Centre de Recherches Mathématiques
CRM Proceedings and Lecture Notes
Volume 19, 1999

Integers Represented by Ternary Quadratic Forms

Ken Ono and K. Soundararajan

ABSTRACT. Some of the most fundamental questions regarding ternary quadratic forms are barely understood. For instance, if $f(x, y, z)$ is an integral positive ternary quadratic form, then the problem of determining all integers of the form $f(x, y, z)$ where $x, y, z \in \mathbb{Z}$ remains open, although it can be handled for exceptional f. We solve this problem, conditional on the *Generalized Riemann Hypothesis*, for Ramanujan's form. The method readily applies to many ternary forms with small discriminant.

A non-negative integer N is *eligible* for a positive ternary quadratic form $f(x, y, z)$ if there are no congruence conditions prohibiting f from representing N. It is well known that a genus of positive ternary forms represents every eligible integer, and so if a genus contains only one class then every form f in that class represents every eligible integer.

L. Dickson, B. Jones, and G. Pall ([**Di1**], [**Jo1**], [**Jo2**], [**JP**]) initiated the study of more general forms, forms in genera with multiple classes. In an effort to describe those integers represented by such forms, two classes emerged: *regular* ternary quadratic forms being those forms which represent all eligible integers, and *irregular* ternary quadratic forms being those which miss some eligible integers.

There are methods for deciding if a form is regular (see [**Di1**], [**Jo2**], [**JP**], [**Ka1**], [**Ka2**]), and if it turns out to be regular, then the problem of determining integers represented by it is solved. However the situation is very different for irregular forms where there is no known effective way of determining the eligible integers which are represented. In fact the problem has never been solved for an irregular form missing at least two eligible integers.

In this direction, W. Duke and R. Schulze-Pillot [**DS-P**] proved that every large *spinor eligible* integer, those represented by the spinor genus of f, is represented by f. Their result depends on Siegel's lower bound for the class number of imaginary quadratic number fields, and so is ineffective. That is their strong result does not give a bound beyond which all spinor eligible integers are represented by f.

1991 *Mathematics Subject Classification.* Primary 11E20; Secondary 11E25.

The first author is supported by NSF grants DMS-9508976 and DMS-9304580.

We thank E. Bombieri, W. K. Chan, A. Earnest, A. Granville, J. Hsia, P. Sarnak, and I. Kaplansky for their helpful comments. We especially thank S. Miller and W. Galway for their extensive computations.

This is the final version of the paper.

Assuming the *Generalized Riemann Hypothesis,* we show that one can effectively determine those integers which are represented by ternary forms with small discriminant. In [**OS**] we investigated those integers represented by *Ramanujan's form*. The purpose of this note is to give a brief exposition of these results.

In [**R**], S. Ramanujan investigated the ternary form

$$\phi_1(x,y,z) := x^2 + y^2 + 10z^2. \tag{1}$$

It is known as Ramanujan's form, and its genus consists of two classes. The form

$$\phi_2(x,y,z) := 2x^2 + 2y^2 + 3z^2 - 2xz \tag{2}$$

is a representative for the other class. Ramanujan stated that [**R,** p. 14] "*... the even numbers which are not of the form $x^2+y^2+10z^2$ are the numbers*

$$4^\lambda(16\mu+6),$$

while the odd numbers that are not of that form, viz.,

$$3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391 \ldots$$

do not seem to obey any simple law."

Every non-negative integer, except those of the form $4^\lambda(16\mu+6)$, is eligible, and in view of the exceptions, Ramanujan's form is irregular. In addition to the integers on Ramanujan's list, the exceptions 679 and 2719 were discovered by B. Jones, G. Pall, and H. Gupta ([**JP**], [**Gu**]) and W. Galway has verified that there are no other exceptions below $2 \cdot 10^{10}$. We are thus led to the following conjecture.

CONJECTURE. *The eligible integers which are not of the form $x^2+y^2+10z^2$ are:*

$$3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391, 679, 2719.$$

Ramanujan noted that it suffices to consider the odd integers, and it will be convenient to make the further reduction to integers prime to 10. Legendre proved that every odd integer is of the form $2n+1 = x^2+y^2+2z^2$ (see [**Di2,** p. 261]). Multiplying by 5 we see that

$$10n+5 = 5(x^2+y^2)+10z^2 = (2x+y)^2+(x-2y)^2+10z^2,$$

which verifies that every integer $N \equiv 5 \pmod{10}$ is represented by Ramanujan's form. Therefore we may restrict our attention to those integers prime to 10.

