TRACES OF SINGULAR MODULI ON HILBERT MODULAR SURFACES

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ABSTRACT. Suppose that $p \equiv 1 \pmod 4$ is a prime, and that $\mathcal{O}_K$ is the ring of integers of $K := \mathbb{Q}(\sqrt{p})$. A classical result of Hirzebruch and Zagier asserts that certain generating functions for the intersection numbers of Hirzebruch-Zagier divisors on the Hilbert modular surface $(\mathfrak{h} \times \mathfrak{h})/\mathrm{SL}_2(\mathcal{O}_K)$ are weight 2 holomorphic modular forms. Using recent work of Bruinier and Funke, we show that the generating functions of traces of singular moduli over these intersection points are often weakly holomorphic weight 2 modular forms. For the singular moduli of $J_1(z) = j(z) - 744$, we explicitly determine these generating functions using classical Weber functions, and we factorize their "norms" as products of Hilbert class polynomials. We also explicitly compute all such generating functions in the "$\mathrm{SL}_2(\mathbb{Z})$ case" for the primes $p = 5, 13$, and 17.

1. Introduction and Statement of Results

For primes $p \equiv 1 \pmod 4$, let $\mathcal{O}_K := \mathbb{Z}\left[\frac{1+\sqrt{p}}{2}\right]$ be the ring of integers of the real quadratic field $K := \mathbb{Q}(\sqrt{p})$. The group $\mathrm{SL}_2(\mathcal{O}_K)$, i.e., the group of 2×2 matrices with entries in $\mathcal{O}_K$ and determinant 1, acts on $\mathfrak{h} \times \mathfrak{h}$, the product of two complex upper half planes $\mathfrak{h}$, by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ (z_1, z_2) := \left(\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta}, \frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'} \right).$$

Here ν' denotes the conjugate of ν in $\mathbb{Q}(\sqrt{p})$. The quotient $X_p := (\mathfrak{h} \times \mathfrak{h})/\mathrm{SL}_2(\mathcal{O}_K)$ is a non-compact surface with finitely many singularities. It can be naturally compactified by adding finitely many points (i.e. cusps), and Hirzebruch showed [6] how to resolve the singularities introduced by adding cusps using cyclic configurations of rational curves. The resulting modular surface Y_p is a nearly smooth compact algebraic surface with quotient singularities supported at those points in $\mathfrak{h} \times \mathfrak{h}$ with a non-trivial isotropy subgroup within $\mathrm{PSL}_2(\mathcal{O}_K)$.

In their famous work [7] on these surfaces, Hirzebruch and Zagier introduced a sequence of algebraic curves $Z_1^{(p)}, Z_2^{(p)}, \cdots \subset X_p$, and studied the generating functions for

Date: August 21, 2005.

2000 *Mathematics Subject Classification.* 11F03, 11F30, 11F37.

The authors thank the National Science Foundation for their generous support. The second author is grateful for the support of the David and Lucile Packard, H. I. Romnes, and John S. Guggenheim Fellowships. The third author is grateful for the support of a NDSEG Fellowship.

their intersection numbers. They proved the striking fact that these generating functions are weight 2 modular forms, an observation which allowed them to identify spaces of modular forms with certain homology groups for Y_p. To define these curves, for a positive integer N, consider the points $(z_1, z_2) \in \mathfrak{h} \times \mathfrak{h}$ satisfying an equation of the form

$$Az_1z_2\sqrt{p} + \lambda z_1 - \lambda' z_2 + B\sqrt{p} = 0, \tag{1.1}$$

where $A, B \in \mathbb{Z}$, $\lambda \in \mathcal{O}_K$, and $\lambda\lambda' + ABp = N$. Each such equation defines a curve in $\mathfrak{h} \times \mathfrak{h}$ isomorphic to $\mathfrak{h}$, and their union is invariant under $\mathrm{SL}_2(\mathcal{O}_K)$. The *Hirzebruch-Zagier divisor* $Z_N^{(p)}$ is defined to be the image of this union in X_p. If $\left(\frac{N}{p}\right) = -1$, then one easily sees from (1.1) that $Z_N^{(p)}$ is empty.

We let $\widetilde{Z_N^{(p)}}$ denote the closure of $Z_N^{(p)}$ in Y_p. If $(\widetilde{Z_m^{(p)}}, \widetilde{Z_n^{(p)}})$ denotes the intersection number of $\widetilde{Z_m^{(p)}}$ and $\widetilde{Z_n^{(p)}}$ in Y_p (see [7] for the precise formulation), then Hirzebruch and Zagier proved in [7], for every positive integer m, that

$$\Phi_m^{(p)}(z) := a_m^{(p)}(0) + \sum_{n=1}^{\infty} (\widetilde{Z_m^{(p)}}, \widetilde{Z_n^{(p)}}) q^n \tag{1.2}$$

(note $q = e^{2\pi i z}$ throughout) is a holomorphic weight 2 modular form on $\Gamma_0(p)$ with Nebentypus $\left(\frac{\cdot}{p}\right)$. Here $a_m^{(p)}(0)$ is a simple constant arising from a volume computation. More precisely, $\Phi_m^{(p)}(z)$ is in the *plus space* $M_2^+\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right)$, the space of holomorphic weight 2 modular forms $F(z) = \sum_{n=0}^{\infty} a(n)q^n$ on $\Gamma_0(p)$ with Nebentypus $\left(\frac{\cdot}{p}\right)$, with the additional property that

$$a(n) = 0 \quad \text{if } \left(\frac{n}{p}\right) = -1. \tag{1.3}$$

In the present paper, we require the space $\mathcal{M}_2\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right)$, the space of weakly holomorphic modular forms of weight 2 on $\Gamma_0(p)$ with Nebentypus $\left(\frac{\cdot}{p}\right)$. Let $\mathcal{M}_2^+\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right)$ be the subspace of those forms in $\mathcal{M}_2\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right)$ that satisfy (1.3). Recall that a function is *weakly holomorphic* if its poles (if there are any) are supported at cusps.

The geometric part of the proof of the modularity of (1.2) provides a concrete description of the intersection points $\widetilde{Z_m^{(p)}} \cap \widetilde{Z_n^{(p)}}$. Loosely speaking, the "finite points" $Z_m^{(p)} \cap Z_n^{(p)}$ are identified with CM points in $\mathfrak{h}$ which are the "roots" of $\Gamma_0(m)$ equivalence classes of binary quadratic forms with negative discriminants of the form $-(4mn - x^2)/p$ (see Section 2.1). The values of modular functions at such CM points are known as *singular moduli*, and in view of the modularity of (1.2), it is natural to consider generating functions for the values of singular moduli over the CM points constituting $Z_m^{(p)} \cap Z_n^{(p)}$.

Suppose that $\ell = 1$ or that ℓ is an odd prime with $\left(\frac{\ell}{p}\right) \neq -1$, and let $\Gamma_0^*(\ell)$ be the projective image of the extension of $\Gamma_0(\ell)$ by the Fricke involution $W_\ell = \left(\begin{smallmatrix} 0 & -1 \\ \ell & 0 \end{smallmatrix}\right)$

in $\mathrm{PSL}_2(\mathbb{R})$. Suppose that $f(z) = \sum_{n \gg -\infty} a(n)q^n \in \mathcal{M}_0(\Gamma_0^*(\ell))$, the space of weakly holomorphic modular functions with respect to $\Gamma_0^*(\ell)$. Furthermore, suppose that $a(0) = 0$. We define the "trace" of $f(z)$ over $Z_\ell^{(p)} \cap Z_n^{(p)}$ by

$$(Z_\ell^{(p)}, Z_n^{(p)})_f^{\mathrm{tr}} := \sum_{\tau \in Z_\ell^{(p)} \cap Z_n^{(p)}} \frac{f(\tau)}{\#\Gamma_0^*(\ell)_\tau}, \tag{1.4}$$

where $\Gamma_0^*(\ell)_\tau$ denotes the stabilizer of τ in $\Gamma_0^*(\ell)$. For these traces, we consider the analog of the generating functions in (1.2) defined by

$$\Phi_{\ell,f}^{(p)}(z) := A_{\ell,f}^{(p)}(z) + B_{\ell,f}^{(p)}(z) + \sum_{n=1}^{\infty} (Z_\ell^{(p)}, Z_n^{(p)})_f^{\mathrm{tr}} q^n. \tag{1.5}$$

