THE PARTITION FUNCTION AND THE ARITHMETIC OF CERTAIN MODULAR L-FUNCTIONS

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1. Introduction and Statement of Results

A *partition* of a positive integer n is any non-increasing sequence of positive integers whose sum is n. As usual, let $p(n)$ denote the number of partitions of size n. One of the fundamental tools used for studying $p(n)$ is Euler's generating function

$$\sum_{n=0}^{\infty} p(n)q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 \cdots = \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \tag{1}$$

(Note: By convention we have that $p(0) = 1$.)

It is well known that partitions and their associated Ferrers-Young diagrams and tableaux play an important role in the study of hypergeometric functions, combinatorics, representation theory, Lie algebras, and statistical mechanics. In some cases the combinatorial properties of the partitions and Ferrers-Young diagrams are important, while in others the content of the theorems rely on identities involving relevant q-series generating functions. In this investigation we show that the values of the partition function $p(n)$, viewed as q-coefficients, play a key role in the arithmetic of several infinite families of modular L-functions. In particular, this suggests that there is a 'correspondence' (in these special cases) between Tate-Shafarevich groups of certain motives of modular forms and sets of partitions.

We begin by fixing notation. If $\ell \geq 5$ is prime, then let $1 \leq \delta_\ell \leq \ell - 1$ and $1 \leq r_\ell \leq 23$ be the unique pair of integers for which

$$24\delta_\ell \equiv 1 \pmod{\ell}, \tag{2}$$

$$r_\ell \equiv -\ell \pmod{24}, \tag{3}$$

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and let $D(\ell, n)$ denote the number given by

$$D(\ell, n) \stackrel{\text{def}}{=} (-1)^{\frac{\ell-3}{2}}(24n + r_\ell). \tag{4}$$

Ramanujan proved that if $\ell = 5, 7$ or 11, then

$$p(\ell n + \delta_\ell) \equiv 0 \pmod{\ell} \tag{5}$$

for every non-negative integer n. There are now many proofs of these congruences which use a variety of techniques. Most notable are those proofs by A. O. L. Atkin and H. P. F. Swinnerton-Dyer [At-SwD], F. Garvan [G], G. E. Andrews and F. Garvan [An-G], and D. Kim, F. Garvan and D. Stanton [G-Ki-St], which are motivated by conjectures of F. Dyson [Dy] on the existence and behavior of combinatorial statistics known as 'ranks' and 'cranks'. These statistics, by design, describe the divisibility of $p(n)$ in terms of the combinatorial structure of the partitions themselves.

If $13 \leq \ell \leq 31$ is prime, then we show that the residue classes $p(\ell n + \delta_\ell) \pmod{\ell}$ play an important role in the arithmetic of certain L-functions. For these primes ℓ, let $G_\ell(z)$ denote the unique newform in the space $S_{\ell-3}(\Gamma_0(6), \chi_1)$ whose Fourier expansion at infinity (see Appendix) begins with the terms

$$G_\ell(z) = \sum_{n=1}^{\infty} a_\ell(n) q^n := q + \left(\frac{2}{\ell}\right) \cdot 2^{\frac{\ell-5}{2}} q^2 + \left(\frac{3}{\ell}\right) \cdot 3^{\frac{\ell-5}{2}} q^3 + \dots. \qquad (\text{here } q := e^{2\pi i z}) \tag{6}$$

Here χ_1 denotes the trivial character and $\left(\frac{\cdot}{\ell}\right)$ denotes the Legendre symbol modulo ℓ.

If D is a fundamental discriminant of a quadratic number field, then let $L(G_\ell \otimes \chi_D, s)$ denote the L-function associated to the D-quadratic twist of $G_\ell(z)$. If D is coprime to 6 and $\chi_D(\cdot) := \left(\frac{D}{\cdot}\right)$ denotes the Kronecker character for $\mathbb{Q}(\sqrt{D})$, then for $\mathrm{Re}(s) > \frac{\ell-2}{2}$ we have that

$$L(G_\ell \otimes \chi_D, s) = \sum_{n=1}^{\infty} \frac{\chi_D(n) a_\ell(n)}{n^s}.$$

If $n \geq 0$ is an integer for which $D(\ell, n)$ is square-free, then it turns out that

$$\frac{L\left(G_\ell \otimes \chi_{D(\ell,n)}, \frac{\ell-3}{2}\right)(24n + r_\ell)^{\frac{\ell-4}{2}}}{L\left(G_\ell \otimes \chi_{D(\ell,0)}, \frac{\ell-3}{2}\right) r_\ell^{\frac{\ell-4}{2}}}$$

is the square of an integer. The first result in this paper is a congruence between these quotients and the square of a quotient of values of the partition function.

Theorem 1. *If $13 \leq \ell \leq 31$ is prime and $n \geq 0$ is an integer for which $D(\ell, n)$ is square-free, then*

$$\frac{L\left(G_\ell \otimes \chi_{D(\ell,n)}, \frac{\ell-3}{2}\right)(24n + r_\ell)^{\frac{\ell-4}{2}}}{L\left(G_\ell \otimes \chi_{D(\ell,0)}, \frac{\ell-3}{2}\right) r_\ell^{\frac{\ell-4}{2}}} \equiv \frac{p(\ell n + \delta_\ell)^2}{p(\delta_\ell)^2} \pmod{\ell}.$$

There have been many important general non-vanishing theorems for the critical values of quadratic twists of modular L-functions by the works of D. Bump, S. Friedberg, J. Hoffstein, H. Iwaniec, N. Katz, M. R. Murty, V. K. Murty, P. Sarnak, among many others. Theorem 1 yields the following curious non-vanishing theorem for these special L-functions.

Corollary 2. *If $13 \leq \ell \leq 31$ is prime, $n \geq 0$ is an integer for which $D(\ell, n)$ is square-free and $p(\ell n + \delta_\ell) \not\equiv 0 \pmod{\ell}$, then $L\left(G_\ell \otimes \chi_{D(\ell,n)}, \frac{\ell-3}{2}\right) \neq 0$.*

Corollary 2 provides a useful criterion for deducing the non-vanishing of these central critical values. To place it in its proper context, we recall that a well known conjecture due to D. Goldfeld [Go] implies that if $F \in S_{2k}(M, \chi_1)$ is a newform and D denotes the fundamental discriminant of the quadratic field $\mathbb{Q}(\sqrt{D})$, then

$$\#\{|D| \leq X \ : \ L(F \otimes \chi_D, k) \neq 0\} \gg X. \tag{7}$$

Throughout, the notation $L(X) \gg R(X)$ shall mean that there is a positive constant c such that for sufficiently large X we have $L(X) \geq c \cdot R(X)$. Although N. Katz and P. Sarnak [Ka-Sa] have conditional proofs of (7), at present the best general result is due to C. Skinner and the second author [O-S]. They prove that if $F(z) \in S_{2k}(M, \chi_1)$ is a newform, then

$$\#\{|D| \leq X \ : \ L(F \otimes \chi_D, k) \neq 0\} \gg \frac{X}{\log X}.$$

To observe the utility of Corollary 2, consider the function $\Psi(\ell, X)$ given by

$$\Psi(\ell, X) \stackrel{\text{def}}{=} \frac{\#\{0 \leq n \leq X \ : \ D(\ell, n) \text{ square-free and } p(\ell n + \delta_\ell) \not\equiv 0 \pmod{\ell}\}}{\#\{0 \leq n \leq X \ : \ D(\ell, n) \text{ square-free}\}}.$$

This function denotes the proportion of non-negative integers $n \leq X$ for which Corollary 2 implies the nonvanishing of $L\left(G_\ell \otimes \chi_{D(\ell,n)}, \frac{\ell-3}{2}\right)$. The authors are indebted to R. Weaver for compiling the following table.

