DIFFERENTIAL ENDOMORPHISMS FOR MODULAR FORMS ON $\Gamma_0(4)$

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Proceedings of Gainesville 1999 Conference.

Abstract. In this note we construct a graded family of endomorphisms on $M_k(\Gamma_0(4), \chi_k)$ using differential operators. These operators are analogous to those constructed by Kaneko and Zagier in the case of modular forms on $\Gamma_0(1)$. We study the arithmetic of these operators. For example, we classify their eigenforms and their kernels.

1. Introduction and Statement of Results

Kaneko and Zagier [K-Z] studied hypergeometric modular forms whose properties are connected to supersingular j-invariants. Recent work by Kaneko and Todaka [K-T] includes a new family of such modular forms. A common thread between these papers is the construction of endomorphisms on spaces of modular forms $M_k(\Gamma_0(1))$, for certain weights k, which are defined using differential operators.

In this simple note we study a similar construction for the spaces of modular forms $M_k(\Gamma_0(4), \chi_k)$ where $k \in \frac{1}{2}\mathbb{N}$ (see [K] for definitions) and

$$\chi_k := \begin{cases} \chi_{triv} & \text{if } k \in 2\mathbb{Z} \text{ or } k \in \frac{1}{2} + \mathbb{Z}, \\ \chi_{-4} & \text{if } k \in 1 + 2\mathbb{Z}. \end{cases} \tag{1}$$

Key words and phrases. Differential operators, Modular forms..

The author is supported by an NSF grant, an Alfred P. Sloan Research Fellowship, and a David and Lucile Packard Research Fellowship.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

We construct, uniformly in k, endomorphisms on $M_k(\Gamma_0(4), \chi_k)$. We characterize some of their intrinsic properties which may be of interest to those working on the combinatorial aspects of q-series and hypergeometric functions.

We shall identify a modular form $f(z) \in M_k(\Gamma_0(4), \chi_k)$ with its Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n \qquad (q := e^{2\pi iz} \text{ throughout}).$$

For convenience, let d denote the *differential operator* defined by

$$d\left(\sum_{n=0}^{\infty} a(n)q^n\right) = \sum_{n=0}^{\infty} na(n)q^n = \frac{1}{2\pi i} \cdot \frac{d}{dz}\left(\sum_{n=0}^{\infty} a(n)e^{2\pi inz}\right). \tag{2}$$

If $k \in \frac{1}{2}\mathbb{N}$, then let C_k denote the map

$$C_k : M_k(\Gamma_0(4), \chi_k) \longrightarrow \mathbb{C}[[q]]$$

given by

$$C_k(f) := \frac{(k+1/2)(f \cdot d(\theta)) - \frac{1}{2}d(f\theta)}{F\theta}, \tag{3}$$

where

$$\theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots,$$

and

$$F(z) = \sum_{n=0}^{\infty} \sigma_1(2n+1)q^{2n+1}.$$

Theorem 1. *If $k \in \frac{1}{2}\mathbb{N}$, then the map C_k*

$$C_k : M_k(\Gamma_0(4), \chi_k) \longrightarrow M_k(\Gamma_0(4), \chi_k)$$

is a $\mathbb{C}$-linear endomorphism. Moreover, the kernel of C_k is

$$\ker(C_k) = \mathbb{C}\theta^{2k}.$$

It is natural to characterize the eigenforms of the C_k. They are values of terminating hypergeometric functions in terms of the modular forms $F(z)$ and $\theta(z)$, which are defined above. Recall that the classical hypergeometric series ${}_pF_q\left(\begin{matrix}\alpha_1, & \alpha_2 & \dots & \alpha_p \\ & \beta_1, & \dots & \beta_q\end{matrix} \mid x\right)$ is defined by

(4)

$$ {}_pF_q\left(\begin{matrix}\alpha_1, & \alpha_2, & \dots & \alpha_p, \\ & \beta_1, & \dots & \beta_q\end{matrix} \mid x\right) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n(\alpha_3)_n\cdots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\cdots(\beta_q)_n n!} x^n $$

where

$$ (\gamma)_n := \begin{cases} 1 & \text{if } n = 0, \\ \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1) & \text{if } n \geq 1. \end{cases} $$

Definition. *A modular form $f \in M_k(\Gamma_0(4), \chi_k)$ is a normalized eigenform if*