Here we have that

$$A_{\ell,f}^{(p)}(z) := -\epsilon(\ell) \sum_{m,n \geq 1} ma(-mn) \left(\sum_{\substack{x \in \mathbb{Z} \\ x^2 \equiv m^2 p \pmod{2\ell}}} q^{\frac{x^2 - m^2 p}{4\ell}} + \sum_{\substack{x \in \mathbb{Z} \\ x \equiv m \pmod{2}}} q^{\frac{x^2 \ell - m^2 p \ell}{4}} \right),$$

$$B_{\ell,f}^{(p)}(z) := 2\epsilon(\ell) \sum_{n \geq 1} (\sigma_1(n) + \ell \sigma_1(n/\ell)) a(-n) \sum_{x \in \mathbb{Z}} q^{\ell x^2},$$

where $\epsilon(\ell) = 1/2$ for $\ell = 1$, and is 1 otherwise. As usual, $\sigma_1(x)$ denotes the sum of the positive divisors of x if x is an integer, and is zero if x is not an integer.

Using recent works of Zagier [13], and Bruinier and Funke [4], we show that these generating functions are also modular forms of weight 2. In particular, we obtain a linear map:

$$\Phi_{\ell,\star}^{(p)} : \mathcal{M}_0(\Gamma_0^*(\ell)) \to \mathcal{M}_2 \left(\Gamma_0(p\ell^2), \left(\frac{\cdot}{p} \right) \right)$$

(where the map is defined for the subspace of those functions with constant term 0).

Theorem 1.1. *Suppose that $p \equiv 1 \pmod 4$ is prime, and that $\ell = 1$ or is an odd prime with $\left(\frac{\ell}{p}\right) \neq -1$. If $f(z) = \sum_{n \gg -\infty} a(n)q^n \in \mathcal{M}_0(\Gamma_0^*(\ell))$, with $a(0) = 0$, then the generating function $\Phi_{\ell,f}^{(p)}(z)$ is in $\mathcal{M}_2\left(\Gamma_0(p\ell^2), \left(\frac{\cdot}{p}\right)\right)$.*

Remark. In [4], Bruinier and Funke establish that the generating functions for traces of singular moduli are modular forms in great generality. In the case of modular curves, for simplicity they work out the details for $X_0^*(\ell)$ for $\ell = 1$ and for odd primes ℓ. We follow their lead by making this assumption as well in Theorem 1.1.

Remark. If we allow the constant term of $f(z)$ to be non-zero, non-holomorphic terms would be included, as in [4]. In particular, if $f(z) = 1$, then we obtain the Hirzebruch-Zagier modular forms restricted to the "finite" points of intersection.

We turn to the problem of explicitly computing natural examples of these modular forms $\Phi_{\ell,f}^{(p)}(z)$. Let $J_1(z) = j(z) - 744$, where $j(z)$ is the usual elliptic modular function

$$j(z) = \frac{E_4(z)^3}{\eta(z)^{24}} = q^{-1} + 744 + 196884q + \cdots, \tag{1.6}$$

where $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is Dedekind's eta-function and

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sum_{d|n} d^3q^n$$

is the usual Eisenstein series of weight 4. The modular forms $\Phi_{1,J_1}^{(p)}(z)$ can be described in terms of $\eta(z)$, $E_4(z)$, and the classical Weber functions

$$\mathfrak{f}_1(z) = \frac{\eta(z/2)}{\eta(z)} \quad \text{and} \quad \mathfrak{f}_2(z) = \sqrt{2}\cdot\frac{\eta(2z)}{\eta(z)}. \tag{1.7}$$

Theorem 1.2. *If $p \equiv 1 \pmod 4$ is prime, then*

$$\Phi_{1,J_1}^{(p)}(z) = \frac{\eta(2z)\eta(2pz)E_4(pz)\mathfrak{f}_2(2z)^2\mathfrak{f}_2(2pz)^2}{4\eta(pz)^6}\cdot\left(\mathfrak{f}_1(4z)^4\mathfrak{f}_2(z)^2 - \mathfrak{f}_1(4pz)^4\mathfrak{f}_2(pz)^2\right).$$

Remark. Using the classical theta functions $\Theta(z)$ and $\Theta_{\mathrm{odd}}(z)$ (see (3.1) and (3.3)), the formula in Theorem 1.2 may be reformulated as

$$\Phi_{1,J_1}^{(p)}(z) = -\frac{2E_4(pz)}{\eta(pz)^6}\cdot\left(\Theta(pz)\Theta_{\mathrm{odd}}(z/4) - \Theta(z)\Theta_{\mathrm{odd}}(pz/4)\right).$$

It turns out that the forms $\Phi_{1,J_1}^{(p)}(z)$, the generating functions for the traces of singular moduli on X_p, are closely related to Hilbert class polynomials. The singular moduli $j(\tau)$, as τ ranges over $\mathcal{C}_D$, the equivalence classes of CM points with discriminant $-D$, are the roots of the *Hilbert class polynomial*

$$H_D(x) = \prod_{\tau\in\mathcal{C}_D}(x - j(\tau)) \in \mathbb{Z}[x]. \tag{1.8}$$

Each $H_D(x)$ is an irreducible polynomial in $\mathbb{Z}[x]$ which generates a class field extension of $\mathbb{Q}(\sqrt{-D})$. To relate the forms $\Phi_{1,J_1}^{(p)}(z)$ to Hilbert class polynomials, define $N_p(z)$ as the "multiplicative norm" of $\Phi_{1,J_1}(z)$

$$N_p(z) := \prod_{M\in\Gamma_0(p)\backslash\mathrm{SL}_2(\mathbb{Z})}\Phi_{1,J_1}^{(p)}|M. \tag{1.9}$$

If $N_p^*(z)$ is the normalization of $N_p(z)$ with leading coefficient 1, then

$$N_p^*(z) = \begin{cases} \Delta(z)H_{75}(j(z)) & \text{if } p = 5,\\ E_4(z)\Delta(z)^2H_3(j(z))H_{507}(j(z)) & \text{if } p = 13,\\ \Delta(z)^3H_4(j(z))H_{867}(j(z)) & \text{if } p = 17,\\ \Delta(z)^5H_7(j(z))^2H_{2523}(j(z)) & \text{if } p = 29,\end{cases}$$

where $\Delta(z) = \eta(z)^{24}$ is the usual Delta-function. These examples illustrate a general phenomenon in which $N_p^*(z)$ is essentially a product of certain Hilbert class polynomials. Before we state the general result, we fix some notation. Define integers $a(p)$, $b(p)$, and $c(p)$ by

$$a(p) := \frac{1}{2}\left(\left(\frac{3}{p}\right) + 1\right), \tag{1.10}$$

$$b(p) := \frac{1}{2}\left(\left(\frac{2}{p}\right) + 1\right), \tag{1.11}$$

$$c(p) := \frac{1}{6}\left(p - \left(\frac{3}{p}\right)\right). \tag{1.12}$$

Furthermore, let $\mathcal{D}_p$ be the set of negative discriminants $-D \neq -3, -4$ of the form $\frac{x^2-4p}{16f^2}$ with $x, f \geq 1$.