X	$\Psi(13, X)$	$\Psi(17, X)$	$\Psi(19, X)$	$\Psi(23, X)$	$\Psi(29, X)$	$\Psi(31, X)$
50000	0.900722	0.937501	0.945341	0.950661	0.964081	0.968021
150000	0.905447	0.938867	0.944167	0.950140	0.962894	0.967267
250000	0.907372	0.938756	0.943944	0.950308	0.963168	0.967296

It is well known (for example, see [Fr]) that if $E/\mathbb{Q}$ is an elliptic curve with a rational point of odd prime order ℓ, then there is a close relationship between the ℓ-Selmer groups of E_D, the D-quadratic twist of E, and the ℓ-part of the ideal class group of $\mathbb{Q}(\sqrt{D})$. By a theorem of Mazur, this only occurs for primes $\ell \leq 7$. In general, apart from some modest results by the second author and W. Kohnen and K. James [Ko-O, J-O], very little is known about the distribution of elements of odd prime order in the Tate-Shafarevich groups of a family of quadratic twists E_D.

In view of Theorem 1 and the Bloch-Kato conjecture, it is natural to suspect that there is a close relationship between the values of the partition function modulo ℓ, for $13 \leq \ell \leq 31$, and the orders of the Tate-Shafarevich groups of a quadratic twists of certain motives. Here we show that there is indeed such a relation assuming the truth of the Bloch-Kato conjecture.

If $13 \leq \ell \leq 31$ is prime, then let $M^{(\ell)}$ be the $\frac{\ell-3}{2}$-th Tate twist of the motive $G^{(\ell)}$ associated to the newform G_ℓ by the work of A. Scholl [Sch]. Let $\mathcal{M}^{(\ell)}$ be any fixed integral structure of $M^{(\ell)}$ which satisfies property (P2) at ℓ (see section 3 for definitions and a 'canonical choice'). If n is a non-negative integer, then let $D^{(\ell,n)}$ denote the motive associated to the Dirichlet character $\chi_{D(\ell,n)}$, and let $M^{(\ell,n)}$ denote the twisted motive

$$M^{(\ell,n)} \stackrel{\text{def}}{=} M^{(\ell)} \otimes_{\mathbb{Q}} D^{(\ell,n)} \tag{8}$$

with the integral structure $\mathcal{M}^{(\ell,n)}$ induced by $\mathcal{M}^{(\ell)}$. Let $Ш(\mathcal{M}^{(\ell,n)})$ denote the Tate-Shafarevich group of $M^{(\ell,n)}$ with respect to the integral structure $\mathcal{M}^{(\ell,n)}$.

Theorem 3. *Suppose that $13 \leq \ell \leq 31$ is prime and $n \geq 0$ is an integer for which*

$$\begin{aligned} &(i) \quad n \not\equiv -\left[\frac{\ell+1}{12}\right] \pmod{\ell}, \\ &(ii) \quad D(\ell,n) \text{ is square-free}, \\ &(iii)\ L\left(G_\ell \otimes \chi_{D(\ell,n)}, \frac{\ell-3}{2}\right) \neq 0. \end{aligned}$$

Assume the truth of the Bloch-Kato Conjecture for $M^{(\ell,n)}$ and $M^{(\ell,0)}$. Then we have that

$$\begin{aligned} &(a) \quad \mathrm{ord}_\ell\left(\frac{\#Ш(\mathcal{M}^{(\ell,n)})}{\#Ш(\mathcal{M}^{(\ell,0)})}\right) \geq 0, \\ &(b) \quad \mathrm{ord}_\ell\left(\frac{\#Ш(\mathcal{M}^{(\ell,n)})}{\#Ш(\mathcal{M}^{(\ell,0)})}\right) > 0 \iff p(\ell n + \delta_\ell) \equiv 0 \pmod{\ell}. \end{aligned}$$

(*Note:* $[\cdot]$ *denotes the greatest integer function.*)

The Bloch-Kato Conjecture is a precise formula for certain special values of L-function in terms of periods, Tamagawa numbers, and the order of a Tate-Shafarevich group. To obtain Theorem 3 from Theorem 1, it suffices to show that the only factor in the Bloch-Kato formula which might not be an ℓ-adic unit is the term corresponding to the order of the Tate-Shafarevich group. This argument is carried out in §3 and is analogous to results for modular elliptic curves appearing in [J, J-O].

A straightforward generalization of the argments in [J-O, Ko-O] yields the following result.

Corollary 4. *Let $13 \leq \ell \leq 31$ be prime. If the Bloch-Kato Conjecture holds for $M^{(\ell,0)}$ and $M^{(\ell,n)}$ for every non-negative integer n satisfying conditions $(i - iii)$ in Theorem 3, then*

$$\#\left\{n \leq X \ : \ \mathrm{ord}_\ell(\#\text{Ш}(\mathcal{M}^{(\ell,n)})) = \mathrm{ord}_\ell(\#\text{Ш}(\mathcal{M}^{(\ell,0)}))\right\} \gg_\ell \frac{\sqrt{X}}{\log X}.$$

Theorem 3 implies that if the ℓ-part of $\text{Ш}(\mathcal{M}^{(\ell,0)})$ is non-trivial, then the ℓ-part of $\text{Ш}(\mathcal{M}^{(\ell,n)})$ is non-trivial for every n satisfying $(i - iii)$. Since it is unlikely that this is ever the case, we record the following corollary.

Corollary 5. *Let $13 \leq \ell \leq 31$ be prime and suppose that $n \geq 0$ is an integer satisfying the conditions in Theorem 3. If the Bloch-Kato Conjecture holds for $M^{(\ell,n)}$ and $M^{(\ell,0)}$ and the ℓ-part of $\text{Ш}(\mathcal{M}^{(\ell,0)})$ is trivial, then*

$$\ell \mid \#\text{Ш}(\mathcal{M}^{(\ell,n)}) \iff p(\ell n + \delta_\ell) \equiv 0 \pmod{\ell}.$$

L. Sze and the second author [O-Sz] have defined a correspondence between 4-core partitions of size n and ideal class groups of the imaginary quadratic fields $\mathbb{Q}(\sqrt{-8n-5})$. In view of this correspondence and recent observations by J. Cremona and B. Mazur [Cr-Ma] on the 'visualization' of elements of Tate-Shafarevich groups of elliptic curves, it is natural to raise the following questions:

Questions. 1) Is there a 'correspondence' explaining Theorem 3 and Corollary 5?
2) Do generalizations of ranks and cranks reveal structural properties of $\text{Ш}(\mathcal{M}^{(\ell,n)})$?

