GORDON'S ϵ-CONJECTURE ON THE LACUNARITY OF MODULAR FORMS

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Abstract. In this note we prove B. Gordon's ϵ-conjecture regarding the lacunarity of modular forms. We show that if $f = \sum_{n=0}^{\infty} a(n)q^n \in M_k(N,\chi)$ has the property that there exists an $\epsilon > 0$ for which

$$\#\{n < X \mid a(n) \neq 0\} = O\left(X^{1-\epsilon}\right),$$

then $f(z)$ is a finite linear combination of theta series of weight 1/2 or 3/2.

ABSTRACT. Ici, on démontre une conjecture de B. Gordon qui s'agit de la lacunarité des formes modulaire. Soit $f = \sum_{n=1}^{\infty} a(n)q^n \in M_k(N,\chi)$ une forme modulaire. On montre que si il existe un $\epsilon > 0$ pour que

$$\#\{n < X \mid a(n) \neq 0\} = O(X^{1-\epsilon}),$$

puis f est une combination des series theta du poids 1/2 ou 3/2.

A formal power series $P(q) := \sum_{n \geq N_0} a(n)q^n$ is called *lacunary* if

$$\lim_{X \to \infty} \frac{\#\{n < X \mid a(n) = 0\}}{X} = 1.$$

These power series have the property that "almost all" of their coefficients are zero. Many important q-series in the theory of partitions are lacunary. For instance the following well known identities are examples of lacunary power series:

(Euler) $$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{3n^2+n}{2}},$$

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(Jacobi) $$\prod_{n=1}^{\infty}(1-q^n)^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n^2+n}{2}}.$$

For each $k \in \frac{1}{2}\mathbb{Z}$, let $M_k(N,\chi)$ be the space of modular forms of weight k on $\Gamma_0(N)$ (if k is half-integral then $4|N$) with Nebentypus character χ, and let $S_k(N,\chi)$ denote its subspace of cusp forms. In this note we are interested in those $f(z) \in M_k(N,\chi)$ whose Fourier expansions $f(z)=\sum_{n=0}^{\infty}a(n)q^n$ (throughout $q := e^{2\pi iz}$) are lacunary. Serre [S] proved a "basis theorem" for lacunary integral weight forms. He proved that an integral weight $f(z)$ is lacunary if and only if it is a finite linear combination of forms with complex multiplication. Using this description, Serre [S2] and Gordon, Hughes and Robins [G-H, G-R] have classified all the lacunary integer weight modular forms in certain special families of forms whose Fourier expansions are given by infinite products. V. K. Murty [M] has obtained an intriguing alternative description of the lacunary integer weight forms.

The characterization of lacunary half-integral weight modular forms remains open. Elementary theta functions serve as convenient examples of lacunary half-integral weight forms. If $i=0$ or 1, $0 \le r < t$, and $a \ge 1$, then the elementary theta function $\theta_{a,i,r,t}(z)$ is given by

$$\theta_{a,i,r,t}(z) := \sum_{n \equiv r \pmod{t}} n^i q^{an^2}.$$

Each function $\theta_{a,i,r,t}(z)$ is a holomorphic form of weight $i+\frac{1}{2}$, and any $f(z)=\sum_{n=0}^{\infty}a(n)q^n$ that is a finite linear combination of such series is called *superlacunary*. In particular every superlacunary form has weight 1/2 or 3/2. By a theorem of Serre and Stark [S-Sta], it is well known that every weight 1/2 modular form is superlacunary. Clearly every superlacunary $f(z)$ is lacunary since there exists a non-zero constant c_f for which

$$\#\{n < X \mid a(n) \neq 0\} \sim c_f\sqrt{X}.$$

Recalling Dedekind's eta-function $\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$, we find that the identities above are examples by Euler and Jacobi obtained from

$$\eta(24z)=\sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2},$$

$$\eta^3(8z)=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{(2n+1)^2}.$$

It is widely believed that every lacunary half-integral weight modular form is superlacunary, i.e. is a finite linear combination of elementary theta series. A proof of this conjecture seems to be well beyond current methods. In view of these technical difficulties, Gordon posed the following unpublished conjecture.

Gordon's ϵ-Conjecture. *If $f(z) = \sum a(n)q^n$ belongs to $M_k(N,\chi)$ and has the property that there exists an $\epsilon > 0$ for which*

$$\#\{n < X \mid a(n) \neq 0\} = O(X^{1-\epsilon}),$$

then $f(z)$ is superlacunary.

In this note we prove:

Theorem 1. *If $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in M_k(N,\chi)$ is not superlacunary, then*

$$\#\{n < X \mid a(n) \neq 0\} \gg_f X/\log X.$$

Corollary 1. *Gordon's ϵ-conjecture is true.*

It is well known that Lehmer speculated that Ramanujan's function $\tau(n)$ is non-zero for every positive integer n. Recall that $\tau(n)$ is defined by

$$\sum_{n=1}^{\infty} \tau(n)q^n := q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

In view of Lehmer's conjecture and Serre's paper on the lacunarity of even powers of the eta-function, we record an elementary corollary that contains estimates on the number non-zero coefficients of all the powers of the eta-function. Although one can make better estimates in many cases, we have sacrificed this for a clear and comprehensive statement.