Theorem 1.3. *Assume the notation above. If $p \equiv 1 \pmod 4$ is prime, then*

$$N_p^*(z) = (E_4(z)H_3(j(z)))^{a(p)} \cdot H_4(j(z))^{b(p)} \cdot \Delta(z)^{c(p)} \cdot H_{3\cdot p^2}(j(z)) \cdot \prod_{-D\in\mathcal{D}_p} H_D(j(z))^2.$$

For the primes $p = 5, 13$, and 17, and when $\ell = 1$, work of Bruinier and Bundschuh [3] make it possible to obtain explicit formulas for the traces of every weakly holomorphic level 1 function, with constant term 0, on X_p. There is a natural sequence of modular functions $J_m(z)$ which forms a basis of such functions. For every positive integer m let $J_m(z)$ be the unique modular function on $\mathrm{SL}_2(\mathbb{Z})$ which is holomorphic on $\mathfrak{h}$ with a Fourier expansion of the form

$$J_m(z) = q^{-m} + \sum_{n=1}^{\infty} c_m(n)q^n. \tag{1.13}$$

In particular, note that

$$J_1(z) = j(z) - 744 = q^{-1} + 196884q + \cdots.$$

Each $J_m(z)$ is a monic degree m polynomial in $j(z)$ with integer coefficients, and its generating function is given by

$$\sum_{m=0}^{\infty} J_m(x)q^m = \frac{E_4(z)^2E_6(z)}{\Delta(z)} \cdot \frac{1}{j(z)-x},$$

where $E_6(z) = 1 - 504\sum_{n=1}^{\infty}\sum_{d|n} d^5q^n$.

To describe $\Phi_{1,J_m}^{(p)}$ for $p = 5, 13$, and 17 we give a basis for $\mathcal{M}_2^+\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right)$. In particular, for $m \geq 0$ with $\left(\frac{m}{p}\right) \neq -1$ there is a unique function $K_m^{(p)}(z)$ with Fourier expansion of the form

$$K_m^{(p)}(z) = q^{-m} + O(q) \in \mathcal{M}_2^+\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right). \tag{1.14}$$

In Section 4, we will provide formulas for $K_0^{(p)}(z)$ and $K_1^{(p)}(z)$. Furthermore, we provide a description of each $K_m^{(p)}(z)$, for $m \geq 1$ in terms of the action of Hecke operators on $K_1^{(p)}(z)$.

We have the following connection between the functions $K_m^{(p)}(z)$ and $\Phi_{1,J_m}^{(p)}(z)$.

Theorem 1.4. *If $p = 5, 13$, or 17, and $m \geq 1$, then*

$$\Phi_{1,J_m}^{(p)}(z) = 2\sigma_1(m) K_0^{(p)}(z) - \sum_{d|m} d \sum_{\substack{x \equiv d \pmod 2 \\ x^2 < d^2 p}} K_{(d^2p - x^2)/4}^{(p)}(z).$$

Remark. Theorem 1.4 shows that the linear map described in Theorem 1.1 is not necessarily surjective. Specifically, the leading term of $\Phi_{1,J_m}^{(p)}(z)$ is $-mq^{\lfloor \frac{m^2 p}{4} \rfloor}$, and this easily implies that for $p = 5$, $K_4^{(p)}(z)$ is not in the image of the linear map.

However, the linear map is always injective. If $f \in M_0(\Gamma_0^*(\ell))$ is a nonconstant modular function, then since $\Gamma_0^*(\ell)$ has only one cusp, f has a pole at ∞. Suppose that

$$f(z) = cq^{-m} + O(q^{-m+1}),$$

with $c \neq 0$ and $m \geq 1$. The formulas for $\Phi_{\ell,f}^{(p)}(z)$ obtained in the proof of Theorem 1.1 easily imply that

$$\Phi_{\ell,f}^{(p)}(z) = \begin{cases} -c\ell m q^{-pm^2/4} + O(q^{(4-pm^2)/4}) & \text{if } m \text{ is even,} \\ -2c\ell m q^{(1-pm^2)/4} + O(q^{(5-pm^2)/4}) & \text{if } m \text{ is odd.} \end{cases}$$

and hence $\Phi_{\ell,f}^{(p)}(z) \neq 0$.

In Section 2.1, we recall the exact relation between the points in $Z_m^{(p)} \cap Z_n^{(p)}$ and CM points (see Definition 2.1), and in Section 2.2 we recall works of Zagier, and Bruinier and Funke which describe generating functions for traces of singular moduli on modular curves as weight 3/2 weakly holomorphic modular forms. Using these facts, we prove Theorem 1.1 in Section 2.3. In Section 3 we investigate the traces of $J_1(z) = j(z) - 744$, and we prove Theorems 1.2 and 1.3. In Section 4, for $p = 5, 13$, and 17 we compute each $\Phi_{1,J_m}^{(p)}(z)$ using works of Bruinier and Bundschuh, and we prove Theorem 1.4.

Acknowledgements

The authors thank J. Bruinier, J. Funke, W. Kohnen and the referee for their helpful comments.

2. The modularity of $\Phi_{\ell,f}^{(p)}(z)$

2.1. **Intersection points on Hilbert modular surfaces as CM points.** The goal of this section is to provide (for $\ell = 1$ or an odd prime with $\left(\frac{\ell}{p}\right) \neq -1$) an interpretation of $Z_\ell^{(p)} \cap Z_n^{(p)}$ as a union of $\Gamma_0^*(\ell)$ equivalence classes of CM points. This is given by Definition 2.1 below.

For $-D \equiv 0, 1 \pmod 4$, $D > 0$ we denote by $\mathcal{Q}_D$ the set of all (not necessarily primitive) binary quadratic forms

$$Q(x,y) = [a,b,c](x,y) := ax^2 + bxy + cy^2$$

with discriminant $b^2 - 4ac = -D$. To each such form Q, we let the CM point α_Q be the unique point in $\mathfrak{h}$ that satisfies $Q(\alpha_Q, 1) = 0$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{Q}_D$ in the usual way, i.e., for $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ we define

$$[a,b,c] \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (x,y) := [a,b,c](\alpha x + \beta y, \gamma x + \delta y).$$

It is easy to see that $\mathcal{Q}_D$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$.

For $\ell = 1$ or an odd prime and $D > 0$, $-D \equiv 0, 1 \pmod 4$ we define $\mathcal{Q}_D^{[\ell]}$ to be the subset of $\mathcal{Q}_D$ with the additional condition that $\ell | a$. It is easy to show that $\mathcal{Q}_D^{[\ell]}$ is invariant under $\Gamma_0^*(\ell)$.

Suppose that $\ell = 1$ or ℓ is an odd prime with $\left(\frac{\ell}{p}\right) \neq -1$. Then, there exists a prime ideal $\mathfrak{p} \subseteq \mathcal{O}_K$ with norm ℓ. Define

$$\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{p}) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(K) : \alpha, \delta \in \mathcal{O}_K, \gamma \in \mathfrak{p}, \beta \in \mathfrak{p}^{-1} \right\}.$$

In this case there is a matrix $A \in \mathrm{GL}_2^+(K)$ such that $A^{-1}\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{p})A = \mathrm{SL}_2(\mathcal{O}_K)$. Define

$$\phi : (\mathfrak{h} \times \mathfrak{h})/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{p}) \to (\mathfrak{h} \times \mathfrak{h})/\mathrm{SL}_2(\mathcal{O}_K)$$

by

$$\phi((z_1, z_2)) := (Az_1, A'z_2).$$

Let Γ be the stabilizer of $\{(z,z) : z \in \mathfrak{h}\} \subseteq \mathfrak{h} \times \mathfrak{h}$ in $\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{p})$. Then $\Gamma = \Gamma_0(\ell)$ if $\ell \neq p$ and $\Gamma = \Gamma_0^*(\ell)$ if $\ell = p$. The image of $\{(z,z) : z \in \mathfrak{h}\}$ under ϕ is $Z_\ell^{(p)}$. Hence, we have a natural map $\psi : \mathfrak{h}/\Gamma \to Z_\ell^{(p)}$. Using work of Hirzebruch and Zagier, we make the following definition.

