rj}D\left(\frac{j}{t};z\right).$$

Theorem 1.3 follows since each $S_1\left(\frac{j}{t};z\right)$ is non-holomorphic.

4. Ramanujan congruences for ranks

Here we use Theorem 1.2 to prove that many of Dyson's partition functions satisfy Ramanujan-type congruences. To prove this, we first show that "sieved" generating functions are indeed already weakly holomorphic modular forms. This observation is the content of Theorem 1.4. Armed with this observation, it is not difficult to prove Theorem 1.5. The proof is a generalization of an argument employed by the second author which proved the existence of infinitely many Ramanujan-type congruences for the partition function $p(n)$ [25].

4.1. **Sieved generating functions.** To prove Theorem 1.4, we first explicitly calculate the Fourier expansions of the Maass forms $D\left(\frac{a}{c}; z\right)$. To give these expansions, we require the incomplete Gamma-function

$$\Gamma(a;x) := \int_x^\infty e^{-t} t^{a-1}\, dt. \tag{4.1}$$

Proposition 4.1. *For integers $0 < a < c$, we have*

$$\begin{aligned} D\left(\frac{a}{c}; z\right) &= q^{-\frac{\ell_c}{24}} + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n) \zeta_c^{am} q^{\ell_c n - \frac{\ell_c}{24}} \\ &+ \frac{i \sin\left(\frac{\pi a}{c}\right) \ell_c^{\frac{1}{2}}}{\sqrt{3}} \sum_{m \pmod{f_c}} (-1)^m \sin\left(\frac{a\pi(6m+1)}{c}\right) \sum_{n \equiv 6m+1 \pmod{6f_c}} \gamma(c,y;n) q^{-\widetilde{\ell}_c n^2}, \end{aligned}$$

where

$$\gamma(c,y;n) := \frac{i \cdot \operatorname{sign}(n)}{\sqrt{2\pi \widetilde{\ell}_c}} \cdot \Gamma\left(\frac{1}{2}; 4\pi \widetilde{\ell}_c n^2 y\right).$$

Proof. It suffices to compute the Fourier expansion of the period integral $S_1\left(\frac{a}{c}; z\right)$. By definition, we find that

$$\begin{aligned} -S_1\left(\frac{a}{c}; z\right) &= \frac{i \sin\left(\frac{\pi a}{c}\right) \ell_c^{\frac{1}{2}}}{\sqrt{3}} \sum_{m \pmod{f_c}} (-1)^m \sin\left(\frac{a\pi(6m+1)}{c}\right) \\ &\qquad \times \sum_{n \equiv 6m+1 \pmod{6f_c}} \int_{-\overline{z}}^{i\infty} \frac{n e^{2\pi i n^2 \widetilde{\ell}_c \tau}}{\sqrt{-i(\tau+z)}}\, d\tau. \end{aligned}$$

To complete the proof, one observes that

$$\int_{-\overline{z}}^{i\infty} \frac{n e^{2\pi i n^2 \widetilde{\ell}_c \tau}}{\sqrt{-i(\tau+z)}}\, d\tau = \gamma(c,y;n) \cdot q^{-\widetilde{\ell}_c n^2}.$$

This integral identity follows by the following changes of variable

$$\begin{aligned} \int_{-\overline{z}}^{i\infty} \frac{n e^{2\pi i n^2 \widetilde{\ell}_c \tau}}{\sqrt{-i(\tau+z)}}\, d\tau &= \int_{2iy}^{i\infty} \frac{n e^{2\pi i n^2 \widetilde{\ell}_c n^2 (\tau - z)}}{\sqrt{-i\tau}}\, d\tau \\ &= i \int_{2y}^{\infty} \frac{n e^{2\pi i n^2 \widetilde{\ell}_c (iu - z)}}{\sqrt{u}}\, du = inq^{-\widetilde{\ell}_c n^2} \int_{2y}^{\infty} \frac{e^{-2\pi n^2 \widetilde{\ell}_c u}}{\sqrt{u}}\, du. \end{aligned}$$

□

Proof of Theorem 1.4. If $f(z)$ is a function on the upper half-plane, $\lambda \in \frac{1}{2}\mathbb{Z}$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$, then we define the usual slash operator by

$$f(z)\big|_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} := (ad-bc)^{\frac{\lambda}{2}}(cz+d)^{-\lambda} f\left(\frac{az+b}{cz+d}\right). \tag{4.2}$$

Suppose that $0 < a < c$ are integers, where c is odd. Since $S_1\left(\frac{a}{c};z\right)$ is the period integral of a cusp form, and since $R(\zeta_c^a;q)$ has no poles in the upper half of the complex plane (which is easily seen by comparing with (1.3), a function with no poles in the upper half plane), it follows that $D\left(\frac{a}{c};z\right)$ has no poles on the upper half of the complex plane.