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2. Proof of Theorem 1

The proof of Theorem 1 depends on theorems of Shimura and Waldspurger which relate the central critical values of quadratic twists of weight $2k$ modular L-functions to the Fourier coefficients of certain weight $k + \frac{1}{2}$ cusp forms. We begin by defining the relevant half integral weight forms. Recall that Dedekind's eta-function is defined by the infinite product

$$\eta(z) \overset{\text{def}}{=} q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{9}$$

It is well known that $\eta(24z)$ is a weight 1/2 cusp form in the space $S_{\frac{1}{2}}(\Gamma_0(576), \chi_{12})$. Furthermore, let $E_4(z)$ and $E_6(z)$ be the usual weight 4 and 6 Eisenstein series for $SL_2(\mathbb{Z})$

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,$$

where $\sigma_\nu(n) := \sum_{1\leq d|n} d^\nu$.

For each prime $13 \leq \ell \leq 31$, define the modular form $g_\ell(z) = \sum_{n=0}^{\infty} b_\ell(n)q^n \in \mathbb{Z}[[q]]$ by

$$g_\ell(z) = \sum_{n=1}^{\infty} b_\ell(n)q^n \stackrel{\text{def}}{=} \begin{cases} \eta^{11}(24z) & \text{if } \ell = 13, \\ \eta^{7}(24z)E_4(24z) & \text{if } \ell = 17, \\ \eta^{5}(24z)E_6(24z) & \text{if } \ell = 19, \\ \eta(24z)E_4(24z)E_6(24z) & \text{if } \ell = 23, \\ \eta^{19}(24z)E_4(24z) & \text{if } \ell = 29, \\ \eta^{17}(24z)E_6(24z) & \text{if } \ell = 31. \end{cases} \tag{10}$$

Note that if $n \not\equiv r_\ell \pmod{24}$, then $b_\ell(n)$ is zero.

Proposition 6. *If $13 \leq \ell \leq 31$ is prime, then $g_\ell(z)$ is in the space $S_{\frac{\ell-2}{2}}(\Gamma_0(576), \chi_{12})$. Moreover, $g_\ell(z)$ is an eigenform of the half integral weight Hecke operators and its image under the Shimura correspondence is $G_\ell \otimes \chi_{12}$, a newform in the space $S_{\ell-3}(\Gamma_0(144), \chi_1)$.*

Proof. That each cusp form g_ℓ lies in the space $S_{\frac{\ell-2}{2}}(\Gamma_0(576), \chi_{12})$ follows immediately from the fact that $\eta(24z)$ is a cusp form in $S_{1/2}(\Gamma_0(576), \chi_{12})$, and the assertion that each g_ℓ is an eigenform is easily verified by a straightforward case by case computation (for example, see [Prop. 1.3, Fr2]).

If $F_\ell(z)$ denotes the image of g_ℓ under the Shimura correspondence [Sh], then by a theorem of Niwa [Ni] it turns out that $F_\ell(z)$ is an eigenform in the space $S_{\ell-3}(\Gamma_0(288), \chi_1)$. However, one easily checks (see Appendix) that the initial segments of the q-expansions of F_ℓ and $G_\ell \otimes \chi_{12}$ agree for more than $(\ell-3)[\Gamma_0(1) : \Gamma_0(288)]/12$ terms. Consequently, by the standard dimension counting argument we have that $F_\ell = G_\ell \otimes \chi_{12}$. It is simple to check that $G_\ell \otimes \chi_{12}$ is a newform in the space $S_{\ell-3}(\Gamma_0(144), \chi_1)$.

Q.E.D.

By Waldspurger's work on the Shimura correspondence [Wal], it turns out that many of the coefficients of the cusp forms g_ℓ 'interpolate' the central values of certain quadratic twists. More precisely, we have the following proposition.

Proposition 7. *If $13 \leq \ell \leq 31$ is prime and $n \geq 0$ is an integer for which $D(\ell, n)$ is square-free, then*

$$\frac{L\left(G_\ell \otimes \chi_{D(\ell,n)}, \frac{\ell-3}{2}\right)(24n + r_\ell)^{\frac{\ell-4}{2}}}{L\left(G_\ell \otimes \chi_{D(\ell,0)}, \frac{\ell-3}{2}\right) r_\ell^{\frac{\ell-4}{2}}} = b_\ell(24n + r_\ell)^2.$$

Proof. This follows from Proposition 6, [Cor. 2, Wal], and the fact that $b_\ell(r_\ell) = 1$.

Q.E.D.

Proof of Theorem 1. To prove the theorem it suffices to prove that

$$\sum_{n=0}^{\infty} p(\ell n+\delta_\ell)q^{24n+r_\ell} \equiv \begin{cases} 11g_{13}(z) \pmod{13} & \text{if } \ell=13,\\ 7g_{17}(z) \pmod{17} & \text{if } \ell=17,\\ 5g_{19}(z) \pmod{19} & \text{if } \ell=19,\\ g_{23}(z) \pmod{23} & \text{if } \ell=23,\\ 8g_{29}(z) \pmod{29} & \text{if } \ell=29,\\ 10g_{31}(z) \pmod{31} & \text{if } \ell=31. \end{cases} \tag{11}$$

Recall that if M is a positive integer, then the $U(M)$ operator is defined by

$$\sum_{n=0}^{\infty} A(n)q^n \mid U(M) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} A(Mn)q^n. \tag{12}$$

Using Euler's generating function from (1), one easily finds that

$$\begin{aligned} \frac{\eta^{\ell}(24\ell z)}{\eta(24z)} \mid U(\ell) &= \left\{ \sum_{n=0}^{\infty} p(n)q^{24n+\ell^2-1} \cdot \prod_{n=1}^{\infty}(1-q^{24\ell n})^{\ell} \right\} \mid U(\ell)\\ &= \sum_{n=0}^{\infty} p(\ell n+\delta_\ell)q^{24n+\beta_\ell} \cdot \prod_{n=1}^{\infty}(1-q^{24n})^{\ell}, \end{aligned}$$

where $\beta_\ell = \ell + \frac{24\delta_\ell - 1}{\ell} = \ell + r_\ell$. Since $(1-X^\ell)^\ell \equiv (1-X)^{\ell^2} \pmod{\ell}$, we find that

$$\sum_{n=0}^{\infty} p(\ell n+\delta_\ell)q^{24n+r_\ell} \equiv \frac{\Delta^{\frac{\ell^2-1}{24}}(24z) \mid U(\ell)}{\eta^{\ell}(24z)} \pmod{\ell}. \tag{13}$$