1. *The first non-zero coefficient in the q-series expansion is 1.*
2. *There is a complex number λ_f for which*

$$ C_k = \lambda_f f. $$

Theorem 2. *If $k \in \frac{1}{2}\mathbb{N}$, then there are exactly*

$$ e := \dim(M_k(\Gamma_0(4), \chi_k) - 1 $$

normalized eigenforms, say $f_{1,k}, f_{2,k}, \dots f_{e,k}$, of the endomorphism C_k. If $e \geq 1$, then for all $1 \leq m \leq e$ we have

$$ (i) \qquad f_{m,k} := \left(1 - \frac{16F}{\theta^4}\right)^m \cdot \theta^{2k} = {}_1F_0\left(-m \mid \frac{16F}{\theta^4}\right) \cdot \theta^{2k}, $$

$$ (ii) \qquad C_k(f_{m,k}) = 8m f_{m,k}. $$

It is natural to ask which cusp forms are in the image of C_k. We prove:

Theorem 3. *If $k \in \frac{1}{2}\mathbb{N}$, then we have*

$$ S_k(\Gamma_0(4), \chi_k) \subseteq \mathrm{Im}(C_k). $$

2. Preliminaries

We consider the spaces $M_k(\Gamma_0(4), \chi_k)$ and $S_k(\Gamma_0(4), \chi_k)$ where $k \in \frac{1}{2}\mathbb{N}$. We recall the following well known proposition (see [C]).

Proposition 2.1. *Using the notation above, we have the following:*

(1)

$$\dim(M_k(\Gamma_0(4), \chi_k) = \begin{cases} 0 & \text{if } k < 0, \\ 1 + [k/2] & \text{if } k \geq 0. \end{cases}$$

$$\dim(S_k(\Gamma_0(4), \chi_k) = \begin{cases} 0 & \text{if } k \leq 2, \\ [k/2] - 1 & \text{if } k > 2 \text{ and } k \notin 2\mathbb{Z}, \\ [k/2] - 2 & \text{if } k > 2 \text{ and } k \in 2\mathbb{Z}. \end{cases}$$

(2) *As a graded algebra, we have*

$$\oplus_{k \in \frac{1}{2}\mathbb{Z}} M_k(\Gamma_0(4), \chi_k) \tilde{=} \mathbb{C}[F, \theta].$$

In [C], Cohen proved that certain bilinear forms in derivatives of modular forms are again modular forms. The following is a special case of [Th. 7.1, C]:

Theorem 2.2. *For $i = 1$ and 2 suppose that $f_i(z) \in M_{k_i}(\Gamma_0(4), \chi_{k_i})$. The function*

$$C(f_1, f_2; z) := (k_1 + k_2)(f_2 \cdot d(f_1)) - k_1 d(f_2 \cdot f_1)$$

is a modular form in $S_{k_1+k_2+2}(\Gamma_0(4), \chi_{k_1+k_2+2})$.

3. Proof of results

Recall the definition of C_k from (3).

Theorem 3.1. *If a and b are non-negative integers, then*

$$C_{2a+\frac{b}{2}}(F^a\theta^b) = \begin{cases} 0 & \text{if } a = 0, \\ -\frac{a}{2}F^{a-1}\theta^{b+4} + 8aF^a\theta^b & \text{if } a > 0. \end{cases}$$

Proof. It suffices to prove that

$$C(\theta, F^a\theta^b; z) = \begin{cases} 0 & \text{if } a = 0, \\ -\frac{a}{2}F^a\theta^{b+5} + 8aF^{a+1}\theta^{b+1} & \text{if } a > 0. \end{cases} \tag{5}$$

Using Theorem 2.2 and the standard rules for differentiation, it is simple to verify that if a is a positive integer and b and c are non-negative integers, then

$$C(\theta, F^b\theta^{c+1}; z) = \theta C(\theta, F^b\theta^c; z), \tag{6}$$
$$C(\theta, F^a\theta^b; z) = aF^{a-1}C(\theta, F\theta^b; z). \tag{7}$$

It suffices to compute $C(\theta, \theta; z) \in M_3(\Gamma_0(4), \chi_{-4})$ and $C(\theta, F; z) \in M_{9/2}(\Gamma_0(4), \chi_{9/2})$, and one easily verifies that

$$C(\theta, \theta; z) = 0,$$
$$C(\theta, F; z) = -\frac{1}{2}F\theta^5 + 8F^2\theta.$$

These identities together with (6) and (7) implies (5).