By the work of J. Benham and J. Hsia [**BHs**] it is known that all eligible integers not of the form $x^2+y^2+10z^2$ are square-free. We prove this by a completely different argument, one which leads to our attack on the general problem.

THEOREM 1. *Every eligible integer not of the form $x^2+y^2+10z^2$ is square-free.*

PROOF. Let $r_i(N)$ denote the number of representations of N by ϕ_i, and let $R_i(N)$ denote the number of *primitive* representations of N by ϕ_i. Recall that a representation $\phi_i(x,y,z) = N$ is primitive if $\gcd(x,y,z) = 1$. Let A_i be the matrices representing the forms ϕ_i. A 3×3 matrix B with determinant 1 is an automorph of A_i if $B^T A_i B = A_i$. Then it is easy to verify that there are 8 automorphs of A_1, and there are four automorphs of A_2.

Two representations of N by ϕ_i, say (x,y,z) and (x',y',z'), are called *essentially distinct* if there is no automorph B of A_i with the property that $(x',y',z') =$

$(x, y, z)B$. If $G(N)$ denotes the number of essentially distinct primitive representation of N by the genus of Ramanujan's form, then for square-free N, since no two distinct automorphs of representations of N by ϕ_1 (resp. ϕ_2) are equal, we obtain

$$G(N) = R_1(N)/8 + R_2(N)/4.$$

By [**Jo1**, Th. 86], one can relate $G(N)$ to class numbers. In particular, if $N > 1$ is a positive eligible integer which is relatively prime to 10, then

$$G(N) = \frac{1}{4}h(-40N). \tag{3}$$

The function $f(z)$

$$f(z) = \sum_{n=1}^{\infty} a(n)q^n = \frac{1}{4}\sum_{n=1}^{\infty}(r_1(n) - r_2(n))q^n = q - q^3 - q^7 - q^9 + 2q^{13} + .. \in S_{\frac{3}{2}}(40, \chi_{10}), \tag{4}$$

where $q := e^{2\pi i z}$, is an eigenform of the half-integral weight Hecke operators $T(p^2)$. Moreover its Shimura lift [**Sh**] is the weight 2 cusp form

$$F(z) = \sum_{n=1}^{\infty} A(n)q^n = q - 2q^3 - q^5 + 2q^7 + q^9 + 2q^{13} + 2q^{15} - 6q^{17} - \cdots \in S_2(20). \tag{5}$$

It is important to note that $F(z)$ is an eigenform of the Hecke operators, and its inverse Mellin transform is the Hasse-Weil L–function $L(E, s)$ for the elliptic curve over $\mathbb{Q}$

$$E: \quad y^2 = x^3 + x^2 + 4x + 4. \tag{6}$$

Since $f(z)$ is an eigenform, for every prime p, there exists a complex number $\alpha(p)$ such that for every positive integer n

$$\alpha(p)a(n) = a(p^2n) + \chi_{10}(p)\left(\frac{-n}{p}\right)a(n) + \chi_{10}(p^2)pa(n/p^2). \tag{7}$$

Further, we find that $\alpha(p) = A(p)$, and since $a(n) = \frac{1}{4}(r_1(n) - r_2(n))$ it follows from (7) that for square-free integers n

$$r_1(np^2) - r_2(np^2) = \left(A(p) - \chi_{10}(p)\left(\frac{-n}{p}\right)\right) \cdot (r_1(n) - r_2(n)). \tag{8}$$

Assume that $N > 1$ is a square-free integer prime to 10 for which $r_1(N) = 0$. Let $p \neq 2, 5$ be prime. If $r_1(Np^2) = 0$, then by (8) we find that

$$\frac{r_2(Np^2)}{r_2(N)} = \left(A(p) - \chi_{10}(p)\left(\frac{-N}{p}\right)\right) \leq A(p) + 1. \tag{9}$$