Definition 2.1. *If $\ell = 1$ or an odd prime with $\left(\frac{\ell}{p}\right) \neq -1$, and $n \geq 1$, then define*

$$Z_\ell^{(p)} \cap Z_n^{(p)} := \bigcup_{\substack{x \in \mathbb{Z} \\ x^2 < 4\ell n \\ x^2 \equiv 4\ell n \pmod p}} \left\{ \alpha_Q : Q \in \mathcal{Q}^{[\ell]}_{(4\ell n - x^2)/p} \,/\, \Gamma_0^*(\ell) \right\}.$$

Here the repetition of x and $-x$ indicates that $Z_\ell^{(p)} \cap Z_n^{(p)}$ is a multiset where a CM point α_Q occurs twice if $Q \in \mathcal{Q}^{[\ell]}_{(4\ell n - x^2)/p}$ for $x \neq 0$. In addition, if $\ell > 1$ and $\ell | n$, then we include

$$\bigcup_{\substack{x \in \mathbb{Z} \\ x^2 < 4n/\ell \\ x^2 \equiv 4n/\ell \pmod p}} \left\{ \alpha_Q : Q \in \mathcal{Q}^{[\ell]}_{(4n/\ell - x^2)/p} \,/\, \Gamma_0^*(\ell) \right\},$$

where each point with non-zero x is taken with multiplicity 2ℓ, and a point where $x = 0$ is taken with multiplicity ℓ.

To justify our definition we argue as follows. Hirzebruch and Zagier ([7], p. 66) show that if $t \in \mathfrak{h}$, $n \geq 1$ and $\psi(t) \in Z_\ell^{(p)} \cap Z_n^{(p)}$, then

$$a\ell t^2 + \frac{\ell\lambda - \ell\lambda'}{\sqrt{p}} t + b = 0$$

for $(a, b, \lambda) \in \mathbb{Z} \oplus \mathbb{Z} \oplus \mathfrak{p}^{-1}$ with $\ell\lambda\lambda' + abp = n$. This follows as a result of considering the inverse image $\phi^{-1}(Z_\ell^{(p)}) \subseteq (\mathfrak{h} \times \mathfrak{h})/\mathrm{SL}_2(\mathcal{O}_K, \mathfrak{p})$.

Write $\ell\lambda = c + d\frac{1+\sqrt{p}}{2}$, for $c, d \in \mathbb{Z}$. We have that the discriminant of the equation above is $d^2 - 4ab\ell$. However, this implies that

$$\frac{(2c+d)^2 - 4n\ell}{p} = d^2 - 4ab\ell.$$

Thus, the discriminant is of the form $(x^2 - 4n\ell)/p$. From Hirzebruch and Zagier's Theorem 3 ([7], p. 77), computing the number of transverse intersections of $Z_\ell^{(p)}$ and $Z_n^{(p)}$, we see that each $z \in \mathfrak{h}$ with discriminant of the form $(x^2 - 4n\ell)/p$ occurs with the appropriate multiplicity.

2.2. **Generating functions for traces of singular moduli on modular curves.** Throughout we let ℓ be 1 or an odd prime. Motivated by Borcherds' work [1] on the infinite product expansions of certain automorphic forms on orthogonal groups, Zagier [13] computed the generating functions for the "traces" of the $J_m(z)$ singular moduli, as well as several other classes of modular functions. If m, D are positive integers and $-D$ is a discriminant, then Zagier defined the trace of the singular moduli of discriminant $-D$ for $J_m(z)$ by

$$t_m(D) := \sum_{Q \in \mathcal{Q}_D/\mathrm{PSL}_2(\mathbb{Z})} \frac{1}{\omega_Q} \cdot J_m(\alpha_Q), \tag{2.1}$$

where ω_Q is the order of the stabilizer of Q in $\mathrm{PSL}_2(\mathbb{Z})$. He proved the striking fact that these generating functions are essentially weight $3/2$ weakly holomorphic modular forms.

We now recall some of Zagier's generating functions. Following Kohnen [8], for integers k let $\mathcal{M}^+_{k+\frac{1}{2}}(\Gamma_0(4))$ be the space of weakly holomorphic weight $k + \frac{1}{2}$ modular forms on $\Gamma_0(4)$ with a Fourier expansion of the form

$$\sum_{\substack{n \gg -\infty \\ (-1)^k n \equiv 0,1 \pmod 4}} a(n) q^n. \tag{2.2}$$

Zagier's trace generating functions are described in terms of a special sequence of weight $3/2$ forms $g_m(z)$. For positive $m \equiv 0, 1 \pmod 4$, $g_m(z)$ is the unique form in

$\mathcal{M}^+_{\frac{3}{2}}(\Gamma_0(4))$ with a Fourier expansion of the form

$$g_m(z) = q^{-m} + \sum_{n=0}^{\infty} B(m,n)q^n. \tag{2.3}$$

Zagier proved that the forms $g_m(z)$ determine the generating functions for traces and "twisted traces" of singular moduli on $\mathrm{SL}_2(\mathbb{Z})$. For example, his work shows that

$$g_1(z) = \frac{\eta(z)^2 E_4(4z)}{\eta(2z)\eta(4z)^6} = q^{-1} - 2 - \sum_{D>0} t_1(D)q^D = q^{-1} - 2 + 248q^3 - \cdots. \tag{2.4}$$

To state his more general result, for positive integers m, let

$$B_m(1,D) := \text{ the coefficient of } q^D \text{ in } g_1(z)|T(m^2), \tag{2.5}$$

where $T(m^2)$ is the usual Hecke operator on $\mathcal{M}^+_{\frac{3}{2}}(\Gamma_0(4))$. Zagier's formulae for the traces $t_m(D)$ are given by the following theorem (Theorem 5 of [13]).

Theorem 2.2. *If $m \geq 1$ and $0 < D \equiv 0, 3 \pmod 4$, then $t_m(D) = -B_m(1,D)$.*

Recently, Bruinier and Funke [4] have generalized Zagier's results to include traces of singular moduli of modular functions on groups which do not necessarily possess a Hauptmodul. A particularly elegant example of their work applies to modular functions on $\Gamma_0^*(\ell)$. Suppose that $f(z) = \sum_{n \gg -\infty} a(n)q^n \in \mathcal{M}_0(\Gamma_0^*(\ell))$ has constant term $a(0) = 0$. The discriminant $-D$ trace is given by

$$t_f^*(D) := \sum_{Q \in \mathcal{Q}_{D,\ell}/\Gamma_0^*(\ell)} \frac{1}{\#\Gamma_0^*(\ell)_Q} \cdot f(\alpha_Q). \tag{2.6}$$

Here $\Gamma_0^*(\ell)_Q$ is the stabilizer of Q in $\Gamma_0^*(\ell)$. Following Kohnen [8], we let for $\epsilon \in \{\pm 1\}$ $\mathcal{M}^{+,\epsilon}_{k+\frac{1}{2}}(\Gamma_0(4\ell))$ be the space of those weight $k + \frac{1}{2}$ weakly holomorphic modular forms $f(z) = \sum_{n \gg -\infty} a(n)q^n$ on $\Gamma_0(4\ell)$ whose Fourier coefficients satisfy

$$a(n) = 0 \;\text{ whenever }\; (-1)^k n \equiv 2, 3 \pmod 4 \;\text{ or }\; \left(\frac{(-1)^k n}{\ell}\right) = -\epsilon. \tag{2.7}$$

Bruinier and Funke's generalization of Zagier's work (Theorem 1.1 of [4]) gives the following theorem.

Theorem 2.3. *If $\ell = 1$ or is an odd prime and $f(z) = \sum_{n \gg -\infty} a(n)q^n \in \mathcal{M}_0(\Gamma_0^*(\ell))$, with $a(0) = 0$, then*

$$G_\ell(f,z) := -\sum_{m,n \geq 1} m a(-mn) q^{-m^2} + \sum_{n \geq 1} \left(\sigma_1(n) + \ell\sigma_1(n/\ell)\right) a(-n) + \sum_{D>0} t_f^*(D)q^D$$

is an element of $\mathcal{M}^{+,+}_{\frac{3}{2}}(\Gamma_0(4\ell))$.

Remark. Theorem 2.3 recovers Zagier's Theorem 2.2 when $\ell = 1$ and $f = J_m$.

2.3. **Proof of Theorem 1.1.** To prove Theorem 1.1 we require the classical Jacobi theta function

$$\Theta(z) = \sum_{x\in\mathbb{Z}} q^{x^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots . \tag{2.8}$$

It is well known that $\Theta(z) \in M_{\frac{1}{2}}(\Gamma_0(4))$.