□

Proof of Theorem 1. At the outset, there is no guarantee that $C_k(f)$ is a modular form for each $f \in M_k(\Gamma_0(4), \chi_k)$. However, Theorem 3.1 and Proposition 2.1 indeed implies that the image of C_k is in $M_k(\Gamma_0(4), \chi_k)$. The $\mathbb{C}$-linearity follows immediately from

$$\begin{aligned} C_k(\alpha f_1 + \beta f_2) &= \frac{\left(k + \frac{1}{2}\right)\left((\alpha f_1 + \beta f_2) \cdot d(\theta)\right) - \frac{1}{2}d((\alpha f_1 + \beta f_2)\theta)}{F\theta} \\ &= \frac{(k + \frac{1}{2})\alpha f_1 d(\theta)}{F\theta} + \frac{(k + \frac{1}{2})\beta f_2 d(\theta)}{F\theta} - \alpha\frac{\frac{1}{2}d(f_1\theta)}{F\theta} - \beta\frac{\frac{1}{2}d(f_2\theta)}{F\theta} \\ &= \alpha C_k(f_1) + \beta C_k(f_2). \end{aligned}$$

□

Now we characterize the eigenforms of the endomorphisms C_k.

Proof of Theorem 2. Since the $\ker(C_k) = \mathbb{C}\theta^{2k}$ is one dimensional, there are at most e distinct normalized eigenforms. Using the formulas in Theorem 3.1, it is simple to verify that these e forms are indeed eigenforms with the designated eigenvalue.

□

We conclude with a study of those cusp forms which are in the image of the differential operators C_k. We recall the following important proposition.

Proposition 3.2. *The fundamental domain of $\mathfrak{h}/\Gamma_0(4)$ has 3 cusps, and canonical representatives are $\{0, -\frac{1}{2}, \infty\}$. The values of F and θ at these three points are*

$$F(\infty) = 0, \quad F(0) = -\frac{1}{64}, \quad F(-1/2) = \frac{1}{16},$$
$$\theta(\infty) = 1, \quad \theta(0) = \frac{1-i}{2}, \quad \theta(-1/2) = 0.$$

As an immediate corollary we obtain:

Corollary 3.3. *Suppose that $k \in \frac{1}{2}\mathbb{N}$ and $f \in M_k(\Gamma_0(4), \chi_k)$ has the following canonical decomposition*

$$f = \sum_{j=0}^{[k/2]} \alpha_j F^j \theta^{2k-4j}.$$

Then f is a cusp form if and only if

$$(i) \qquad \alpha_0 = 0,$$
$$(ii) \qquad \alpha_{k/2} = 0 \quad \textit{when } k \in 2\mathbb{Z},$$
$$(iii) \qquad \sum_{j=0}^{[k/2]} \alpha_j \, (1/16)^j = 0.$$

The next theorem proves Theorem 3.

Theorem 3.4. *If $k \in \frac{1}{2}\mathbb{N}$ and j is a positive integer, then*

$$S_k(\Gamma_0(4), \chi_k) \subseteq \mathrm{Im}(C_k).$$

Moreover, every $f \in \mathrm{Im}(C_k)$ has the property that $f(z_0) = 0$ for

$$z_0 \in \begin{cases} \{0, -\frac{1}{2}\} & \text{if } k \notin 2\mathbb{Z}, \\ \{0\}, & \text{otherwise.} \end{cases}$$

Proof. By Theorem 3.1, we see that $\mathrm{Im}(C_k)$ is generated by the modular forms

$$\begin{aligned} &\theta^{2k} - 16F\theta^{2k-4}, \\ &\theta^{2k-4}F - 16F^2\theta^{2k-8}, \\ &\qquad\vdots \end{aligned}$$

The last generator is of the form $\theta^4 F^{a-1} - 16F^a$ if and only if $k = 2a$ is an even integer.

By Corollary 3.3, it is easy to see that the first form is not a cusp form. However, all the remaining forms (with the exception of the last one when $k \in 2\mathbb{N}$) is a cusp form. This follows from Corollary 3.3 (*iii*). It is easy to see that the number of cusp forms equals the dimension of $S_k(\Gamma_0(4), \chi_k)$ from Proposition 2.1. Moreover, it clear that these forms are linearly independent over $\mathbb{C}$.

The conclusion about the z_0 follows immediately by noting that $\theta^a - 16F\theta^{a-4}$ vanishes at $z = 0$ and $-\frac{1}{2}$ while the form $F^{a-1}\theta^4 - 16F^a$ vanishes at $z = 0$.

□

Corollary 3.5. *Suppose that $k \in \frac{1}{2}\mathbb{N}$ and that $f \in M_k(\Gamma_0(4), \chi_k)$ has canonical decomposition*

$$f = \sum_{j=0}^{[k/2]} \alpha_j F^j \theta^{2k-4j}.$$

1. *If $k \notin 2\mathbb{Z}$, then $C_k(f) \in S_k(\Gamma_0(4), \chi_k)$ if and only if $\alpha_1 = 0$.*
2. *If $k \in 2\mathbb{Z}$, then $C_k(f) \in S_k(\Gamma_0(4), \chi_k)$ if and only if $\alpha_1 = \alpha_{k/2} = 0$.*

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