Therefore, the truth of (11) and Theorem 1 follows from the truth of

$$\Delta^{\frac{\ell^2-1}{24}}(24z) \mid U(\ell) \equiv h_\ell(z) \stackrel{\text{def}}{=} \begin{cases} 11\Delta(24z) \pmod{13} & \text{if } \ell=13,\\ 7\Delta(24z)E_4(24z) \pmod{17} & \text{if } \ell=17,\\ 5\Delta(24z)E_6(24z) \pmod{19} & \text{if } \ell=19,\\ \Delta(24z)E_4(24z)E_6(24z) \pmod{23} & \text{if } \ell=23,\\ 8\Delta^2(24z)E_4(24z) \pmod{29} & \text{if } \ell=29,\\ 10\Delta^2(24z)E_6(24z) \pmod{31} & \text{if } \ell=31. \end{cases} \tag{14}$$

Since the Hecke operator $T(\ell)$ and the operator $U(\ell)$ are equivalent on the reduction modulo ℓ of the Fourier expansion of an integer weight modular form with integer Fourier coefficients, for each ℓ the form $\Delta^{\frac{\ell^2-1}{24}}(24z)$ is congruent modulo ℓ to a cusp form with integer coefficients in the space $S_{\frac{\ell^2-1}{2}}(\Gamma_0(24),\chi_1)$. The forms on the right hand side of the alleged congruences in (14) lie in the space $S_{\ell-1}(\Gamma_0(24),\chi_1)$.

For each such ℓ, it is well known that the normalized Eisenstein series $E_{\ell-1}(z)$ satisfies the congruence $E_{\ell-1}(z)\equiv 1 \pmod{\ell}$ [SwD]. Therefore, $E_{\ell-1}^{\frac{\ell-1}{2}}(z)\cdot h_\ell(z)$ is a cusp form which is in the space $S_{\frac{\ell^2-1}{2}}(\Gamma_0(24),\chi_1)$, along with $\Delta^{\frac{\ell^2-1}{24}}(24z)\mid U(\ell)$, and has the same Fourier expansion modulo ℓ as $h_\ell(z)$. For each ℓ a simple computation verifies that (14) holds for more than the first $2\ell^2-2$ Fourier coefficients. By a theorem of J. Sturm [Theorem 1, St], this implies the truth of (14).

Q.E.D.

In view of Theorem 1, all of the central critical values of the relevant quadratic twists are uniquely determined by $L\left(G_\ell\otimes\chi_{D(\ell,0)},\frac{\ell-3}{2}\right)$. The table below contains approximations for these L-values. These values were calculated using the first few hundred terms of the Fourier expansions of the G_ℓ using the following well known result which is easily deduced from the integral representation of modular L-functions and the behavior of newforms under the Atkin-Lehner involution.

Theorem 8. *Suppose that $f(z)=\sum_{n=1}^{\infty}a(n)q^n\in S_{2k}(\Gamma_0(N),\chi_1)$ is a newform with integer coefficients. If $\epsilon=\pm1$ is the eigenvalue of $f(z)$ with respect to the Atkin-Lehner involution W_N, then*

$$L(f,k)=\frac{(2\pi)^k\cdot(1+\epsilon)\sum_{n=1}^{\infty}a(n)\Phi(2\pi n/\sqrt{N})}{(k-1)!N^{k/2}}.$$

Here $\Phi(x)$ is defined by

$$\Phi(x)\overset{\text{def}}{=}\frac{(k-1)!}{x^k}\cdot e^{-x}\cdot\left(1+x+\frac{x^2}{2!}+\cdots\frac{x^{k-1}}{(k-1)!}\right).$$

ℓ	$L\left(G_\ell\otimes\chi_{D(\ell,0)},\frac{\ell-3}{2}\right)$	$p(\delta_\ell)\pmod{\ell}$
13	10.169	11
17	4.396	7
19	5.261	5
23	3.774	1
29	6.360	8
31	8.949	10

Example. Here we illustrate the first three cases of Theorem 1 when $\ell=13$. Using Theorem 8, one easily obtains the following numerical data using the first 500 Fourier coefficients of G_{13}.

n	$b_{13}(24n+11)^2$	$\frac{L(G_{13}\otimes\chi_{D(13,n)},5)\cdot(24n+11)^{9/2}}{L\left(G_{13}\otimes\chi_{-11},5\right)\cdot 11^{9/2}}$	$\frac{p(13n+6)^2}{p(6)^2} \pmod{13}$
1	$121\equiv 4 \pmod{13}$	120.9998	4
2	$1936\equiv 12 \pmod{13}$	1935.9998	12
3	$3025\equiv 9 \pmod{13}$	3024.9997	9

3. Proof of Theorem 3

If $13\le \ell_0\le 31$ is prime and n is a non-negative integer, then let $M^{(\ell_0,n)}$ denote the motive defined in (8). By construction, $M^{(\ell_0)}$ and each $M^{(\ell_0,n)}$ admits a premotivic structure in the sense of Fontaine and Perrin-Riou (for details see [F-PR]). This structure consists of Betti, de Rham, and ℓ-adic realizations, together with comparison isomorphisms which relate these realizations. If M is $M^{(\ell_0)}$ or one of the $M^{(\ell_0,n)}$, then these realizations and isomorphisms satisfy:

(R1) The Betti realization M_B is a 2-dimensional $\mathbb{Q}$-vector space with a $G_{\mathbb{R}}$ action.

(R2) The de Rham realization M_{dr} is a 2-dimensional $\mathbb{Q}$-vector space with a finite decreasing filtration Fil^i.

(R3) For each finite prime ℓ of $\mathbb{Q}$, the ℓ-adic realization M_ℓ is a 2-dimensional $\mathbb{Q}_\ell$ vector space with a continuous pseudo-geometric $G_{\mathbb{Q}}$ action.

(R4) There is an $\mathbb{R}$-linear isomorphism

$$I_B:\mathbb{C}\otimes M_B\to\mathbb{C}\otimes M_{\mathrm{dr}}$$

respecting the $G_{\mathbb{R}}$ action and inducing a $\mathbb{Q}$-Hodge structure over $\mathbb{R}$ on M_B.

(R5) For each finite prime ℓ of $\mathbb{Q}$ there is a $\mathbb{Q}_\ell$-linear isomorphism

$$I_\ell^B:\mathbb{Q}_\ell\otimes_{\mathbb{Q}} M_B\to M_\ell$$

respecting the action of $G_{\mathbb{R}}$.

(R6) Define the integer N_M by

$$N_M:=\begin{cases}(\ell_0-3)! & \text{if } M=M^{(\ell_0)},\\ (\ell_0-3)!\cdot(24n+r_{\ell_0}) & \text{if } M=M^{(\ell_0,n)}.\end{cases}$$

For each prime $\ell\nmid N_M$ there is a $B_{\mathrm{dr},\ell}$-linear isomorphism

$$I_\ell: B_{\mathrm{dr},\ell}\otimes_{\mathbb{Q}_\ell} M_\ell\to B_{\mathrm{dr},\ell}\otimes_{\mathbb{Q}_\ell} M_{\mathrm{dr}}$$

respecting filtrations and the action of $G_{\mathbb{Q}_\ell}$, where $B_{\mathrm{dr},\ell}$ is the filtered $\mathbb{Q}_\ell[G_{\mathbb{Q}_\ell}]$-algebra constructed by Fontaine.