Suppose that $f(z) = \sum_{n\gg-\infty} a(n)q^n \in \mathcal{M}_0(\Gamma_0^*(\ell))$ satisfies the hypotheses of Theorem 1.1. By Definition 2.1 and Theorem 2.3, a straightforward calculation reveals that

$$\Phi_{\ell,f}^{(p)}(z) = \epsilon(\ell)\left(G_\ell(f,pz)\Theta(z)\right) \mid U(4) \mid \left(U(\ell) + \ell V(\ell)\right), \tag{2.9}$$

where for $d \geq 1$ the operators $U(d)$ and $V(d)$ are defined on formal power series by

$$\left(\sum a(n)q^n\right) \mid U(d) := \sum a(dn)q^n, \tag{2.10}$$

and

$$\left(\sum a(n)q^n\right) \mid V(d) := \sum a(n)q^{dn}. \tag{2.11}$$

It is well-known (for example, see [8, 12]) that $V(p)$ maps $G_\ell(f,z) \in \mathcal{M}_{\frac{3}{2}}(\Gamma_0(4\ell))$ to the space $\mathcal{M}_{\frac{3}{2}}\left(\Gamma_0(4p\ell), \left(\frac{\cdot}{p}\right)\right)$. Since $\Theta(z) \in M_{\frac{1}{2}}(\Gamma_0(4))$, it follows that $G_\ell(f,pz)\Theta(z)$ is in $\mathcal{M}_2\left(\Gamma_0(4p\ell), \left(\frac{\cdot}{p}\right)\right)$.

Now, we apply the operator $U(2)$ twice to $G_\ell(f,pz)\Theta(z)$. Since $2^2 \mid 4p\ell$ and $\left(\frac{\cdot}{p}\right)$ has conductor p, Lemma 1 of [9] implies that $(G_\ell(f,pz)\Theta(z)) \mid U(2)$ is in $\mathcal{M}_2\left(\Gamma_0(2p\ell), \left(\frac{\cdot}{p}\right)\right)$. Since the non-zero coefficients of $G_\ell(f,z)$ are supported on exponents $n \equiv 0, 3 \pmod 4$, it follows that the non-zero coefficients of $G_\ell(f,pz)\Theta(z)$ are supported on exponents $n \equiv 0,1,3 \pmod 4$. In particular, the non-zero coefficients of

$$(G_\ell(f,pz)\Theta(z)) \mid U(2)$$

are supported on exponents $n \equiv 0 \pmod 2$. Lemma 4 (i) of [9] then implies that

$$(G_\ell(f,pz)\Theta(z)) \mid U(2) \mid U(2) = (G_\ell(f,pz)\Theta(z)) \mid U(4)$$

is in $\mathcal{M}_2\left(\Gamma_0(p\ell), \left(\frac{\cdot}{p}\right)\right)$. The theorem follows since it is well-known that $U(\ell) + \ell V(\ell)$ maps $(G_\ell(f,pz)\Theta(z)) \mid U(4)$ to $\mathcal{M}_2\left(\Gamma_0(p\ell^2), \left(\frac{\cdot}{p}\right)\right)$.

Remark. Strictly speaking, Lemmas 1 and 4 of [9] are only stated for integer weight cusp forms. However, it is simple to check that their proofs also hold for weakly holomorphic integer weight forms.

3. The special case of $J_1(z) = j(z) - 744$

Here we examine the modular forms $\Phi^{(p)}_{1,J_1}(z)$, the trace generating functions for $J_1(z) = j(z) - 744$ on the Hilbert modular surface X_p. In particular, we prove Theorem 1.2 which describes these forms in terms of the classical Weber functions, and Theorem 1.3 which relates these forms to products of Hilbert class polynomials.

3.1. **Proof of Theorem 1.2.** To prove Theorem 1.2, we work directly with Zagier's identity (2.4). We recall the following classical theta function identities:

$$\Theta(z) = \frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2} = \sum_{x\in\mathbb{Z}} q^{x^2} = 1 + 2q + 2q^4 + \cdots, \tag{3.1}$$

$$\Theta_0(z) = \frac{\eta(z)^2}{\eta(2z)} = \sum_{x\in\mathbb{Z}} (-1)^x q^{x^2} = 1 - 2q + 2q^4 - 2q^9 + \cdots, \tag{3.2}$$

and

$$\Theta_{\mathrm{odd}}(z) = \frac{\eta(16z)^2}{\eta(8z)} = \sum_{x\geq 0} q^{(2x+1)^2} = q + q^9 + q^{25} + q^{49} + \cdots. \tag{3.3}$$

Proof of Theorem 1.2. By (2.4), (2.9), and (3.2), we have that

$$\begin{aligned}\Phi^{(p)}_{1,J_1}(z) &= -\left(g_1(pz)\Theta(z)\right) \mid U(4)\\ &= -\left(\frac{\Theta_0(pz)E_4(4pz)}{\eta(4pz)^6}\cdot\Theta(z)\right) \mid U(4).\end{aligned}$$

By the definition of $U(4)$, it is straightforward to rewrite this expression as

$$\begin{aligned}\Phi^{(p)}_{1,J_1}(z) &= -\frac{1}{4}\sum_{\nu=0}^{3}\left(\frac{\Theta_0(pz)E_4(4pz)}{\eta(4pz)^6}\cdot\Theta(z)\right) \Bigg| \begin{pmatrix}1 & \nu\\ 0 & 4\end{pmatrix}\\ &= -\frac{1}{4}\sum_{\nu=0}^{3}\left(\frac{\Theta_0(p(z+\nu)/4)E_4(p(z+\nu))}{\eta(p(z+\nu))^6}\cdot\Theta((z+\nu)/4)\right).\end{aligned}$$

Using the fact that $E_4(p(z+\nu)) = E_4(pz)$, and that

$$\eta(p(z+\nu))^6 = i^{\nu}\eta(pz)^6,$$

we obtain

$$\Phi^{(p)}_{1,J_1}(z) = -\frac{E_4(pz)}{4\eta(pz)^6}\sum_{\nu=0}^{3} i^{-\nu}\Theta_0(p(z+\nu)/4)\Theta((z+\nu)/4).$$

By (3.2) and (3.3), one finds that

$$\Phi^{(p)}_{1,J_1}(z) = -\frac{E_4(pz)}{4\eta(pz)^6}\cdot\sum_{x,y\in\mathbb{Z}} q^{(px^2+y^2)/4}\cdot(-1)^x\left(\sum_{\nu=0}^{3} i^{p\nu x^2+y^2\nu-\nu}\right).$$

Since we have that

$$\sum_{\nu=0}^{3} i^{p\nu x^2+y^2\nu-\nu} = \begin{cases} 0 & \text{if } x \equiv y \pmod{2}, \\ 4 & \text{if } x \not\equiv y \pmod{2}, \end{cases}$$

it follows that

$$\begin{aligned} \Phi_{1,J_1}^{(p)}(z) &= -\frac{E_4(pz)}{\eta(pz)^6} \cdot \left(\sum_{x,y\in\mathbb{Z}} q^{((2y+1)^2+4px^2)/4} - \sum_{x,y\in\mathbb{Z}} q^{(4y^2+(2x+1)^2p)/4} \right) \\ &= -\frac{2E_4(pz)}{\eta(pz)^6} \cdot \left(\Theta(pz)\Theta_{\text{odd}}(z/4) - \Theta(z)\Theta_{\text{odd}}(pz/4)\right). \end{aligned}$$

The claimed formula now follows easily from (1.7), (3.1), and (3.3). □

3.2. **Proof of Theorem 1.3.** Here we prove Theorem 1.3, the description of $N_p^*(z)$ in terms of products of Hilbert class polynomials.