Furthermore, suppose that $\mathcal{P} \nmid 6c$ is prime. For this prime $\mathcal{P}$, let

$$g := \sum_{v=1}^{\mathcal{P}-1} \left(\frac{v}{\mathcal{P}}\right) e^{\frac{2\pi i v}{\mathcal{P}}}$$

be the usual Gauss sum with respect to $\mathcal{P}$. Define the function $D\left(\frac{a}{c};z\right)_{\mathcal{P}}$ by

$$D\left(\frac{a}{c};z\right)_{\mathcal{P}} := \frac{g}{\mathcal{P}} \sum_{v=1}^{\mathcal{P}-1} \left(\tfrac{v}{\mathcal{P}}\right) D\left(\frac{a}{c};z\right)\ \big|_{\frac{1}{2}} \left(\begin{smallmatrix} 1 & -\frac{v}{\mathcal{P}} \\ 0 & 1 \end{smallmatrix}\right). \tag{4.3}$$

By construction, $D\left(\frac{a}{c};z\right)_{\mathcal{P}}$ is the $\mathcal{P}$ quadratic twist of $D\left(\frac{a}{c};z\right)$. In other words, the nth coefficient in the q-expansion of $D\left(\frac{a}{c};z\right)_{\mathcal{P}}$ is $\left(\frac{n}{\mathcal{P}}\right)$ times the nth coefficient of $D\left(\frac{a}{c};z\right)$. That this holds for the non-holomorphic part follows from the fact that the factors $\gamma(c,y;n)$ appearing in Proposition 4.1 are fixed by the transformations in (4.3).

Generalizing the classical argument on twists of modular forms in the obvious way (for example, see Proposition 17 of [22]), $D\left(\frac{a}{c};z\right)_{\mathcal{P}}$ is a weak Maass form of weight 1/2 on $\Gamma_1(144 f_c^2 \widetilde{\ell}_c \mathcal{P}^2)$. By Proposition 4.1, it follows that

$$D\left(\frac{a}{c};z\right) - \left(\frac{-\widetilde{\ell}_c}{\mathcal{P}}\right) D\left(\frac{a}{c};z\right)_{\mathcal{P}} \tag{4.4}$$

is a weak Maass form of weight 1/2 on $\Gamma_1(144 f_c^2 \widetilde{\ell}_c \mathcal{P}^2)$ with the property that its non-holomorphic part is supported on summands of the form $*q^{-\widetilde{\ell}_c \mathcal{P}^2 n^2}$. These terms are annihilated by taking the $\mathcal{P}$-quadratic twist of this Maass form. Consequently, by the discussion above, we obtain a weakly holomorphic modular form of weight 1/2 on $\Gamma_1(144 f_c^2 \widetilde{\ell}_c \mathcal{P}^4)$. Thanks to (4.4), the conclusion of Theorem 1.4 follows easily by arguing as in the proof of Theorem 1.3. $\square$

4.2. **Ramanujan-type Congruences.** Here we use Theorem 1.4 and facts about eigenvalues of Hecke operators to prove Theorem 1.5. Basically, the result follows from the general phenomenon that coefficients of weakly holomorphic modular forms satisfy Ramanujan-type congruences. This phenomenon was first observed by the second author in his first work on Ramanujan congruences for $p(n)$ [25]. Subsequent generalizations of this argument appear in [1, 2, 24, 31]. Since this strategy is now quite well known, for brevity we only offer sketches of proofs.

To prove Theorem 1.5, we shall employ a recent general result of Treneer [31], which generalizes earlier works by Ahlgren and the second author on weakly holomorphic modular forms. In short, Theorem 1.4, combined with her result, reduces the proof of Theorem 1.5 to the fact that any finite number of half-integral weight cusp forms with integer coefficients are annihilated modulo a fixed prime power by a positive proportion of half-integral weight Hecke operators.

The following theorem is easily obtained by generalizing the proof of Theorem 2.2 of [26].

Theorem 4.2. *Suppose that $f_1(z), f_2(z), \ldots, f_s(z)$ are half-integral weight cusp forms where*

$$f_i(z) \in S_{\lambda_i+\frac{1}{2}}(\Gamma_1(4N_i)) \cap \mathcal{O}_K[[q]],$$

and where $\mathcal{O}_K$ is the ring of integers of a fixed number field K. If Q is prime and $j \geq 1$ is an integer, then the set of primes L for which

$$f_i(z) \mid T_{\lambda_i}(L^2) \equiv 0 \pmod{Q^j},$$

for each $1 \leq i \leq s$, has positive Frobenius density. Here $T_{\lambda_i}(L^2)$ denotes the usual L^2 index Hecke operator of weight $\lambda_i + \frac{1}{2}$.

