On cooperative parabolic systems: Harnack inequalities and asymptotic symmetry

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Abstract

We consider fully nonlinear weakly coupled systems of parabolic equations on a bounded reflectionally symmetric domain. Assuming the system is cooperative we prove the asymptotic symmetry of positive bounded solutions. To facilitate an application of the method of moving hyperplanes, we derive Harnack type estimates for linear cooperative parabolic systems.

Keywords: Parabolic cooperative systems, positive solutions, asymptotic symmetry, Harnack inequality.

Contents

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ for some $N \geq 1$. We consider systems of nonlinear parabolic equations of the form

$$\partial_t u_i = F_i(t, x, u, Du_i, D^2 u_i), \quad (x, t) \in \Omega \times (0, \infty), \; i = 1, \dots, n, \tag{1.1}$$

where $n \geq 1$ is an integer, $u := (u_1, \cdots, u_n)$, and Du_i, $D^2 u_i$ denote, respectively, the gradient and Hessian matrix of u_i with respect to x. The hypotheses on the nonlinearities F_i, which we formulate precisely in the next section, include, in particular, regularity (Lipschitz continuity), ellipticity, and the cooperativity condition requiring the derivatives $\partial F_i / \partial u_j$ to be nonnegative for $i \neq j$. A model problem to which our results apply is the reaction-diffusion system

$$u_t = D(t)\Delta u + f(t, u), \quad (x, t) \in \Omega \times (0, \infty), \tag{1.2}$$

where $D(t) = \text{diag}(d_1(t), \dots, d_n(t))$ is a diagonal matrix whose diagonal entries are continuous functions bounded above and below by positive constants, and $f = (f_1, \dots, f_n) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function which is Lipschitz continuous in u and which satisfies $\partial f_i(t, u)/\partial u_j \geq 0$ whenever $i \neq j$ and the derivative exists.

Observe that while the more general system (1.1) is fully nonlinear, it is still only weakly coupled in the sense that the arguments of F_i do not involve the derivatives of u_j for $j \neq i$.

We complement the system with Dirichlet boundary conditions

$$u_i(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, \infty), \; i = 1, \dots, n. \tag{1.3}$$

Our main goal is to investigate symmetry properties of positive solutions of (1.1), (1.3). More specifically, assuming that Ω is convex in x_1 and symmetric with respect to the hyperplane $H_0 := \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 = 0\}$ and that the nonlinearities F_i satisfy suitable symmetry conditions, we prove the asymptotic symmetry of positive global bounded solutions. Our theorems extend earlier symmetry results for scalar parabolic and elliptic equations and for elliptic systems. We now summarize the previous results briefly.

In the celebrated paper [22], Gidas, Ni and Nirenberg considered the semilinear problem

$$\begin{aligned} \Delta u + f(u) &= 0, \qquad x \in \Omega \\ u &= 0, \qquad x \in \partial\Omega, \end{aligned}$$

where f is Lipschitz, the domain Ω is reflectionally symmetric as above, and $\partial\Omega$ is of class C^2. They proved that any positive solution u is even in x_1 and monotone in x_1 for $x_1 > 0$: $\partial_{x_1} u < 0$ in $\{x \in \Omega : x_1 > 0\}$. Of course, if Ω is also symmetric and convex in other directions, then so is the solution. In particular, positive solutions on a ball are radially symmetric. The method of moving hyperplanes, which, combined with the maximum principle, is the basic tool in the paper, was introduced earlier by Alexandrov [1] and further developed by Serrin [38] ([38] also contains a related result on radial symmetry).

Similar symmetry results were later proved for fully nonlinear elliptic equations $F(x, u, Du, D^2u) = 0$, first for smooth domains by Li [29], later for general bounded symmetric domains by Berestycki and Nirenberg [8] (see also [18]). Da Lio and Sirakov [17] considered an even more general class of elliptic equations, including equations involving Pucci operators, and they proved the symmetry of positive viscosity solutions. Many authors, starting again with Gidas, Ni and Nirenberg [23], established symmetry and monotonicity properties of positive solutions of elliptic equations on unbounded domains under various conditions on the equations and the solutions. For surveys of these theorems, as well as additional results on bounded domains, we refer the readers to [7, 27, 33].