Since N is square-free we obtain $r_2(Np^2) = R_2(Np^2) + R_2(N) = R_2(Np^2) + r_2(N)$. Observe that, since N is square-free, $4G(N) = R_1(N)/2 + R_2(N) = r_2(N)$. Also, since $Np^2 \neq 0$, every primitive essentially distinct representation of Np^2 by ϕ_2 has at least 2 different automorphs hence $2G(Np^2) \leq R_2(Np^2)$. Consequently

$$\frac{r_2(Np^2)}{r_2(N)} = 1 + \frac{R_2(Np^2)}{r_2(N)} \geq 1 + \frac{2G(Np^2)}{4G(N)} = 1 + \frac{G(Np^2)}{2G(N)}. \tag{10}$$

By (3) and the index formula for $h(-D)$ (see [**Co**]) it follows that

$$\frac{G(Np^2)}{G(N)} = \frac{h(-40Np^2)}{h(-40N)} = p - \left(\frac{-40N}{p}\right) \geq p-1$$

which upon substitution in (10) yields

$$\frac{r_2(Np^2)}{r_2(N)} \geq 1 + \frac{G(Np^2)}{2G(N)} \geq \frac{p+1}{2}. \tag{11}$$

From (9) and (11) we find that $(p-1)/2 \leq A(p)$ which, by Hasse's bound $|A(p)| \leq 2\sqrt{p}$, is impossible for $p \geq 19$. For $p = 3, 7, 11, 13, 17$ we find that $A(3) = -2, A(7) = 2, A(11) = 0, A(13) = 2$, and $A(17) = -6$, and none satisfy $A(p) \geq (p-1)/2$. □

Now we have restricted the problem to those integers N that are square-free and prime to 10. For every integer D let $E(D)$ denote the $D-$quadratic twist elliptic curve (over $\mathbb{Q}$) of E :

$$E(D): \qquad y^2 = x^3 + Dx^2 + 4D^2x + 4D^3. \tag{12}$$

If N is a square-free integer prime to 10, then $E(-10N)$ has conductor $1600N^2$ [**Cr**, p. 49].

THEOREM 2. *If N is an eligible integer not of the form $x^2 + y^2 + 10z^2$, then*

$$h^2(-40N) = \frac{4\sqrt{N}}{\Omega(E(-10))} L(E(-10N), 1),$$

where $\Omega(E(-10)) \sim 0.71915$ is the real period of $E(-10)$.

PROOF. Theorem 2 is derived using a deep theorem due to J.-L. Waldspurger [**Wal**] that relates the Fourier coefficients of half-integer weight cusp forms to central values of the quadratic twists of the L-function of Shimura lifts.

Let $\mathcal{M}$ denote the set $\mathcal{M} := \{1, 3, 7, 13, 19, 21, 31, 33\}$, representatives of all the square classes modulo 40. If $m \in \mathcal{M}$ which belongs to the same square class as N then, by Waldspurger's theorem

$$\frac{a^2(N)}{\sqrt{N}L(E(-10N), 1)} = \frac{a^2(m)}{\sqrt{m}L(E(-10m), 1)}.$$

For each m, it turns out that

$$\frac{a^2(m)}{\sqrt{m}L(E(-10m), 1)} = \frac{1}{4\Omega(E(-10))}.$$

Hence N, which necessarily belongs to one of these classes, satisfies

$$a^2(N) = \frac{1}{16}(R_1(N) - R_2(N))^2 = \frac{\sqrt{N}}{4\Omega(E(-10))} L(E(-10N), 1). \tag{13}$$

If $R_1(N) = 0$ then, we see that $a(N) = (R_1(N) - R_2(N))/4 = -R_2(N)/4 = -h(-40N)/4$ so that Theorem 2 is an immediate consequence of (13). □

Although Theorem 2 is stated in terms of the elliptic curves $E(-10N)$, we note that the method in no way depends on the arithmetic of these special curves. In general the $L(E(-10N),1)$ are replaced by central critical values of suitable automorphic $L-$functions which are guaranteed to exist by the work of Waldspurger [**Wal**].

If a square-free eligible integer N is not represented by Ramanujan's form then, $|a(N)| = r_2(N)/4 = R_2(N)/4 = h(-40N)/4$. The known effective lower bounds for class numbers, due to D. Goldfeld [**Go**] and B. Gross and D. Zagier [**GZ**], implies that $|a(N)| \gg \log N$, a bound which cannot unconditionally solve the conjecture.