To make sense of Theorem 3, we must first specify the nature of the integral structures $\mathcal{M}^{(\ell_0,n)}$ for the motives $M^{(\ell_0,n)}$ which are used to define the relevant Tate-Shafarevich groups. These structures will be induced from a fixed choice of integral structure $\mathcal{M}^{(\ell_0)}$ for $M^{(\ell_0)}$.

First we begin by defining an essential property for our investigation, motivated **by** the definition of motivic pairs in [B-K]. If M is a motive, then an integral structure, say $\mathcal{M}$, consists of a $\mathbb{Z}$-lattice $\mathcal{M}_B$ in M_B and a $\mathbb{Z}$-lattice $\mathcal{M}_{\mathrm{dr}}$ in M_{dr} such that the $\mathbb{Z}_\ell$-lattice

$$\mathcal{M}_\ell \overset{\text{def}}{=} I_\ell^B(\mathbb{Z}_\ell \otimes \mathcal{M}_B) \subseteq \mathcal{M}_\ell$$

is invariant under the action of $G_{\mathbb{Q}}$. In addition, there must be a finite set of primes of $\mathbb{Q}$ containing the infinite prime, say S, such that for every prime $\ell \notin S$ we have:

(C1) M_ℓ is a crystalline representation of G_ℓ.
(C2) There are integers $i \leq 0$ and $j \geq 1$ with $j - i < \ell$ such that $\mathrm{Fil}^i DR(M_\ell) = DR(M_\ell)$ and $\mathrm{Fil}^j DR(M_\ell) = 0$.
(C3) $\mathbb{Z}_\ell \otimes \mathcal{M}_{\mathrm{dr}}(j-1)$ is a strongly divisible lattice of $\mathbb{Q}_\ell \otimes_{\mathbb{Q}} M_{\mathrm{dr}}(j-1)$.
(C4) The isomorphism

$$\mathrm{Fil}^0(B_{\mathrm{crys}} \otimes_{\mathbb{Q}_\ell} M_{\mathrm{dr}}(j-1))^{F=1} = M_\ell(j-1)$$

induced by I_ℓ restricts to an isomorphism

$$\mathrm{Fil}^0(A_{\mathrm{crys}} \otimes_{\mathbb{Z}_\ell} \mathcal{M}_{\mathrm{dr}}(j-1))^{F=1} = \mathcal{M}_\ell(j-1).$$

The ring $A_{\mathrm{crys},\ell}$ is defined by Fontaine in [F]. For convenience, we make the following definition.

Definition (Property (P2)). *An integral structure $\mathcal{M}$ for M has property* (P2) *at a prime ℓ if* (C1-C4) *hold for ℓ.*

There is a canonical integral structure $\mathcal{M}^{(\ell_0)}$ for $M^{(\ell_0)}$ that has property (P2) for every prime $\ell \nmid (\ell_0 - 3)!$ (see [D-F-G,Ne,Sch] for further details). Specifically, let $s : E_6 \to X_6$ be the universal elliptic curve on the modular curve X_6 parameterizing elliptic curves with level 6 structure. One may choose the desired integral structure $\mathcal{M}^{(\ell_0)}$ to be the integral structure coming from the premotivic structure defined by the parabolic cohomology groups associated to the universal elliptic curve, followed by the $\frac{\ell_0-3}{2}$-th Tate twist.

For every non-negative integer n, let $D^{(\ell_0,n)}$ denote for the Dirichlet motive associated to the Dirichlet character $\chi_{D(\ell_0,n)}$ (for example, see [De2]), and let $\mathcal{D}^{(\ell_0,n)}$ denote the canonical integral structure of $D^{(\ell_0,n)}$ given by the canonical bases of the realizations. If $\ell \nmid (24n + r_{\ell_0})$, then $\mathcal{D}^{(\ell_0,n)}$ has property (P2) at ℓ. After all this, one may simply define $\mathcal{M}^{(\ell_0,n)}$, the integral structure for $M^{(\ell_0,n)}$, by

$$\mathcal{M}^{(\ell_0,n)} \overset{\text{def}}{=} \mathcal{M}^{(\ell_0)} \otimes \mathcal{D}^{(\ell_0,n)}. \tag{16}$$

By construction, $\mathcal{M}^{(\ell_0,n)}$ has property (P2) at every prime $\ell \nmid (\ell_0-3)!(24n+r_{\ell_0})$.

Proof of Theorem 3. Given an integral structure $\mathcal{M}^{(\ell_0,n)}$ for $M^{(\ell_0,n)}$, let $(M^{(\ell_0,n)})^*$ be the dual motive of $M^{(\ell_0,n)}$ with the integral structure $(\mathcal{M}^{(\ell_0,n)})^*$ which is dual to $\mathcal{M}^{(\ell_0,n)}$. If n satisfies hypothesis (*iii*) in the statement of Theorem 3, then the Bloch-Kato conjecture for $M^{(\ell_0,n)}$ (with respect to $\mathcal{M}^{(\ell_0,n)}$) is given by the formula

Conjecture (Bloch-Kato).
(17)
$$\frac{L(M^{(\ell_0,n)},0)}{\mu_\infty(\mathcal{M}^{(\ell_0,n)}(\mathbb{R}))}=\frac{\#\text{Ш}(\mathcal{M}^{(\ell_0,n)})\ \prod_{p<\infty}c_p(\mathcal{M}^{(\ell_0,n)})}{\#H^0(\mathbb{Q},M_f^{(\ell_0,n)}/\mathcal{M}_f^{(\ell_0,n)})\#H^0(\mathbb{Q},(M^{(\ell_0,n)})^*(1)_f/(\mathcal{M}^{(\ell_0,n)})^*(1)_f)}.$$

With respect to the integral structure $\mathcal{M}^{(\ell_0,n)}$, recall that
- $\mu_\infty(\mathcal{M}^{(\ell_0,n)}(\mathbb{R}))$ *is the Bloch-Kato period,*
- $\text{Ш}(\mathcal{M}^{(\ell_0,n)})$ *is the Tate-Shafarevich group of $M^{(\ell_0,n)}$,*
- $c_p(\mathcal{M}^{(\ell_0,n)})$ *is the Tamagawa factor of $M^{(\ell_0,n)}$ at p.*

Moreover, we have that $M_f^{(\ell_0,n)}=\bigoplus_{\ell<\infty}M_\ell^{(\ell_0,n)}$ and $\mathcal{M}_f^{(\ell_0,n)}=\bigoplus_{\ell<\infty}\mathcal{M}_\ell^{(\ell_0,n)}$. The quantities $(M^{(\ell_0,n)})^*(1)_f$ and $(\mathcal{M}^{(\ell_0,n)})^*(1)_f$ are defined in an analogous manner.