Extensions of the symmetry results to cooperative elliptic systems were first made by Troy [40], then by Shaker [39] (see also [15]) who considered semilinear equations on smooth bounded domains. In [20], de Figueiredo removed the smoothness assumption on the domain in a similar way as Berestycki and Nirenberg [8] did for the scalar equation. For cooperative systems on the whole space, a general symmetry result was proved by Busca and Sirakov [11], an earlier more restrictive result can be found in [21]. The cooperative structure of the system assumed in all these references is in some sense unavoidable. Without it, neither is the maximum principle applicable nor do the symmetry result hold in general (see [12] and [39] for counterexamples).

For parabolic equations, first symmetry results in the same spirit appeared in the nineties. In [19], Dancer and Hess proved the spatial symmetry of periodic solutions of time-periodic reaction diffusion equations. Then, in [25], reaction diffusion equations with general time dependence were considered and the asymptotic spatial symmetry of positive bounded solutions was established. In an independent work, Babin [4, 5] considered fully nonlinear autonomous equations on bounded domains and proved the spatial symmetry of bounded solutions defined for all $t \in \mathbb{R}$ (henceforth we refer to such solutions as entire solutions). Using this, in combination with a version of invariance principle, he also obtained the asymptotic symmetry of suitably bounded solutions defined for $t \geq 0$ (global solutions). Babin and Sell [6] then extended these results to time-dependent fully nonlinear equations. Unlike in [25], no smoothness of the domain Ω was assumed in [4, 5, 6]. On the other hand, rather strong positivity hypotheses were made in these papers on the nonlinearity and the solution, requiring in particular the solution to stay away from zero at all (interior) points of Ω. This and other shortcomings of the existing results (see the introduction of [36] for an account) motivated the second author of the present paper for writing [36]. Using techniques originally developed for symmetry theorems for parabolic equations on $\mathbb{R}^N$, see [34, 35], he removed these restrictions and made further improvements. We wish to emphasize that when the solution is allowed to converge to zero along a sequence of times, as in [36] (and also in the present paper), the proof of the asymptotic symmetry gets significantly more complicated (for comparison, the reader can refer to the appendix of [36], where a much simpler proof for solutions staying away from zero is given).

Other types of parabolic symmetry results can be found in [16], where the asymptotic roundness of hypersurfaces evolving under a geometric flow is established and in [26], where the asymptotic roundness of a (rescaled) traveling front is proved for a reaction diffusion equation. A variant of this argument and its application in a multidimensional Stefan problem appeared in [32].

To the best of our knowledge, the present paper is the first to contain symmetry results similar to those in [4, 5, 6, 25, 36] for cooperative parabolic systems. We prove theorems extending the main results of [36]. To do so, we follow the basic approach and some key ideas of [36], extending the needed technical results to the present setting. Not all these extensions are straightforward. In particular, new difficulties arise is connection with the fact that different components of the solution may be small at different times. This situation has to be handled carefully using Harnack type estimates which we develop for this purpose. As these estimates are of independent interest, we have devoted a part of the paper to linear cooperative systems, see Subsection 3.2. The estimates derived there extend similar results for elliptic cooperative systems as given in [3, 14, 12]. See Remark 3.7 for additional bibliographical comments.

Our general symmetry theorems are formulated in Section 2. To give the reader their flavor, we now state a simpler result dealing with the semilinear system (1.2).