Therefore we aim to obtain an effective solution to the problem under suitable hypotheses. Assuming the Riemann hypothesis for Dirichlet's L-functions, J. E. Littlewood [**L**] effectively proved that $h(-40N) \gg \sqrt{N}/\log\log N$. Unfortunately the bound for N, beyond which every eligible integer is represented, obtained in this way is enormous. To see this assume the best conceivable bound for class numbers on GRH, $h(-40N) \geq \sqrt{N}$ (in reality such a strong bound is false) so that $|a(N)| \geq \sqrt{N}/4$. Also suppose that the Iwaniec-Duke ([**Du**],[**I**]) results give $|a(N)| \leq \tau(N)N^{3/7}(\log 2N)^2$. Then we require $N \geq (4\tau(N)\log^2(2N))^{14}$ to obtain a contradiction. This occurs only if $N \geq 10^{75}$; a bound which is infeasible.

In addition to the Riemann hypothesis for Dirichlet L-functions, suppose we assume the Riemann hypothesis for $L(E(-10N),s)$. Since the Riemann hypothesis for $L(E(-10N),s)$ implies the Lindelöf bound, $|L(E(-10N),1)| \ll N^{\epsilon}$, it seems plausible that one can obtain a feasible solution to the problem. However the familiar deduction of the Lindelöf bound from the Riemann hypothesis (see Theorem 13.2 of E. C. Titchmarsh [**T**] for a proof in the case of $\zeta(s)$; the ideas generalize easily) leads, at best, to a bound of the form

$$|L(E(-10N),1)| \leq \exp\left(\frac{3}{2}\frac{\log q}{\log\log q}\right),$$

where $q = 1600N^2$. Assuming the very strong bound $h(-40N) \geq \sqrt{N}$, this requires $N \geq 10^{85}$ before Theorem 2 yields a contradiction. Again this is infeasible.

Thus we require a completely different attack. Our attack involves explicit formulae and Hadamard's factorization formula. This contrasts sharply with the traditional method for deducing the Lindelöf bound from the Riemann hypothesis which uses the Borel-Caratheodory theorem and Hadamard's three circles theorem [**Ru**]. A noteworthy feature of our method is that we exploit the fact that both the Hasse-Weil L-function $L(E(-10N),s)$ and the Dirichlet L-function for the number field $\mathbb{Q}(\sqrt{-40N})$ are twists by the same quadratic character $\chi = -40N/\cdot$.

THEOREM 3. *Suppose the non-trivial zeros of all Dirichlet L-functions, $L(s,\chi)$, with χ a primitive, real character, have real part $1/2$. Further suppose that the non-trivial zeros of th Hasse-Weil L-functions $L(E(-10N),s)$ (with N a square-free integer prime to* 10*) have real part* 1. *Then the only eligible integers which are not of the form $x^2+y^2+10z^2$ are:*

$$3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391, 679, 2719.$$

SKETCH OF PROOF. W. Galway has verified that the only eligible integers $N < 2\cdot 10^{10}$ not of the form $x^2+y^2+10z^2$ are the eighteen known missed eligible integers. Therefore there is no loss of generality in assuming that $N \geq 2\cdot 10^{10}$. Let $\chi = -40N/\cdot$ denote the Kronecker-Legendre symbol. For brevity let

$$L(s) := L(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s},$$

and

$$L_a(s) := L(E(-10N), s) = \sum_{n=1}^{\infty} \frac{A(n)\chi(n)}{n^s}.$$

Let q be the conductor of $E(-10N)$ so $q = 1600N^2$. It is well-known that $L_a(s)$ satisfies the functional equation

$$\left(\frac{\sqrt{q}}{2\pi}\right)^{s} \Gamma(s)L_a(s) = \pm \left(\frac{\sqrt{q}}{2\pi}\right)^{2-s} \Gamma(2-s)L_a(2-s).$$

Since χ is a primitive character to the modulus $40N = \sqrt{q}$ and since $\chi(-1) = -1$ it follows (see Chapter 12 of [**Da2**]) that $L(s)$ obeys the functional equation

$$\left(\frac{\sqrt{q}}{\pi}\right)^{s/2} \Gamma\left(\frac{s+1}{2}\right) L(s) = \pm \left(\frac{\sqrt{q}}{\pi}\right)^{(1-s)/2} \Gamma\left(\frac{2-s}{2}\right) L(1-s)$$

where the sign of the functional equation depends on the sign of the Gauss sum $\tau(\chi)$. Apart from the trivial zeros at $0, -1, -2, \ldots$ our assumption ensures that the zeros of $L_a(s)$ lie on the line $\sigma = 1$. Similarly, apart from the trivial zeros at $-1, -3, \ldots$, the zeros of $L(s)$ are guaranteed by GRH to lie on the line $\sigma = 1/2$.