Proposition 3.1. *If* $p \equiv 1 \pmod{4}$ *is prime, then*

$$N_p^*(z) = E_4(z)^{a(p)} \cdot \Delta(z)^{c(p)} \cdot F_p(j(z)), \tag{3.4}$$

where $F_p(x) \in \mathbb{Z}[x]$ *is a monic polynomial with*

$$\deg(F_p(x)) = \begin{cases} (5p-5)/12 & \text{if } p \equiv 1 \pmod{12}, \\ (5p-1)/12 & \text{if } p \equiv 5 \pmod{12}. \end{cases}$$

Proof. From Theorem 1.1 one easily sees that $N_p^*(z) \in M_{2p+2}(\mathrm{SL}_2(\mathbb{Z}))$. Lemma 2.34 of [10] then implies that $N_p^*(z)$ has the desired factorization (3.4). Since

$$\Phi_{1,J_1}^{(p)}(z) \in \mathcal{M}_2^+\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right),$$

Lemma 3 of [3] implies that

$$\Phi_{1,J_1}^{(p)}(z) \mid W_p = \frac{1}{\sqrt{p}} \cdot \Phi_{1,J_1}^{(p)}(z) \mid U(p). \tag{3.5}$$

This implies in particular that $N_p^*(z)$ has integer coefficients with leading coefficient one. Since $j(z)$ has integer coefficients with leading coefficient 1, it follows that $F_p(x)$ is a monic polynomial with integer coefficients.

To complete the proof, it suffices to compute the degree of $F_p(x)$ which is equivalent to computing the order of $N_p(z)$ at $z = \infty$. For this notice that (2.4) and (2.9) give

$$\begin{aligned} \Phi_{1,J_1}^{(p)}(z) &= -\left(\left(q^{-p} - 2 + \cdots\right) \cdot \Theta(z)\right) \mid U(4) \\ &= -\left(q^{-p} - 2q^{-p+1} + \cdots\right) \mid U(4) = 2q^{-(p-1)/4} + \cdots. \end{aligned} \tag{3.6}$$

We now use as a set of representatives for the coset space $\Gamma_0(p)\backslash\mathrm{SL}_2(\mathbb{Z})$ the matrices

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & -1 \\ 1 & s \end{pmatrix} \ : \ 0 \le s \le p-1 \right\}.$$

Moreover we have

$$\begin{pmatrix} 0 & -1 \\ 1 & s \end{pmatrix} = W_p \cdot \begin{pmatrix} 1/p & s/p \\ 0 & 1 \end{pmatrix}.$$

Therefore, by (2.4), (2.9), and (3.5) it follows, since $U(p)U(4) = U(4)U(p)$, that

$$\Phi_{1,J_1}^{(p)}(z) \mid W_p = \frac{1}{\sqrt{p}} \cdot \Phi_{1,J_1}^{(p)}(z) \mid U(p) = \frac{2}{\sqrt{p}} + \cdots .$$

Together with (3.6), this implies that

$$N_p^*(z) = q^{-(p-1)/4} + \cdots .$$

Using (3.4), we get

$$F_p(j(z)) = \begin{cases} q^{-(5p-5)/12} + \cdots & \text{if } p \equiv 1 \pmod{12}, \\ q^{-(5p-1)/12} + \cdots & \text{if } p \equiv 5 \pmod{12}. \end{cases}$$

The conclusion about $\deg(F_p(x))$ follows from the fact that $j(z) = q^{-1} + 744 + \cdots$. $\square$

To prove Theorem 1.3, it suffices to compute the factorization of $F_p(x)$ over $\mathbb{Z}[x]$. Loosely speaking, $F_p(x)$ captures the divisor of the modular form $N_p^*(z)$ in $\mathfrak{h}$. To compute the points in the divisor, we shall make use of Theorem 1.2. Since $\eta(z)$ is non-vanishing on $\mathfrak{h}$, the factors of $F_p(x)$ only arise from the zeros of the "norm" of $E_4(pz)$ and of

$$\mathfrak{f}_1(4z)^4\mathfrak{f}_2(z)^2 - \mathfrak{f}_1(4pz)^4\mathfrak{f}_2(pz)^2.$$

To determine these zeros and their corresponding multiplicities, we first recall some classical facts about class numbers. Let $h(-D)$ denote the class number of primitive positive definite binary quadratic forms with discriminant $-D$. These class numbers have the property that $h(-D) = \deg(H_D(x))$. The following well known fact shall play an important role (for example, see page 69 of [7]).

Proposition 3.2. *If $-D_0$ is a fundamental discriminant and $f \geq 1$, then*

$$\frac{h(-D_0 f^2)}{\omega(-D_0 f^2)} = \frac{h(-D_0)}{\omega(-D_0)} \cdot f \prod_{\substack{p \mid f \\ \text{prime}}} \left(1 - \frac{\left(\frac{-D_0}{p}\right)}{p}\right),$$

where $\omega(-D)$ is half the number of units in the imaginary quadratic order of discriminant $-D$.

We shall require the following class number relation to prove Theorem 1.3.

Lemma 3.3. *If $p \equiv 1 \pmod 4$ is prime, then*

$$\sum_{\substack{-2\sqrt{p} < s < 2\sqrt{p} \\ f \mid t(s,p)}} \frac{h\left(\frac{s^2-4p}{16f^2}\right)}{\omega\left(\frac{s^2-4p}{16f^2}\right)} = \frac{p-5}{12},$$

where $t(s,p) := t$ is the largest integer for which $t^2 \mid (s^2 - 4p)$ and $(s^2-4p)/t^2 \equiv 0, 1 \pmod 4$.

Proof. We use the Eichler-Selberg trace formula (for example, see [5], [11]), giving the trace of T_p, where T_p is the usual Hecke operator, on $S_2(\Gamma_0(16))$. This trace is zero, since $S_2(\Gamma_0(16)) = 0$.

The Eichler-Selberg trace formula for this case gives the identity

(3.7)

$$p+1-\frac{1}{p-1}\sum_{f|p-1}\phi((p-1)/f)\cdot c(p+1,f,16,p)-\frac{1}{2}\sum_{\substack{s^2<4p\\ f|t(s,p)}} b(s,f,p)c(s,f,16,p)=0,$$

where ϕ is Euler's totient. The number $b(s,f,p) = h((s^2-4p)/f^2)/\omega((s^2-4p)/f^2)$, and the $c(s,f,16,p)$ count solutions to quadratic congruences modulo powers of 2 (these are explicitly given in [11]). It is straightforward to prove that $c(s,f,16,p)$ is equal to 2, 6, 8, 4, 6 if $\frac{s^2-4p}{f^2} \equiv 1 \pmod 8, 4 \pmod{32}, 16 \pmod{128}, 80 \pmod{128}, 0 \pmod{64}$, respectively. Otherwise it is equal to 0. From this it is straightforward to see that

$$\frac{1}{p-1}\sum_{f|p-1}\phi((p-1)/f)c(p+1,f,16,p)=6.$$

Hence, it suffices to consider the third term of (3.7). We have that

$$\sum_{\substack{s^2<4p\\ f|t(s,p)}} b(s,f,p)c(s,f,16,p)=2(p-5).$$

From the consideration above it follows that $c(0,f,16,p) = 0$. Also, $b(s,f,p)c(s,f,16,p) = b(-s,f,p)c(-s,f,16,p)$ and hence

$$\sum_{\substack{0<s<2\sqrt{p}\\ f|t(s,p)}} b(s,f,p)c(s,f,16,p)=p-5.$$

Now, if we write $t(s,p) = 2^{\alpha(s)}m(s)$ where m is odd and $\alpha \geq 1$, we have

$$\sum_{0<s<2\sqrt{p}}\ \sum_{\substack{0\leq i\leq\alpha(s)\\ g|m(s)}} b(s,2^ig,p)c(s,2^ig,16,p)=p-5.$$

It suffices to prove for all g that

$$\sum_{i=0}^{\alpha} b(s,2^ig,p)c(s,2^ig,16,p)=24\sum_{i=0}^{\alpha-2} h\left(\frac{s^2-4p}{16\cdot 2^{2i}g^2}\right)/\omega\left(\frac{s^2-4p}{16\cdot 2^{2i}g^2}\right).$$