For $\lambda \in \mathbb{R}$ we denote

$$H_\lambda := \{x \in \mathbb{R}^N : x_1 = \lambda\},$$
$$\Omega_\lambda := \{x \in \Omega : x_1 > \lambda\}.$$

Assume that

(D1) Ω is convex in x_1 and symmetric with respect to the hyperplane H_0;

(D2) for each $\lambda > 0$ the set Ω_λ has only finitely many connected components.

Denoting by $\mathbb{R}^n_+$ the cone in $\mathbb{R}^n$ consisting of vectors with nonnegative components, we further assume that

(S1) $D(t) = \operatorname{diag}(d_1(t), \dots, d_n(t))$, where $d_i : [0, \infty) \to \mathbb{R}$ are continuous functions satisfying $\alpha_0 \le d_i(t) \le \beta_0$ $(t \ge 0,\ i = 1, \dots, n)$ for some positive constants α_0 and β_0;

(S2) $f = (f_1, \dots, f_n) : [0, \infty) \times \mathbb{R}^n_+ \to \mathbb{R}^n$ is a continuous function which is locally Lipschitz in u uniformly with respect to t: for each M there exists a constant $\beta_1 = \beta_1(M) > 0$ such that

$$|f(t,u) - f(t,v)| \le \beta_1 |u - v| \quad (t \ge 0,\ u, v \in \mathbb{R}^n_+, |u|, |v| \le M);$$

(S3) for each $i \ne j$ and $t \in [0, \infty)$ one has $\partial f_i(t,u)/\partial u_j \ge 0$ for each $u \in \mathbb{R}^n_+$ such that the derivative exists (which is for almost every u by (S2)).

The following strong cooperativity condition will allow us to relax the positivity assumptions on the considered solutions. In Section 2 we formulate a different condition, the irreducibility of the system, which together with (S3) can be used in place of (S4).

(S4) For each M there is a constant $\sigma = \sigma(M) > 0$ such that for all $i \ne j$ and $t \in [0, \infty)$ one has $\partial f_i(t,u)/\partial u_j \ge \sigma$ for each $u \in \mathbb{R}^n_+$ with $|u| \le M$ such that the derivative exists.

We consider a global classical solution u of (1.2), (1.3) which is nonnegative (by which we mean that all its components are nonnegative) and bounded:

$$\sup_{x \in \Omega, t \ge 0} |u(x,t)| < \infty.$$

Moreover, we require that u assume the Dirichlet boundary condition uniformly with respect to time:

$$\lim_{\mathrm{dist}(x,\partial\Omega)\to 0} \sup_{t\geq 0} |u(x,t)| = 0. \tag{1.4}$$

We formulate the asymptotic symmetry of u in terms of its ω-limit set:

$$\omega(u) = \{z : z = \lim_{k\to\infty} u(\cdot, t_k) \text{ for some } t_k \to \infty\},$$

where the limit is taken in the space $E := (C(\bar{\Omega}))^n$ equipped with the supremum norm. It follows from standard parabolic interior estimates and the assumptions on u, specifically the boundedness and (1.4), that the orbit $\{u(\cdot,t) : t \geq 0\}$ is contained in a compact set in E. Therefore $\omega(u)$ is nonempty and compact in E and it attracts u in the sense that

$$\lim_{t\to\infty} \mathrm{dist}_E(u(\cdot,t), \omega(u)) = 0.$$

Theorem 1.1. *Assume (D1), (D2), (S1)–(S3) and let u be a bounded nonnegative global solution of* (1.2) *satisfying* (1.4). *Assume in addition that one of the following conditions holds:*

(i) there exists $\varphi = (\varphi_1, \ldots, \varphi_n) \in \omega(u)$ such that $\varphi_i > 0$ in Ω for all $i \in \{1, \ldots, n\}$;

(ii) (S4) holds and there is $\varphi \in \omega(u)$ such that $\varphi_i > 0$ in Ω for some $i \in \{1, \ldots, n\}$.

Then for each $z = (z_1, \ldots, z_n) \in \omega(u)\setminus\{0\}$ and each $i = 1, \ldots, n$, the function z_i is even in x_1 and it is strictly decreasing in x_1 on $\{x \in \Omega : x_1 > 0\}$.