(Note: Although all of the quantities in the conjectured formula except $L(M^{(\ell_0,n)},0)$ depend on the choice of integral structure, Bloch and Kato [B-K] have shown that the truth of the conjectured formula is independent of this choice.)

It follows from (17) that

(18)
$$\ell_0\text{-part of}\frac{L(M^{(\ell_0,n)},0)}{\mu_\infty(\mathcal{M}^{(\ell_0,n)}(\mathbb{R}))}$$
$$=\frac{\ell_0\text{-part of }\#\text{Ш}(\mathcal{M}^{(\ell_0,n)})\ \times\ \prod_{p<\infty}\ell_0\text{-part of }c_p(\mathcal{M}^{(\ell_0,n)})}{\#H^0(\mathbb{Q},M_{\ell_0}^{(\ell_0,n)}/\mathcal{M}_{\ell_0}^{(\ell_0,n)})\ \#H^0(\mathbb{Q},(M^{(\ell_0,n)})^*(1)_{\ell_0}/(\mathcal{M}^{(\ell_0,n)})^*(1)_{\ell_0})}.$$

It turns out that $\bar{\rho}$, the residual ℓ_0-Galois representation associated to G_{ℓ_0} [De1]

$$\bar{\rho}:\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\to\mathrm{Aut}(\frac{1}{\ell_0}\mathcal{M}_{\ell_0}^{(\ell_0)}/\mathcal{M}_{\ell_0}^{(\ell_0)})\to GL_2(\mathbb{F}_{\ell_0}),$$

is irreducible. To see this, recall that if $p\nmid 6\ell_0$ is prime, then

$$\mathrm{tr}(\bar{\rho}(\mathrm{Frob}(p)))\equiv a_{\ell_0}(p)\pmod{\ell_0},$$
$$\det(\bar{\rho}(\mathrm{Frob}(p))\equiv p^{\ell_0-4}\pmod{\ell_0}.$$

Using the formulas in the Appendix and [Prop. 19, Se], it is simple to check that the image of $\bar{\rho}$ contains $SL_2(\mathbb{F}_{\ell_0})$. Therefore, for every n the mod ℓ_0 residual representations of

$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\frac{1}{\ell_0}\mathcal{M}_{\ell_0}^{(\ell_0,n)}/\mathcal{M}_{\ell_0}^{(\ell_0,n)}$ and $\frac{1}{\ell_0}(\mathcal{M}^{\ell_0,n})^*(1)_{\ell_0}/(\mathcal{M}^{\ell_0,n})^*(1)_{\ell_0}$ are both irreducible, and so $H^0(\mathbb{Q}, M_{\ell_0}^{\ell_0,n}/\mathcal{M}_{\ell_0}^{\ell_0,n})$ and $H^0(\mathbb{Q}, (M^{\ell_0,n})^*(1)_{\ell_0}/(\mathcal{M}^{\ell_0,n})^*(1))_{\ell_0}$ are both trivial.

Hypothesis (i) in Theorem 3 implies $\ell_0 \nmid (24n + r_{\ell_0})$. So $M^{(\ell_0,n)}$ has good reduction at ℓ_0 [Sch] and $\mathcal{M}^{(\ell_0,n)}$ has (P2) at $\ell = \ell_0$. Therefore, by [Theorem 4.1, B-K], the ℓ_0-part of $c_p(\mathcal{M}^{(\ell_0,n)})$ is 1 for every prime $p < \infty$. Consequently, (17) may be replaced by

$$\mathrm{ord}_{\ell_0}\left(\frac{L(M^{(\ell_0,n)},0)}{\mu_\infty(\mathcal{M}^{(\ell_0,n)}(\mathbb{R}))}\right) = \mathrm{ord}_{\ell_0}(\#\text{Ш}(\mathcal{M}^{(\ell_0,n)})). \tag{19}$$

Since $M^{(\ell_0)}$ is a critical motive in the sense of Deligne [De2], the comparison isomorphism $I_B : \mathbb{C}\otimes_{\mathbb{Q}} M_B^{(\ell_0)} \to \mathbb{C}\otimes_{\mathbb{Q}} M_{\mathrm{dr}}^{(\ell_0)}$ induces an isomorphism

$$I^+(M^{(\ell_0)}):\ \mathbb{R}\otimes_{\mathbb{Q}} (M_B^{(\ell_0)})^{G_{\mathbb{R}}} \to \mathbb{R}\otimes_{\mathbb{Q}} (M_{\mathrm{dr}}^{(\ell_0)}/\mathrm{Fil}^0 M_{\mathrm{dr}}^{(\ell_0)}).$$

With respect to the lattices $\mathcal{M}_B^{(\ell_0)} \le M_B^{(\ell_0)}$ and $\mathcal{M}_{\mathrm{dr}}^{(\ell_0)} \le M_{\mathrm{dr}}^{(\ell_0)}$ given by the integral structure $\mathcal{M}^{(\ell_0)}$, the determinant of $I^+(M^{(\ell_0)})$ defines $c^+(\mathcal{M}^{(\ell_0)}) \in \mathbb{R}^\times$, the Deligne period for $M^{(\ell_0)}$, up to a choice of sign. Since

$$M^{(\ell_0,n)} = M^{(\ell_0)} \otimes_{\mathbb{Q}} D^{(\ell_0,n)},$$

it is easy to see that $I^+(M^{(\ell_0,n)}) = I^+(M^{(\ell_0)}) \otimes I^+(D^{(\ell_0,n)})$. Since the determinant of

$$I^+(D^{(\ell_0,n)}) = I_B(D^{(\ell_0,n)}) : \mathbb{R}\otimes_{\mathbb{Q}} D_B^{(\ell_0,n)} \to \mathbb{R}\otimes_{\mathbb{Q}} D_{\mathrm{dr}}^{(\ell_0,n)}$$

is g_n^{-1}, where g_n is the Gauss sum $\displaystyle\sum_{a \pmod{D(\ell_0,n)}} \chi_{D(\ell_0,n)}(a) e^{\frac{2\pi i a}{|D(\ell_0,n)|}}$, we have

$$c^+(\mathcal{M}^{(\ell_0,n)}) = c^+(\mathcal{M}^{(\ell_0)}) g_n^{-1}.$$

A calculation [De2] shows that the Bloch-Kato period $\mu_\infty(\mathcal{M}^{(\ell_0,n)}(\mathbb{R}))$ for a critical motive equals the Deligne period up to a power of 2. Also as is well-known, the motivic L-function $L(M^{(\ell_0,n)}, s)$ equals to $L(G_{\ell_0}\otimes\chi_{D(\ell_0,n)}, s + \frac{\ell_0-3}{2})$. So (19) becomes

$$\mathrm{ord}_{\ell_0}\left(\frac{L(G_{\ell_0}\otimes\chi_{D(\ell_0,n)}, \frac{\ell_0-3}{2})}{C^+(\mathcal{M}^{(\ell_0)})g_n^{-1}}\right) = \mathrm{ord}_{\ell_0}(\#\text{Ш}(\mathcal{M}^{(\ell_0,n)})). \tag{20}$$

Since $\chi_{D(\ell_0,n)}$ is real, the Gauss sum g_n is

$$g_n = \begin{cases} i\sqrt{|D(\ell_0,n)|} = i\sqrt{24n + r_{\ell_0}} & \text{if } \ell_0 \equiv 1 \pmod 4, \\ \sqrt{D(\ell_0,n)} = \sqrt{24n + r_{\ell_0}} & \text{if } \ell_0 \equiv 3 \pmod 4. \end{cases}$$

We then have

$$\frac{g_0}{g_n}\left(\frac{24n+r_{\ell_0}}{r_{\ell_0}}\right)^{\frac{\ell_0-4}{2}} = \left(\frac{24n+r_{\ell_0}}{r_{\ell_0}}\right)^{\frac{\ell_0-3}{2}}. \tag{21}$$

In view of Hypothesis (i), this fraction is an ℓ_0-unit.