By Theorem 2 and Dirichlet's class number formula (see [**Da2**]) we obtain

$$\frac{4\sqrt{N}}{\Omega(E(-10))} L_a(1) = h(-40N)^2 = \left(\frac{\sqrt{40N}L(1)}{\pi}\right)^2 = \frac{40N}{\pi^2} L(1)^2,$$

so that, since $\Omega(E(-10)) \geq 0.7191$ and $q = 1600N^2$,

$$\frac{L_a(1)}{L(1)^2} \geq \frac{7.191\sqrt{N}}{\pi^2} \geq 0.1152 q^{1/4} \geq \frac{2}{7}\left(\frac{q}{4\pi^2}\right)^{1/4}. \tag{14}$$

If the functional equation for $L_a(s)$ has a negative sign then $L_a(1) = 0$, contradicting (14). Thus we may suppose, without loss of generality, that the sign is positive. We prove Theorem 3 by showing that (14) is violated under the GRH.

Define $F(s)$ by

$$F(s) = \left(\frac{\sqrt{q}}{2\pi}\right)^{s-1} \frac{L_a(s)\Gamma(s)}{L(s)L(2-s)}. \tag{15}$$

$F(s)$ is regular in the strip $1/2 < \sigma < 3/2$ and, because of the functional equation of $L_a(s)$, satisfies the functional equation $F(s) = F(2-s)$. Using the Phragmen-Lindelöf principle, see [**Ru**] for example, to the vertical strip bounded by the lines with real part σ and $2-\sigma$, for $1 \leq \sigma < 3/2$, we see that

$$F(1) = \frac{L_a(1)}{L(1)^2} \leq \max_t \max(|F(\sigma+it)|, |F(2-\sigma+it)|) = \max_t |F(\sigma+it)|.$$

From the perspective of attaining numerically feasible bounds it is desirable to fix, at the outset, a value for σ. We take $\sigma = 7/6$ and thus concentrate on bounding $|F(7/6+it)|$, a choice which is admittedly somewhat arbitrary. At any rate, it suffices for our purposes. In the sequel θ will denote a complex number, not necessarily the same at each occurence, with $|\theta| \leq 1$.

To obtain an upper bound for $\max_t |F(7/6+it)|$, we obtain an upper bound for $\log|L_a(7/6+it)|$, and similarly obtain a lower bound for $\log|L(5/6+it)|$. For brevity, we only discuss the automorphic case here with the understanding that an analogous treatment holds in the Dirichlet case (see [**OS**]).

We first obtain explicit formulae for $-L_a'(s)/L_a(s)$. From the Euler product for $L_a(s)$, we obtain the Dirichlet series expansion, for $\sigma > 3/2$

$$-\frac{L_a'}{L_a}(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)\chi(n)}{n^s}.$$

It is easy to check that $\lambda(n) = 0$ unless n is the power of a prime p. Further, if we write $A(p) = \alpha + \overline{\alpha}$ with $|\alpha| = \sqrt{p}$, then $\lambda(p^m) = (\alpha^m + \overline{\alpha}^m)\log p$ for all $m \geq 1$. Consequently $|\lambda(p^m)| \leq 2p^{m/2}\log p$ and so $|\lambda(n)| \leq 2\sqrt{n}\Lambda(n)$ for all n.