The additional assumption in the theorem means that along a sequence of times all components of u (or at least some components, if (S4) is assumed) stay away from 0 at every $x \in \Omega$. Although this assumption can be relaxed somewhat, it cannot be removed completely even if u is required to be strictly positive and $n = 1$ (see [36, Example 2.3]). Without this assumption, a weaker symmetry theorem is valid if (S4) holds. Namely, the components of u symmetrize, as $t \to \infty$, around a hyperplane $\{x : x_1 = \mu\}$ with $\mu \geq 0$ possibly different from zero. The strong cooperativity condition (S4), or more general cooperativity and irreducibility conditions given in Section 2, are needed to guarantee that all the components symmetrize around the same hyperplane.

Theorem 1.1 follows from more general Theorem 2.1 given in the next section. The equicontinuity condition (2.4) assumed in Theorem 2.1 is easily verified in the semilinear setting using (1.4), the boundedness of u, and standard parabolic interior estimates, as mentioned above.

We remark that the technical assumption (D2) on Ω can be removed if

$$\liminf_{t\to\infty} u(x,t) > 0$$

for each $x \in \Omega$.

The remainder of the paper is organized as follows. In Section 2, after fixing general notation, we formulate our main theorems. Section 3 is devoted to linear systems. We prove there Harnack type estimates for positive solutions and a related result for sign-changing solutions. The proofs of our symmetry results are given in Section 4.

2 Statements of the symmetry results

As in the introduction, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, satisfying (D1), (D2), and for any $\lambda \in \mathbb{R}$ we denote

$$\begin{aligned} H_\lambda &= \{x \in \mathbb{R}^N : x_1 = \lambda\}, \\ \Omega_\lambda &= \{x \in \Omega : x_1 > \lambda\}. \end{aligned}$$

Also we let

$$\begin{aligned} x^\lambda &:= (2\lambda - x_1, x') \quad (x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1}), \\ \Omega'_\lambda &:= \{x^\lambda : x \in \Omega_\lambda\}, \\ \ell &:= \sup\{\lambda \geq 0 : \Omega \cap H_\lambda \neq \emptyset\}, \\ S &:= \{1, \dots, n\}. \end{aligned} \tag{2.1}$$

Note that (D1) implies that $\Omega'_\lambda \subset \Omega$ for each $\lambda \geq 0$.

We consider the fully nonlinear parabolic system (1.1) assuming that for each $i \in S$ the function

$$F_i : (t, x, u, p, q) \mapsto F_i(t, x, u, p, q) \in \mathbb{R}^n$$

is defined on $[0,\infty)\times\bar\Omega\times\mathcal{O}_i$, where $\mathcal{O}_i$ is an open convex subset of $\mathbb{R}^{n+N+N^2}$ invariant under the transformation

$$Q:(u,p,q)\mapsto(u,-p_1,p_2,\dots,p_N,\tilde q),$$
$$\tilde q_{ij}=\begin{cases}-q_{ij} & \text{if exactly one of } i,j \text{ equals } 1,\\ \ \ q_{ij} & \text{otherwise.}\end{cases}$$

We further assume that $F=(F_1,\dots,F_n)$ satisfies the following hypotheses:

(N1) *Regularity.* For each $i\in S$ the function $F_i:[0,\infty)\times\bar\Omega\times\mathcal{O}_i\to\mathbb{R}^n$ is continuous, differentiable with respect to q and Lipschitz continuous in (u,p,q) uniformly with respect to $(x,t)\in\bar\Omega\times\mathbb{R}^+$. This means that there is $\beta>0$ such that

$$|F_i(t,x,u,p,q)-F_i(t,x,\tilde u,\tilde p,\tilde q)|\le\beta|(u,p,q)-(\tilde u,\tilde p,\tilde q)|$$
$$((x,t)\in\bar\Omega\times\mathbb{R}^+,\ (u,p,q),(\tilde u,\tilde p,\tilde q)\in\mathcal{O}_i). \quad (2.2)$$