By Theorem 1, (19) and (20), for any n satisfying $(i-iii)$ we have

$$\begin{aligned}
&\operatorname{ord}_{\ell_0}\left(\frac{p\left(\ell_0 n+\delta_{\ell_0}\right)^2}{p\left(\delta_{\ell_0}\right)^2}\right)\\
&=\operatorname{ord}_{\ell_0}\left(\frac{L(G_{\ell_0}\otimes\chi_{D(\ell_0,n)},\frac{\ell_0-3}{2})(24n+r_{\ell_0})^{\frac{\ell_0-4}{2}}}{L(G_{\ell_0}\otimes\chi_{D(\ell_0,0)},\frac{\ell_0-3}{2})r_{\ell_0}^{\frac{\ell_0-4}{2}}}\right)\\
&=\operatorname{ord}_{\ell_0}\left(\frac{\#\text{Ш}(\mathcal{M}^{(\ell_0,n)})\ g_0(24n+r_{\ell_0})^{\frac{\ell_0-4}{2}}}{\#\text{Ш}(\mathcal{M}^{(\ell_0,0)})\ g_n\ r_{\ell_0}^{\frac{\ell_0-4}{2}}}\right)\\
&=\operatorname{ord}_{\ell_0}\left(\frac{\#\text{Ш}(\mathcal{M}^{(\ell_0,n)})}{\#\text{Ш}(\mathcal{M}^{(\ell_0,0)})}\right).
\end{aligned}$$

Theorem 3 now follows easily from the fact that $\ell_0 \nmid p(\delta_{\ell_0})$.

Q.E.D.

Appendix: q-expansions of the $G_\ell(z)$

We begin with the following proposition whose proof is a standard exercise.

Proposition. *If $13 \le \ell \le 31$ is prime, then let $\mathfrak{S}_\ell$ denote the set*

$$\mathfrak{S}_\ell := \{\text{newforms in } S_{\ell-3}(\Gamma_0(6),\chi_1)\} \ \cap\ \mathbb{Z}[[q]].$$

(i) *There is exactly one newform in $\mathfrak{S}_{13}$ and its Fourier expansion begins with*

$$q-16q^2+81q^3+256q^4+2694q^5-\dots.$$

(ii) *There is exactly one newform in $\mathfrak{S}_{17}$ and its Fourier expansion begins with*

$$q+64q^2-729q^3+4096q^4+54654q^5-\dots.$$

(iii) *There are exactly three newforms in $\mathfrak{S}_{19}$ and their Fourier expansions begin with*

$$\begin{gathered}
q-128q^2-2187q^3+16384q^4-314490q^5+\dots,\\
q+128q^2-2187q^3+16384q^4-114810q^5-\dots,\\
q+128q^2+2187q^3+16384q^4+77646q^5+\dots,
\end{gathered}$$

(*iv*) *There are exactly three newforms in* $\mathfrak{S}_{23}$ *and their Fourier expansions begin with*

$$\begin{aligned}&q+512q^2+19683q^3+262144q^4+1953390q^5+\dots,\\&q-512q^2+19683q^3+262144q^4-5849490q^5-\dots,\\&q-512q^2-19683q^3+262144q^4-3732474q^5+\dots.\end{aligned}$$

(*v*) *There are exactly three newforms in* $\mathfrak{S}_{29}$ *and their Fourier expansions begin with*

$$\begin{aligned}&q-4096q^2-531441q^3+16777216q^4-292754850q^5+\dots,\\&q+4096q^2-531441q^3+16777216q^4-799327650q^5-\dots,\\&q-4096q^2+531441q^3+16777216q^4+590425734q^5+\dots.\end{aligned}$$

(*vi*) *There are exactly three newforms in* $\mathfrak{S}_{31}$ *and their Fourier expansions begin with*

$$\begin{aligned}&q+8192q^2-1594323q^3+67108864q^4+2904255750q^5-\dots,\\&q+8192q^2+1594323q^3+67108864q^4+1220703150q^5+\dots,\\&q-8192q^2+1594323q^3+67108864q^4+1992850350q^5-\dots.\end{aligned}$$

Remark. It is easy to see that the forms G_ℓ are well defined by (6).

Now we present formulas for the newforms G_ℓ. Since the coefficients $a_\ell(2)$ and $a_\ell(3)$ are well defined, we simplify our formulas by giving 'closed' expressions for some cusp forms $G^*_\ell(z)=\sum_{n=1}^{\infty}a^*_\ell(n)q^n\in S_{\ell-3}(\Gamma_0(6),\chi_1)$. for which

$$\sum_{\gcd(n,6)=1}a^*_\ell(n)q^n=\sum_{\gcd(n,6)=1}a_\ell(n)q^n\in S_{\ell-3}(\Gamma_0(36),\chi_1). \tag{22}$$

These identities are verified by checking that the initial segments of the Fourier expansions agree for more than $6\ell-18$ terms.

$$\begin{aligned}G^*_{13}(z)=&-116\eta^8(z)\eta^2(2z)\eta^8(3z)\eta^2(6z)+50\eta^7(z)\eta^7(2z)\eta^3(3z)\eta^3(6z)\\&-450\eta^3(z)\eta^3(2z)\eta^7(3z)\eta^7(6z)+\big(\eta^8(z)\eta^2(2z)\eta^8(3z)\eta^2(6z)\mid U(2)\big)\end{aligned}$$

$$\begin{aligned}
G_{17}^*(z) = {} & \frac{1063420}{1461}\eta^{18}(z)\eta^{2}(3z)\eta^{8}(6z) + \frac{10828}{487}\eta^{14}(z)\eta^{8}(2z)\eta^{6}(3z) \\
& + \frac{66824}{487}\eta^{13}(z)\eta^{13}(2z)\eta(3z)\eta(6z) + \frac{2414760}{487}\eta^{13}(z)\eta(2z)\eta(3z)\eta^{13}(6z) \\
& + \frac{5917}{2922}\left(\eta^{18}(z)\eta^{2}(3z)\eta^{8}(6z) \mid U(2)\right) + \left(\eta^{14}(z)\eta^{8}(2z)\eta^{6}(3z) \mid U(2)\right)
\end{aligned}$$