If $i \leq \alpha-3$, we have that that $c(s,2^ig,16,p) = 6$. In this case, Proposition 3.2 implies that $b(s,2^ig,p) = 4h\left(\frac{s^2-4p}{2^{2i}g^2}\right)/\omega\left(\frac{s^2-4p}{2^{2i}g^2}\right)$. This gives

$$b(s,2^ig,p)c(s,2^ig,16,p)=24h\left(\frac{s^2-4p}{2^{2i}g^2}\right)/\omega\left(\frac{s^2-4p}{2^{2i}g^2}\right).$$

Hence, it suffices to consider $\alpha - 2 \leq i \leq \alpha$. These may be argued on a case by case basis. We give the argument when $D = (s^2 - 4p)/(2^{2i} g^2) \equiv 1 \pmod 8$. The other cases are similar and slightly simpler. Suppose that $D \equiv 1 \pmod 8$. If $\alpha = 0$, then $s^2 - 4p \equiv 1 \pmod 8$ and hence s is odd, so $s^2 \equiv 1 + 4p \equiv 5 \pmod 8$, a contradiction. If $\alpha = 1$, then $s^2 - 4p \equiv 4D \equiv 4 \pmod{16}$. Thus, $s^2 \equiv 4 + 4p \equiv 8 \pmod{16}$, a contradiction. Hence, $\alpha \geq 2$. Now, we have that $c(s, 2^\alpha g, 16, p) = 2$, $c(s, 2^{\alpha-1}, 16, p) = 6$, and $c(s, 2^{\alpha-2} g, 16, p) = 8$. We also have by Proposition 3.2 that $b(s, 2^\alpha g, 16, p) = b(s, 2^{\alpha-1} g, 16, p) = h(-D)/\omega(-D)$ and $b(s, 2^{\alpha-2} g, 16, p) = 2h(-D)/\omega(-D)$. Thus,

$$\sum_{i=\alpha-2}^{\alpha} b(s, 2^i g, p) c(s, 2^i g, 16, p) = \frac{24h(-D)}{\omega(-D)} = 24h\left(\frac{s^2-4p}{16\cdot 2^{2\alpha-4}}\right)/\omega\left(\frac{s^2-4p}{16\cdot 2^{2\alpha-4}}\right),$$

as desired. □

Now we determine the factor of $F_p(x)$ arising from $E_4(pz)$.

Proposition 3.4. *If $p \equiv 1 \pmod 4$ is prime, then*

$$F_p(x) = H_{3\cdot p^2}(x) \cdot I_p(x),$$

where $I_p(x) \in \mathbb{Z}[x]$ has

$$\deg(I_p(x)) = \begin{cases} (p-1)/12 & \text{if } p \equiv 1 \pmod{12}, \\ (p-5)/12 & \text{if } p \equiv 5 \pmod{12}. \end{cases}$$

Proof. Since $E_4(\omega) = 0$ for $\omega = e^{2\pi/3} = \frac{-1+\sqrt{-3}}{2}$ it follows that $E_4(pz)$ is zero for $z_p := \omega/p$. Since z_p has discriminant $-3p^2$, by the integrality of $F_p(x)$ and the irreducibility of $H_{3\cdot p^2}(x)$, it follows that $H_{3\cdot p^2}(x) \mid F_p(x)$ in $\mathbb{Z}[x]$. By Proposition 3.2, we have that

$$\deg(H_{3\cdot p^2}(x)) = \begin{cases} (p-1)/3 & \text{if } p \equiv 1 \pmod{12}, \\ (p+1)/3 & \text{if } p \equiv 5 \pmod{12}, \end{cases}$$

and so the claimed formula for $\deg(I_p(x))$ follows from Proposition 3.1. □

In view of Proposition 3.1 and Proposition 3.4, to prove Theorem 1.3 it suffices to determine the polynomial $I_p(x)$. To this end, we begin by observing that $I_p(x)$ is the polynomial which encodes the divisor of the norm of

$$\mathfrak{f}_1(4z)^4 \mathfrak{f}_2(z)^2 - \mathfrak{f}_1(4pz)^4 \mathfrak{f}_2(pz)^2.$$

To study this divisor, we first recall the following modular transformation properties.

Proposition 3.5. *If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $b \equiv c \equiv 0 \pmod 4$ and $g(z) := \mathfrak{f}_1(4z)^4 \mathfrak{f}_2(z)^2$, then $g\left(\frac{az+b}{cz+d}\right) = g(z)$.*

The proof of Theorem 1.3 is complete once we establish the following lemma.

Lemma 3.6. *If $p \equiv 1 \pmod 4$ is a prime, then we have*

$$I_p(x) = H_3(x)^{a(p)} \cdot H_4(x)^{b(p)} \prod_{-D \in \mathcal{D}_p} H_D(x)^2.$$

Proof. By Proposition 3.5, $z \in \mathbb{H}$ is a root of $g(z) - g(pz)$ if $\frac{az+b}{cz+d} = pz$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $b \equiv c \equiv 0 \pmod{4}$. This leads to the quadratic equation

$$\frac{pc}{4}z^2 + \frac{pd-a}{4}z - \frac{b}{4} = 0.$$

In view of Lemma 3.3 and since the Hilbert class polynomials are irreducible, we simply need to show that for a negative discriminant of the form $-D := \frac{x^2-4p}{16f^2}$ with $x, f \in \mathbb{Z}$ there exist two integral binary quadratic forms

$$\begin{aligned} Q_1 &:= \frac{pc_1}{4f}x^2 + \frac{pd_1 - a_1}{4f}xy - \frac{b_1}{4f}y^2 \\ Q_2 &:= \frac{pc_2}{4f}x^2 + \frac{pd_2 - a_2}{4f}xy - \frac{b_2}{4f}y^2, \end{aligned}$$

which are inequivalent under $\Gamma_0(p)$ with discriminants $-D$ such that $\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $b_1 \equiv b_2 \equiv c_1 \equiv c_2 \equiv 0 \pmod{4}$. We can easily show that there exist a_1, a_2, d_1, and d_2 such that $a_1d_1 \equiv a_2d_2 \equiv 1 \pmod{16f^2}$ and $pd_1 + a_1 = -(pd_2 + a_2) = x$. Moreover we can choose a_1, a_2, d_1, and d_2 such that $pd_1 - a_1 \equiv pd_2 - a_2 \equiv 0 \pmod{4f}$. We let $b_1 = b_2 = -4f$, $c_1 = (a_1d_1 - 1)/(4f)$, and $c_2 = (a_2d_2 - 1)/(4f)$. Then $a_1, a_2, b_1, b_2, c_1, c_2, d_1$, and d_2 are integral with $b_1 \equiv c_1 \equiv b_2 \equiv c_2 \equiv 0 \pmod{4}$ and $a_1d_1 - b_1c_1 = a_2d_2 - b_2c_2 = 1$. It is well-known that if $\alpha_1x^2 + \alpha_2xy + \alpha_3y^2$ and $\beta_1x^2 + \beta_2xy + \beta_3y^2$ are two integral primitive binary quadratic forms with $\alpha_3, \beta_3 > 0$, which are equivalent under $\Gamma_0(p)$, then $\alpha_2 \equiv \beta_2 \pmod{p}$. This fact implies that Q_1 and Q_2 are not equivalent under $\Gamma_0(p)$ since $\frac{pd_1-a_1}{4f} \equiv -\overline{4f}x \pmod{p}$ and $\frac{pd_2-a_2}{4f} \equiv \overline{4f}x \pmod{p}$. Here $\overline{4f}$ denotes the inverse of $4f \pmod{p}$. ☐

4. Traces of $J_m(z)$ for $p = 5, 13,$ and 17

In this section, we give formulas for the $K_m^{(p)}(z)$ and prove Theorem 1.4. For $p = 5, 13,$ and 17 and $m \geq 0$ with $\left(\frac{m}{p}\right) \neq -1$ there is a unique

$$K_m^{(p)}(z) = q^{-m} + O(q) \in \mathcal{M}_2^+\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right).$$