(N2) *Ellipticity.* There is a positive constant α_0 such that for all $i\in S$ and $(t,x,u,p,q)\in[0,\infty)\times\bar\Omega\times\mathcal{O}_i$, and $\xi\in\mathbb{R}^N$ one has

$$\sum_{j,k=1}^{N}\frac{\partial F_i}{\partial q_{jk}}(t,x,u,p,q)\xi_j\xi_k\ \ge\alpha_0|\xi|^2.$$

(N3) *Symmetry and monotonicity.* For each $i\in S$, $(t,u,p,q)\in[0,\infty)\times\mathcal{O}_i$, and any $(x_1,x'),(\tilde x_1,x')\in\Omega$ with $\tilde x_1>x_1\ge0$ one has

$$F_i(t,\pm x_1,x',Q(u,p,q))=F_i(t,x_1,x',u,p,q)\ge F_i(t,\tilde x_1,x',u,p,q)\,.$$

(N4) *Cooperativity.* For all $i,j\in S$, $i\ne j$, $(t,x,u,p,q)\in[0,\infty)\times\bar\Omega\times\mathcal{O}_i$ one has

$$\frac{\partial F_i}{\partial u_j}(t,x,u,p,q)\ge0,$$

whenever the derivative exists.

In some results we need to complement (N4) with the following condition.

(N5) *Irreducibility.* There exists $\sigma > 0$ such that for any nonempty subsets $I, J \subset S$ with $I \cap J = \emptyset$, $I \cup J = S$ there exist $i \in I$, $j \in J$ such that

$$\frac{\partial F_i}{\partial u_j}(t, x, u, p, q) \geq \sigma$$

for all $(t, x, u, p, q) \in [0, \infty) \times \bar{\Omega} \times \mathcal{O}_i$ such the derivative exists.

Note that the derivatives in (N4) and (N5) exist almost everywhere by (N1).

We remark that although the hypotheses are formulated with fixed constants α_0, β, σ, we really need them to be fixed on the range of (u, Du, D^2u) for each considered solution u. Thus, for example, if the solution in question is bounded and has bounded derivatives Du, D^2u, then the Lipschitz continuity can be replaced with the local Lipschitz continuity as in the introduction and similarly one can relax the ellipticity and irreducibility conditions. Note, however, that we do not assume any boundedness of the derivatives of u.

If $n = 2$, condition (N5) is equivalent to $\partial F_i/\partial u_j \geq \sigma$ for all $i, j \in S$, $i \neq j$. For $n \geq 3$, the latter condition is stronger than (N5), for example, consider functions satisfying $\partial F_i/\partial u_j \geq \sigma$ for all $i, j \in S$, $|i - j| = 1$ and $\partial F_i/\partial u_j \equiv 0$ for all $i, j \in S$, $|i - j| > 1$. It is not hard to verify that, aside from the uniformity in all variables, condition (N5) is equivalent to other commonly used notions of irreducibility, see for example [2] and references therein.

Notice that the assumption (N3) implies the following condition

(N3 cor) For each $i \in S$, F_i is even in x_1 and for any $(t, u, p, q) \in [0, \infty) \times \mathcal{O}_i$, $\lambda > 0$ and $(x_1, x') \in \Omega_\lambda$, $\lambda > 0$, one has

$$F_i(t, 2\lambda - x_1, x', Q(u, p, q)) \geq F_i(t, x_1, x', u, p, q).$$

This weaker condition is sufficient for some of our results but for simplicity and consistency we just assume (N3) in all our symmetry theorems.