$$\begin{aligned}
G_{19}^*(z) = {} & -\frac{214540}{477}\eta^{26}(z)\eta^{2}(2z)\eta^{2}(3z)\eta^{2}(6z) - \frac{2633264}{477}\eta^{21}(z)\eta^{3}(2z)\eta(3z)\eta^{7}(6z) \\
& + \frac{32356}{477}\eta^{16}(z)\eta^{16}(2z) - \frac{2835536}{159}\eta^{16}(z)\eta^{4}(2z)\eta^{12}(6z) \\
& - \frac{460452}{53}\eta^{14}(z)\eta^{2}(2z)\eta^{14}(3z)\eta^{2}(6z) + \left(\eta^{26}(z)\eta^{2}(2z)\eta^{2}(3z)\eta^{2}(6z) \mid U(2)\right) \\
& + \frac{6793}{477}\left(\eta^{21}(z)\eta^{3}(2z)\eta(3z)\eta^{7}(6z) \mid U(2)\right)
\end{aligned}$$

$$\begin{aligned}
G_{23}^*(z) = {} & -\frac{22696872}{1613}\eta^{32}(z)\eta^{8}(2z) - \frac{18770227340}{30647}\eta^{28}(z)\eta^{4}(2z)\eta^{4}(3z)\eta^{4}(6z) \\
& - \frac{77831474526}{30647}\eta^{24}(z)\eta^{8}(3z)\eta^{8}(6z) - \frac{65500631328}{30647}\eta^{23}(z)\eta^{5}(2z)\eta^{3}(3z)\eta^{9}(6z) \\
& + \frac{1036323584}{1613}\eta^{14}(z)\eta^{14}(2z)\eta^{6}(3z)\eta^{6}(6z) + \left(\eta^{32}(z)\eta^{8}(2z) \mid U(2)\right) \\
& - \frac{202239385}{245176}\left(\eta^{28}(z)\eta^{4}(2z)\eta^{4}(3z)\eta^{4}(6z) \mid U(2)\right) \\
& + \frac{23975190}{30647}\left(\eta^{24}(z)\eta^{8}(3z)\eta^{8}(6z) \mid U(2)\right) \\
& - \frac{534947419}{61294}\left(\eta^{23}(z)\eta^{5}(2z)\eta^{3}(3z)\eta^{9}(6z) \mid U(2)\right).
\end{aligned}$$

$$
\begin{aligned}
G_{29}^{*}(z) = {} & \frac{4632891951294690053814110128}{424668169983336169005}\eta^{42}(z)\eta^{2}(3z)\eta^{8}(6z) \\
& + \frac{5350461297197237391082527 6749}{114660405895500765631 35}\eta^{38}(z)\eta^{8}(2z)\eta^{6}(3z) \\
& - \frac{3979434854340016630055193224}{674472975855886856655}\eta^{37}(z)\eta^{13}(2z)\eta(3z)\eta(6z) \\
& + \frac{2765340020679851796448266496}{471853522203706854 45}\eta^{37}(z)\eta(2z)\eta(3z)\eta^{13}(6z) \\
& + \frac{8946874199552312339578848 67}{815362886368005444489 60}\left(\eta^{42}(z)\eta^{2}(3z)\eta^{8}(6z) \mid U(2)\right) \\
& + \frac{614128045535015603072820 59}{1467653195462409800081 28}\left(\eta^{38}(z)\eta^{8}(2z)\eta^{6}(3z) \mid U(2)\right) \\
& - \frac{53629842690559658478591 2}{1031943653059506890682 15}\left(\eta^{37}(z)\eta^{13}(2z)\eta(3z)\eta(6z) \mid U(3)\right) \\
& + \frac{4647639878228230505158892}{1031943653059506890682 15}\left(\eta^{42}(z)\eta^{2}(3z)\eta^{8}(6z) \mid U(3)\right) \\
& + \frac{6394927922115472738541 27}{1031943653059506890682 15}\left(\eta^{38}(z)\eta^{8}(2z)\eta^{6}(3z) \mid U(3)\right) \\
& + \frac{8637476930233691991096144 67}{9059587626311171605440}\left(\eta^{37}(z)\eta(2z)\eta(3z)\eta^{13}(6z) \mid U(2)\right) \\
& + \frac{2707461154152141673797 76}{674472975855886856655}\left(\eta^{37}(z)\eta(2z)\eta(3z)\eta^{13}(6z) \mid U(3)\right) \\
& + \frac{1232234862917505262090687 67}{2446088659104016333468 80}\left(\eta^{37}(z)\eta^{13}(2z)\eta(3z)\eta(6z) \; |U(2)\right).
\end{aligned}
$$

$$
\begin{aligned}
G_{31}^{*}(z) = {} & \frac{8764467928168553952519459623}{30891300587851524624}\eta^{50}(z)\eta^{2}(2z)\eta^{2}(3z)\eta^{2}(6z) \\
& + \frac{73419725842368053086155028139}{1287137524493813526}\eta^{45}(z)\eta^{3}(2z)\eta(3z)\eta^{7}(6z) \\
& - \frac{1200930595651799333501307865}{7722825146962881156}\eta^{40}(z)\eta^{16}(2z) \\
& + \frac{92680825489101355891394707648}{214522920748968921}\eta^{40}(z)\eta^{4}(2z)\eta^{12}(6z) \\
& - \frac{1656366896397944526889090235}{31781173444291692}\eta^{38}(z)\eta^{2}(2z)\eta^{14}(3z)\eta^{2}(6z) \\
& + \frac{438375590965076099323490516}{882810373452547}\eta^{36}(z)\eta^{4}(3z)\eta^{16}(6z) \\
& + \frac{50248248964724105324453}{15445650293925762312}\left(\eta^{50}\eta^{2}(2z)\eta^{2}(3z)\eta^{2}(6z) \mid U(2)\right) \\
& + \frac{223338812649201265633277}{834065115871991164848}\left(\eta^{40}(z)\eta^{16}(2z) \mid U(3)\right) \\
& - \frac{85011693687118126027003}{34752713161332965202}\left(\eta^{45}(z)\eta^{3}(2z)\eta(3z)\eta^{7}(6z) \mid U(3)\right) \\
& - \frac{222504747533329274468429}{834065115871991164848}\left(\eta^{50}(z)\eta^{2}(2z)\eta^{2}(3z)\eta^{2}(6z) \mid U(3)\right) \\
& - \frac{49605292349825928145421}{42374897925722256}\left(\eta^{36}(z)\eta^{4}(3z)\eta^{16}(6z) \mid U(2)\right) \\
& - \frac{16588467118194493964012}{17376356580666482601}\left(\eta^{36}(z)\eta^{4}(3z)\eta^{16}(6z) \mid U(3)\right) \\
& - \frac{6229502876006220268425287}{13729466927934010944}\left(\eta^{40}(z)\eta^{4}(2z)\eta^{12}(6z) \mid U(2)\right).
\end{aligned}
$$

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