Corollary 2. *If r is a positive integer, then define $\tau_r(n)$ by*

$$\sum_{n=0}^{\infty} \tau_r(n)q^n := \prod_{n=1}^{\infty}(1-q^n)^{r}.$$

If $r \neq 1$ or 3, then

$$\#\{n < X \mid \tau_r(n) \neq 0\} \gg_r \begin{cases} X & \textit{if } r \neq 2,4,6,8,10,14,26 \quad \textit{is even}, \\ X/\log X & \textit{if } r \quad \textit{odd or } r = 2,4,6,8,10,14,26. \end{cases}$$

Proof of Results

If $f \in S_{k+\frac{1}{2}}(N,\chi)$ has the property that for every prime $p \nmid N$ there exists a complex number $\lambda(p)$ for which

$$T(p^2)|f = \lambda(p)f,$$

then we shall refer to f as an "eigenform." The author and C. Skinner [O-S] proved the following key lemma. For each positive integer r let $P(r)$ denote the set

$$P(r) := \{D \mid D > 1 \ \text{ square-free with exactly } r \ \text{ prime factors}\}.$$

Lemma 1. *Let* $g(z)=\sum_{n=1}^{\infty} b(n)q^n \in S_{k+\frac{1}{2}}(N,\chi)$ *be an eigenform for which*

(*i*) $b(m)\neq 0$ *for at least one square-free* $m>1$ coprime to N,

(*ii*) *the coefficients* $b(n)$ *are algebraic integers contained in a number field* K.

Let v *be a place of* K *over* 2, *and for each* s *let*

$$B_s := \{m \mid m>1 \ \ \textit{square-free},\ (m,N)=1,\ \textit{and}\ \mathrm{ord}_v(b(m))=s\}.$$

Let s_0 *be the smallest integer for which* $B_{s_0}\neq\emptyset$. *If* $B_{s_0}\cap P(r)\neq\emptyset$, *then*

$$\#\{m\in B_{s_0}\cap P(r) \mid m\leq X\} \gg \frac{X}{\log X}(\log\log X)^{r-1}.$$

Proof of Theorem 1. In view of Serre's work [S] it is well known that every integral weight modular form $f(z)=\sum_{n=1}^{\infty} a(n)q^n$ has the property that

$$\#\{n<X \mid a(n)\neq 0\} \gg_f X/\log X.$$

Moreover amongst the half-integral weight forms it is well known that we can without loss of generality assume that $f(z)$ is a cusp form, and by the theorem of Serre and Stark we may assume that its weight $\geq 3/2$.

Let $f(z)=\sum_{n=1}^{\infty} a(n)q^n \in S_{k+\frac{1}{2}}(N,\chi)$ be an eigenform. If $f(z)$ is not superlacunary, then the conclusion of the lemma shows that the number of $n<X$ for which $a(n)\neq 0$ is $\gg_f \frac{X}{\log X}$. It suffices to show that the hypotheses of the lemma are satisfied for a suitable non-trivial scalar multiple of $f(z)$. Since f is in the orthogonal complement of the elementary theta series, its Shimura lift is a weight $2k$ cuspidal eigenform. Hence by Waldspurger theory there exists an arithmetic progression with the property that for every square-free n coprime to N the number $a(n)^2$ is the "algebraic part" of the central critical value of the modular L-function of the Shimura lift of $f(z)$ twisted by a quadratic character. (see [Wal,Corollary 2]).

Verifying (*i*) now follows from a theorem of Friedberg and Hoffstein [F-H] that guarantees that infinitely many such values are non-zero. To show that $f(z)$ satisfies (*ii*) one may consult the theory of modular symbols [G-S, M-T-T], i.e. the existence of uniform periods of modular L-functions of twists so that the "algebraic parts" of these twisted values are algebraic integers in some number field K. Therefore Theorem 1 holds for every non-superlacunary eigenform.

Now we consider the case where f is not an eigenform. This argument is similar to the integral weight argument employed in [S,M]. If $g=\sum b(n)q^n \in S_{k+\frac{1}{2}}(N,\chi)$, then define $M_g(X)$ by

$$M_g(X) := \#\{n<X \mid b(n)\neq 0\}.$$

It is easy to see that

$$M_{g_1+g_2}(X) \leq M_{g_1}(X) + M_{g_2}(X). \tag{1}$$

Suppose that $f(z) \in S_{k+\frac{1}{2}}(N,\chi)$ has the property that $M_f(X) = O(X^{1-\epsilon})$ for some $\epsilon > 0$. If $f = f_\theta + f_1$ where f_θ is superlacunary or trivial, and f_1 is orthogonal to the elementary theta series, then by (1) we find that $M_{f_1(X)} \leq M_f(X) + M_{f_\theta}(X)$. In particular there exists an $\epsilon_1 > 0$ for which $M_{f_1}(X) = O(X^{1-\epsilon_1})$.

Recall that if p is prime and $g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+\frac{1}{2}}(N,\chi)$, then

$$g(z) \mid T_{p^2} := \sum_{n=1}^{\infty}(b(p^2n) + \chi(p)\left(\frac{(-1)^k n}{p}\right)p^{k-1}b(n) + \chi(p^2)p^{2k-1}b(n/p^2))q^n. \tag{2}$$

By a quick examination of (2) one finds that

$$M_{T(p^2)|g}(X) \leq M_g(p^2X) + 2M_g(X). \tag{3}$$

Now let $\mathbb{T}$ be the Hecke algebra and let $\mathbb{X} := \mathbb{T}f_1(z)$. By (1) and (3) we see that for every $h(z) \in \mathbb{X}$ that $M_h(X) = O(X^{1-\epsilon_1})$. Since $\mathbb{T}$ is commutative, every simple $\mathbb{T}$ submodule of $\mathbb{X}$ is of the form $\mathbb{C}h(z)$, but on the other hand $h(z)$ is an eigenform. Therefore by the eigenform case we find that $M_h(X) \gg \frac{X}{\log X}$, and this is a contradiction.

Q.E.D.

Proof of Corollary 2. Serre [S2] proved that the only even r for which $\sum_{n=0}^{\infty} \tau_r(n)q^n$ is lacunary are $r = 2, 4, 6, 8, 10, 14, 26$. The result follows immediately from Theorem 1.

Q.E.D.

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