Let

$$E_1^{(p)}(z) = \sum_{n=1}^{\infty}\left(\sum_{d|n} d \cdot \left(\frac{n/d}{p}\right)\right)q^n,$$

$$E_2^{(p)}(z) = \frac{L\left(-1, \left(\frac{\cdot}{p}\right)\right)}{2} + \sum_{n=1}^{\infty}\left(\sum_{d|n} d \cdot \left(\frac{d}{p}\right)\right)q^n,$$

be the two Eisenstein series in $M_2\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right)$ corresponding to the cusps 0 and ∞. Here $L(s, \chi)$ denotes the usual Dirichlet series with Dirichlet character χ. Then, we can express $K_0^{(p)}$ as a linear combination of these two Eisenstein series.

$$K_0^{(p)}(z) = \frac{2}{L\left(-1, \left(\frac{\cdot}{p}\right)\right)} \left(E_1^{(p)}(z) + E_2^{(p)}(z)\right). \tag{4.1}$$

In addition, for $p = 5$ and $p = 13$, we can express $K_1^{(p)}(z)$ in terms of $E_1^{(p)}(z)$ and $E_2^{(p)}(z)$.

$$K_1^{(5)}(z) = \frac{25E_2^{(5)}(z)^2 - 55E_1^{(5)}(z)E_2^{(5)}(z)}{E_1^{(5)}(z)}, \tag{4.2}$$

and

$$K_1^{(13)}(z) = \frac{E_2^{(13)}(z)^2 - 3E_1^{(13)}(z)E_2^{(13)}(z)}{E_1^{(13)}(z)}. \tag{4.3}$$

For $p = 17$, we need

$$M_{17}(z) = 1 + \frac{3}{2}\sum_{n=1}^{\infty}(\sigma_1(n) - 17\sigma_1(n/17))q^n$$

the Eisenstein series for $M_2(\Gamma_0(17))$ and

$$S_{17}(z) = q - q^2 - q^4 - 2q^5 + \cdots,$$

the cusp form associated to the elliptic curve $X_0(17)$. Then,

$$K_1^{(17)}(z) = -\frac{E_2^{(17)}(z)M_{17}(z)}{2S_{17}(z)} + \frac{E_2^{(17)}(z)}{4} - \frac{17E_1^{(17)}(z)}{4}. \tag{4.4}$$

All the $K_m^{(p)}(z)$ can be expressed in terms of $K_1^{(p)}(z)$ using Hecke operators. For this description, suppose that $m \geq 1$ and $\left(\frac{m}{p}\right) \neq 1$. Let

$$\tilde{K}_m^{(p)}(z) = q^{-m} + \sum_{n=1}^{\infty} a_m(n)q^n \in \mathcal{M}_2\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right)$$

be the unique form such that $a_m(n) = 0$ if $\left(\frac{n}{p}\right) = 1$. It is straightforward to verify the following facts about the $K_m^{(p)}(z)$ and $\tilde{K}_m^{(p)}(z)$.

Lemma 4.1. *If* $\gcd(m,p) = 1$, *then*

$$\frac{1}{m}\left(K_1^{(p)}(z) \mid T_m\right) = \begin{cases} K_m^{(p)}(z) & \left(\frac{m}{p}\right) = 1, \\ -\tilde{K}_m^{(p)}(z) & \left(\frac{m}{p}\right) = -1, \end{cases}$$

where T_m *is the usual Hecke operator on* $\mathcal{M}_2\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right)$.

Lemma 4.2. *If* $\gcd(m,p) = 1$ *and* $b \geq 1$, *then*

$$\frac{1}{2}\left(K_{mp^b}^{(p)}(z) - \tilde{K}_{mp^b}^{(p)}(z)\right) = \begin{cases} K_m^{(p)}(z) \mid U(p^b) & \left(\frac{m}{p}\right) = 1, \\ -\tilde{K}_m^{(p)}(z) \mid U(p^b) & \left(\frac{m}{p}\right) = -1. \end{cases}$$

Lemma 4.3. *If* $\gcd(m,p)=1$ *and* $b \geq 1$, *then*

$$\frac{1}{2}\left(K^{(p)}_{mp^b}(z)+\tilde{K}^{(p)}_{mp^b}(z)\right) \mid U(p^b) = \begin{cases} K^{(p)}_m(z) & \left(\frac{m}{p}\right)=1,\\ \tilde{K}^{(p)}_m(z) & \left(\frac{m}{p}\right)=-1.\end{cases}$$

Using these lemmas, one can determine the q-expansions of all the $K^{(p)}_m(z)$ and $\tilde{K}^{(p)}_m(z)$ in terms of the q-expansion of $K^{(p)}_1(z)$. Now we prove Theorem 1.4.

Proof of Theorem 1.4. From (3.5) it follows that a weakly holomorphic modular form $f \in \mathcal{M}_2^+(\Gamma_0(p), \left(\frac{\cdot}{p}\right))$ is holomorphic at infinity if and only if it is holomorphic at zero. Since $S_2(\Gamma_0(p), \left(\frac{\cdot}{p}\right)) = 0$ for $p = 5, 13$ and 17, this implies that $f \in \mathcal{M}_2^+(\Gamma_0(p), \left(\frac{\cdot}{p}\right))$ is determined by its principal part and constant term.

The principal part of $\Phi^{(p)}_{1,J_m}(z)$ comes from that of $A^{(p)}_{1,J_m}(z)$ by (1.5). The definitions of $J_m(z)$ and $A^{(p)}_{\ell,f}(z)$ imply that

$$\begin{aligned} A^{(p)}_{1,J_m}(z) &= -\frac{1}{2}\sum_{d|m} d\left(\sum_{\substack{x^2\equiv d^2p \pmod 4\\ x^2<d^2p}} q^{\frac{x^2-d^2p}{4}} + \sum_{\substack{x\equiv d \pmod 2\\ x^2<d^2p}} q^{\frac{x^2-d^2p}{4}}\right)\\ &= -\sum_{d|m} d \sum_{\substack{x\equiv d \pmod 2\\ x^2<d^2p}} q^{\frac{x^2-d^2p}{4}}. \end{aligned}$$

It coincides with the principal term of the right hand side of the identity in Theorem 1.4.

Similarly, the constant term of $\Phi^{(p)}_{1,J_m}(z)$ comes from that of $B^{(p)}_{1,J_m}(z)$ and is clearly $2\sigma_1(m)$. Again, this coincides with the constant term on the right hand side of the identity in Theorem 1.4. This proves the theorem. $\square$

References

[1] R. E. Borcherds, *Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products*, Invent. Math. **120** (1995), pages 161-213.

[2] J. H. Bruinier, *Borcherds products on $O(2,\ell)$ and Chern classes of Heegner divisors*, Springer Lect. Notes, **1780**, Springer-Verlag, Berlin, 2002.

[3] J. H. Bruinier and M. Bundschuh, *On Borcherds products associated with lattices of prime discriminant*, Ramanujan J. **7**, (2003), pages 49-61.

[4] J. H. Bruinier and J. Funke, *Traces of CM-values of modular functions*, preprint.

[5] H. Hijikata, A. Pizer, and T. Shemanske, *The basis problem for modular forms on $\Gamma_0(N)$*, Proc. Japan Acad. Ser. A Math. Sci. **56**(6) (1980), pages 280-284.

[6] F. Hirzebruch, *Hilbert modular surfaces*, L'Enseign. Math. **19** (1973), pages 183-281.

[7] F. Hirzebruch and D. Zagier, *Intersection numbers of curves on Hilbert modular surfaces and modular forms with Nebentypus*, Invent. Math. **36** (1976), pages 57-113.

[8] W. Kohnen, *Newforms of half-integral weight,* J. Reine Angew. Math. **333** (1982), pages 32–72.

[9] W. C. Li, *Newforms and functional equations*, Math. Ann. **212** (1975), pages 285-315.

[10] K. Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conference Series in Mathematics, No. 102, Amer. Math. Soc., Providence, R.I., 2004.

[11] J. Rouse, *Vanishing and Non-Vanishing of Traces of Hecke Operators*, to appear in Trans. Amer. Math. Soc.
[12] G. Shimura, *On modular forms of half integral weight*, Ann. of Math. (2) **97** (1973), pages 440-481.
[13] D. Zagier, *Traces of singular moduli*, Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998) (2002), Int. Press Lect. Ser., 3, I, Int. Press, Somerville, MA, pages 211–244.

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