LEMMA 1. *Let $X > 0$ be a real number and put*

$$\mathcal{F}_1(s, X) = \sum_{n=1}^{\infty} \frac{\lambda(n)\chi(n)}{n^s} e^{-n/X}.$$

Let ρ_a denote a typical non-trivial zero of $L_a(s)$. If $L_a(s) \neq 0$ then

$$-\frac{L_a'}{L_a}(s) = \mathcal{F}_1(s, X) + R_{\text{sig}}(s) + R_{\text{tri}}(s) + R_{\text{ins}}(s)$$

where

$$R_{sig}(s) = \sum_{\rho_a} X^{\rho_a - s}\Gamma(\rho_a - s), \qquad R_{tri}(s) = \sum_{n=0}^{\infty} X^{-n-s}\Gamma(-n-s),$$

and

$$R_{ins}(s) = \sum_{n=1}^{\infty} \frac{(-X)^{-n}}{n!} \frac{L_a'}{L_a}(s-n).$$

LEMMA 2. *If $L_a(s) \neq 0$ then*

$$\Re\frac{L_a'}{L_a}(s) = -\frac{1}{2}\log\frac{q}{4\pi^2} - \Re\frac{\Gamma'}{\Gamma}(s) + \sum_{\rho_a}\Re\frac{1}{s-\rho_a}$$

where ρ_a runs over the non-trivial zeros of $L_a(s)$.

With these lemmas we can obtain an upper bound for $\log|L_a(7/6 + it)|$.

PROPOSITION 1. *Let X be a positive real number and put*

$$\mathcal{F}(s, X) = \sum_{n=2}^{\infty} \frac{\lambda(n)\chi(n)}{n^s \log n} e^{-n/X}.$$

Let $s = 7/6 + it$, $s_1 = 11/6 + it$ and $s_2 = 27/20 + it$. Let

$$\beta(X) = -\frac{7}{20}\frac{X^{7/20}}{\Gamma(13/20) + X^{7/20}} \int_{7/6}^{11/6} X^{1-u}\Gamma(1-u)du$$

and

$$\alpha(X) = \max_y \left| \int_{1/6}^{5/6} X^{-u}\Gamma(-u+iy)du - \frac{\beta(X)}{X^{7/20}}\Gamma(-7/20+iy) \right| \left(\frac{7}{20} + \frac{20}{7}y^2\right).$$

If $X \geq \max(500, 5\log(q/4\pi^2))$ *then*

$$\log|L_a(s)| \leq \frac{X}{X+1}\Re\mathcal{F}(s,X) + \frac{1}{4} + \frac{\log(1+t^2)}{75} + \frac{(5\alpha(X)-\beta(X))}{4}\left(\frac{51}{100}\log\frac{q}{4\pi^2} + \frac{3}{4}\log(1+t^2) - \Re\mathcal{F}_1(s_2,X)\right).$$

Further, if $X \geq \max(5000, 5\log(q/4\pi^2))$ *then*

$$\log|L_a(s)| \leq \frac{X}{X+1}\Re\mathcal{F}(s,X) + \frac{1}{4} + \frac{\log(1+t^2)}{8} + \frac{1}{7X^{1/6}}\log\frac{q}{4\pi^2} - \frac{5}{18X^{1/6}}\Re\mathcal{F}_1(s_2,X).$$

To obtain this we begin by integrating both sides of Lemma 1 from $s = 7/6+it$ to $s_1 = 11/6+it$. The contribution of the trivial zeros of $L_a(s)$ and the insignificant poles of the Γ-function are handled in a straight-forward, albeit tedious way. The main difficulties lies in the treatment of the contribution of the non-trivial zeros of $L_a(s)$: that is, $\int_s^{s_1}\Re R_{sig}(w)dw$. Let us first discuss the second assertion of Proposition 1. The contribution of an individual zero $\rho_a = 1+i\gamma_a$ is

$$\begin{aligned}\int_s^{s_1} &\Re X^{\rho_a-w}\Gamma(\rho_a-w)dw \\ &= \theta\left|\int_{7/6}^{11/6} X^{1-u}\Gamma(1-u+i(\gamma_a-t))du\right|\left(\frac{7}{20}+\frac{20}{7}(t-\gamma_a)^2\right)\Re\frac{1}{s_2-\rho_a} \\ &= \theta\gamma(X)\Re\frac{1}{s_2-\rho_a},\end{aligned}$$

where

$$\gamma(X) = \max_y\left|\int_{1/6}^{5/6} X^{-u}\Gamma(-u+iy)du\right|\left(\frac{7}{20}+\frac{20}{7}y^2\right).$$

Summing over all zeros ρ_a we obtain

$$\int_s^{s_1}\Re R_{sig}(w)dw = \theta\gamma(X)\sum_{\rho_a}\Re\frac{1}{s_2-\rho_a}.$$