Sketch of the Proof. By the commutativity of the Hecke operators of integer and half-integral weight under the Shimura correspondence [30], it suffices to show that a positive proportion of primes L have the property that

$$Sh(f_i) \mid T_{2\lambda_i}(L) \equiv 0 \pmod{Q^j},$$

for each $1 \leq i \leq s$. Here $Sh(f_i)$ denotes the image of $f_i(z)$ under the Shimura correspondence, and $T_{2\lambda_i}(L)$ denotes the usual Lth weight $2\lambda_i$ Hecke operator. Theorem 2.2 of [26] ensures that the set of such primes L has positive Frobenius density provided that a single such prime $L \nmid \mathrm{lcm}(4, Q, N_1, \ldots, N_s)$ exists. That such primes L exist is essentially a classical observation of Serre (for example, see §6 of [29]). $\square$

Two remarks.

1) The primes L in Theorem 4.2 may be chosen to lie in the arithmetic progression $L \equiv -1 \pmod{\mathrm{lcm}(4, Q, N_1, \ldots, N_s)}$.

2) Strictly speaking, Serre only states his observations for integer weight modular forms on a congruence subgroup $\Gamma_0(N)$ with fixed Nebentypus and fixed weight. To verify the claim, one examines the Q-adic Galois representation

$$\rho := \oplus_f \rho_f,$$

where the indices f walk over all the weight $2\lambda_i$ newforms with Nebentypus whose levels divide $4N_i$. By the Chebotarev Density Theorem, the claim follows since the number of such f is finite, and the fact that each ρ_f is odd and has the property that their corresponding traces of Frobenius elements Q-adically interpolate the Hecke eigenvalues of f.

Sketch of the Proof of Theorem 1.5. Suppose that $\mathcal{P} \nmid 6tQ$ is prime. By Theorem 1.4, for every $0 \leq r < t$

$$F(r,t,\mathcal{P};z) = \sum_{n=1}^{\infty} a(r,t,\mathcal{P};n)q^n := \sum_{\left(\frac{24\ell_t n-\ell_t}{\mathcal{P}}\right)=-\left(\frac{-24\widetilde{\ell_t}}{\mathcal{P}}\right)} \left(N(r,t;n) - \frac{p(n)}{t}\right) q^{\ell_t n - \frac{\ell_t}{24}} \tag{4.5}$$

is a weakly holomorphic modular form of weight $1/2$ on $\Gamma_1(144 f_t^2 \widetilde{\ell_t} \mathcal{P}^4)$. Furthermore, by the work of Ahlgren and the second author [2], it is known that

$$P(t,\mathcal{P};z) = \sum_{n=1}^{\infty} p(t,\mathcal{P};n)q^n := \sum_{\left(\frac{24\ell_t n-\ell_t}{\mathcal{P}}\right)=-\left(\frac{-24\widetilde{\ell_t}}{\mathcal{P}}\right)} p(n) q^{\ell_t n - \frac{\ell_t}{24}} \tag{4.6}$$

is a weakly holomorphic modular form of weight $-1/2$ on $\Gamma_1(576\widetilde{\ell_t}\mathcal{P}^4)$. In particular, observe that all of these forms are modular with respect to $\Gamma_1(576 f_t^2 \widetilde{\ell_t} \mathcal{P}^4)$.

Now since $Q \nmid 576 f_t^2 \widetilde{\ell_t} \mathcal{P}^4$, a recent result of Treneer (see Theorem 3.1 of [31]), generalizing earlier observations of Ahlgren and Ono [2, 3, 25], implies that there is a sufficiently large integer m for which

$$\sum_{Q \nmid n} a(r,t,\mathcal{P};Q^m n)q^n,$$

for all $0 \leq r < t$, and

$$\sum_{Q \nmid n} p(t,\mathcal{P};Q^m n)q^n$$

are all congruent modulo Q^j to forms in the graded ring of half-integral weight cusp forms with algebraic integer coefficients on $\Gamma_1(576 f_t^2 \widetilde{\ell_t} \mathcal{P}^4 Q^2)$.

Theorem 4.2 applies to these $t+1$ forms, and it guarantees that a positive proportion of primes L have the property that these $t+1$ half-integral weight cusp forms modulo Q^j are annihilated by the index L^2 half-integral weight Hecke operators. Theorem 1.5

now follows *mutatis mutandis* as in the proof of Theorem 1 of [25] (see the top of page 301 of [25]). ☐

Remark. Treneer states her result for weakly holomorphic modular forms on $\Gamma_0(4N)$ with Nebentypus. We are using a straightforward extension of her result to $\Gamma_1(4N)$ which is obtained by decomposing such forms into linear combinations of forms with Nebentypus. It is not difficult to produce such decompositions involving algebraic linear combinations of modular forms whose Fourier coefficients are algebraic integers (which is important when proving congruences). For example, one can multiply each such form by a suitable odd power of $\eta(24z) \in S_{\frac{1}{2}}(\Gamma_0(576), \left(\frac{12}{\cdot}\right))$ to obtain an integer weight cusp form with integer coefficients. One may rewrite such forms as an algebraic linear combination of cusp forms with algebraic integer coefficients using the theory of newforms with Nebentypus. Then divide each resulting summand by the original odd power of $\eta(24z)$, which is non-vanishing on $\mathbb{H}$, to obtain the desired decomposition into weakly holormophic forms with Nebentypus.

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