Note the role played by the positivity of $\Re(1/(s_2-\rho_a))$ in the above argument. Hadamard's factorization formula (Lemma 2) affords an alternate expression for $\sum_{\rho_a}\Re(s_2-\rho_a)^{-1}$ as a sum of $\Re L_a'(s_2)/L_a(s_2)$ and other easily handled terms. The $\Re L_a'(s_2)/L_a(s_2)$ term causes us some difficulties here. We deal with it by using Lemma 1 to essentially reduce the problem to estimating $\Re R_{sig}(s_2)$. This quantitiy is estimated by repeating the argument used above: that is, by bounding each individual term, $\Re X^{\rho_a-s_2}\Gamma(\rho_a-s_2)$, by some function of X times $\Re(s_2-\rho_a)^{-1}$ and then summing and using Hadamard factorization. Residual traces of these complications may be seen in the presence of the terms involving $\Re\mathcal{F}_1(s_2,X)$ in Proposition 1.

The bound obtained in this way is not sufficiently effective for 'small' values of q. The first assertion is used to obtain more economical constants (at the price of greater complications) for these q. We expect that the maximum over y in the expression defining $\alpha(X)$ is attained at $y = 0$. If so, then $\alpha(X)$ would have the value

$$\frac{7}{20}\left|\int_{1/6}^{5/6} X^{-u}\Gamma(-u)du - \frac{\beta(X)}{X^{7/20}}\Gamma(-7/20)\right| = \beta(X).$$

This is not a proof, and we have merely demonstrated that $\alpha(X) \geq \beta(X)$, but this expectation should help motivate our definitions of $\alpha(X)$ and $\beta(X)$. In our application $\alpha(X)$ and $\beta(X)$ will turn out to be very nearly equal. In other words, we obtain further savings by exploiting the fact that if

$$\log(|L_a(s)|/|L_a(s_1)|) = \int_s^{s_1} -\Re L_a'(w)/L_a(w)dw$$

is very large, then $-\Re L_a'(s_2)/L_a(s_2)$ is very large as well. This works in our favor by forcing (see Lemma 2)

$$\sum_{\rho_a} \Re\frac{1}{s_2-\rho_a} = \left(\frac{1}{2}\log\frac{q}{4\pi^2} + \Re\frac{\Gamma'}{\Gamma}(s_2) + \Re\frac{L_a'}{L_a}(s_2)\right)$$

to be small.

To make this more precise, consider the contribution of an indvidual zero $\rho_a = 1 + i\gamma_a$ to $\int_s^{s_1} \Re R_{sig}(w)dw$ with $y = \gamma_a - t$:

$$\begin{aligned}\Re\int_s^{s_1} X^{\rho_a - w}\Gamma(\rho_a - w)dw &= \Re\beta(X)X^{\rho_a - s_2}\Gamma(\rho_a - s_2)\\ &\quad + \Re\left(\int_{1/6}^{5/6} X^{-u+iy}\Gamma(-u+iy)du - \beta(X)X^{-7/20+iy}\Gamma(-7/20+iy)\right)\\ &\leq \beta(X)\Re R_{sig}(s_2) + \alpha(X)\Re\frac{1}{s_2-\rho_a}.\end{aligned}$$

Summing over all zeros ρ_a we obtain

$$\int_s^{s_1} \Re R_{sig}(w)dw \leq \alpha(X)\sum_{\rho_a}\Re\frac{1}{s_2-\rho_a} + \beta(X)\Re R_{sig}(s_2). \tag{16}$$

As usual we use the partial fractions decomposition of Lemma 2 to estimate $\sum_{\rho_a}\Re(s_2-\rho_a)^{-1}$ in terms of $\Re L_a'(s_2)/L_a(s_2)$ and other easy terms and then we use Lemma 1 to reduce the $\Re L_a'(s_2)/L_a(s_2)$ term to $-\Re R_{sig}(s_2)$. In this way we loosely obtain

$$\sum_{\rho_a}\Re\frac{1}{s_2-\rho_a} \leq \text{“known terms”} + \Re\frac{L_a'}{L_a}(s_2) \leq \text{“known terms”} - \Re R_{sig}(s_2).$$

Since $\alpha(X)$ and $\beta(X)$ are expected to be nearly equal we see upon using this in (16) that the meddlesome $\Re R_{sig}(s_2)$ term has been practically eliminated! This plan is executed and the tedious details appear in [§8, **OS**]. The net effect of this trick is to save a factor of $X^{7/20}/(\Gamma(13/20) + X^{7/20})$ which, although negligible for large X, is of vital importance to the 'small' range of q where we apply it.

Proposition 1 and its analog for the Dirichlet case constitute the bulk of our argument for Theorem 3. Using them we establish without too much difficulty, Theorem 3 in the range $50 \leq \log(q/4\pi^2)$. Since the range $N \leq 2\cdot 10^{10}$ includes the range $\log(q/4\pi^2) \leq 50$, we see that the proof of Theorem 3 is complete. □

References

[BHs] J. Benham and J. Hsia, *Spinor equivalence of quadratic forms*, J. Number Theory **17** (1983), 337–342.

[Co] D. Cox, *Primes of the form $x^2 + ny^2$*, Wiley, New York, 1989.

[Di1] L. E. Dickson, *Ternary quadratic forms and congruences*, Ann. Math. **28** (1926), 333–341.

[Di2] ———, *History of the theory of numbers*, G.E. Strechert & Co., 1934.

[Du] W. Duke, *Hyperbolic distribution problems and half-integral weight Maass forms*, Invent. Math. **92** (1988), 73–90.

[DS-P] W. Duke and R. Schulze-Pillot, *Representations of integers by positive ternary quadratic forms and equidistribution of lattice points on ellipsoids*, Invent. Math. **99** (1990), 49–57.

[Go] D. Goldfeld, *Gauss' class number problem for imaginary quadratic fields*, Bull. Amer. Math. Soc. **13** (1985), 23–37.

[GZ] B. Gross and D. Zagier, *Heegner points and derivatives of $L-$series*, Invent. Math. **84** (1986), 225–320.

[Gu] H. Gupta, *Some idiosyncratic numbers of Ramanujan*, Proc. Indian Acad. Sci., Ser. A **13** (1941), 519–520.

[Hs] J. Hsia, *Regular positive ternary quadratic forms*, Mathematika **28** (1981), 231–238.

[I] H. Iwaniec, *Fourier coefficients of modular forms of half-integral weight*, Invent. Math. **87** (1987), 385–401.

[Jo1] B. Jones, *The arithmetic theory of quadratic forms*, Math. Assoc. Amer., Carus 10, 1967.

[Jo2] B. Jones, *The regularity of a genus of positive ternary quadratic forms*, Trans. Amer. Math. Soc. **33** (1931), 111–124.

[JP] B. Jones and G. Pall, *Regular and semi-regular positive ternary quadratic forms*, Acta. Math. **70** (1939), 165–191.

[Ka1] I. Kaplansky, *The first nontrivial genus of positive definite ternary forms*, Math. Comp. **64** (1995), 341–345.

[Ka2] I. Kaplansky, *Ternary positive quadratic forms that represent all odd integers*, Acta. Arith. **70** (1995), 209–214.

[Ko] N. Koblitz, *Introduction to elliptic curves and modular forms*, Springer-Verlag, New York, 1984.

[L] J. E. Littlewood, *On the class number of the corpus $P(\sqrt{-k})$*, Proc. London Math. Soc., Ser. 2 **27** (1927), 358–372.

[OS] K. Ono and K. Soundararajan, *Ramanujan's ternary quadratic form*, Invent. Math. **130** (1997), 415–454.

[R] S. Ramanujan, *On the expression of a number in the form $ax^2 + by^2 + cz^2 + du^2$*, Proc. Camb. Phil. Soc. **19** (1916), 11–21.

[Ru] W. Rudin, *Real and complex analysis*, McGraw Hill, New York, 1987.

[Sh] G. Shimura, *On modular forms of half-integral weight*, Ann. Math. **97** (1973), 440–481.

[Si] J. Silverman, *The arithmetic of elliptic curves*, Springer-Verlag, New York, 1986.

[T] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford Univ. Press, Oxford, 1986.

[Wal] J.-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier*, J. Math. Pures et Appl. **60** (1981), 375–484.

Department of Mathematics, Penn State University, University Park, Pa 16802, USA

E-mail address: `ono@math.psu.edu`

Department of Mathematics, Princeton University, Princeton, New Jersey 08540, U.S.A.

E-mail address: `skannan@math.princeton.edu`