By a *solution* of (1.1), (1.3) we mean a function $u = (u_1, \ldots, u_n)$ such that $u_i \in C^{2,1}(\Omega \times (0, \infty)) \cap C(\bar{\Omega} \times [0, \infty))$, $(u, Du_i, D^2u_i) \in \mathcal{O}_i$ for all $i \in S$ and u satisfies (1.1), (1.3) everywhere. By a *nonnegative (positive) solution* we mean a solution with all components u_i nonnegative (positive) in $\Omega \times (0, \infty)$. All solutions of (1.1) considered in this paper are assumed to be nonnegative, regardless of whether it is explicitly stated or not. We shall consider solutions such that

$$\sup_{t\in[0,\infty)} \max_{i\in S} \|u_i(\cdot,t)\|_{L^\infty(\Omega)} < \infty \tag{2.3}$$

and the functions $u(\cdot, \cdot + s)$, $s \geq 1$, are equicontinuous on $\bar{\Omega} \times [0, 1]$, that is,

$$\lim_{h\to 0} \sup_{\substack{x,\bar{x}\in\bar{\Omega},\, t,\bar{t}\geq 1,\\ |t-\bar{t}|,|x-\bar{x}|<h}} |u(x,t) - u(\bar{x},\bar{t})| = 0. \tag{2.4}$$

We remark that if for each $i \in S$ one has $(0, 0, 0) \in \mathcal{O}_i$ and the function $(x, t) \mapsto F_i(x, t, 0, 0, 0)$ is bounded on $\Omega \times [0, \infty)$, then (2.4) holds if (2.3) and (1.4) hold, or if (2.3) holds and Ω satisfies the exterior cone condition. This follows from the fact that each u_i solves a linear uniformly parabolic equation with bounded coefficients and a bounded right-hand side (see the proof of Proposition 2.7 in [36] for details).

The orbit $\{u(\cdot, t) : t \geq 0\}$ of a solution satisfying (2.3), (2.4) is relatively compact in the space $E = (C(\bar{\Omega}))^n$ and then, as noted in the introduction, the ω-limit set

$$\omega(u) = \{z : z = \lim_{k\to\infty} u(\cdot, t_k) \text{ for some } t_k \to \infty\},$$

is nonempty, compact in E and it attracts $u(\cdot, t)$ as $t \to \infty$.

We now state a more general version of Theorem 1.1.

Theorem 2.1. *Assume (D1), (D2), (N1) - (N4). Let u be a nonnegative solution of* (1.1), (1.3) *satisfying* (2.3) *and* (2.4). *Assume in addition that one of the following conditions holds:*

- *(i) there exists $\varphi = (\varphi_1, \dots, \varphi_n) \in \omega(u)$ such that $\varphi_i > 0$ in Ω for all $i \in \{1, \dots, n\}$;*
- *(ii) (N5) holds and there is $\varphi \in \omega(u)$ such that $\varphi_i > 0$ in Ω for some $i \in \{1, \dots, n\}$.*

Then for each $z = (z_1, \dots, z_n) \in \omega(u)$ and $i \in S$, the function z_i is even in x_1:

$$z_i(x_1, x') = z_i(-x_1, x') \quad ((x_1, x') \in \Omega_0), \tag{2.5}$$

and either $z_i \equiv 0$ on Ω or z_i is strictly decreasing in x_1 on Ω_0. The latter holds in the form $(z_i)_{x_1} < 0$ if $(z_i)_{x_1} \in C(\Omega_0)$.

The last condition, $(z_i)_{x_1} \in C(\Omega_0)$, is satisfied if $\{u_{x_1}(\cdot, t) : t \geq 1\}$ is relatively compact in $C(\bar{\Omega})$. This is the case if, for example, $D^2 u$ is bounded (which we do not assume).

Remark 2.2. We will prove (see the proof of Theorems 2.1 and 2.4 in Section 4) that if (N5) holds, in addition to all the other hypotheses of Theorem 2.1, then for any $\varphi \in \omega(u)$ the relation $\varphi_i(x) > 0$ holds *for all* $x \in \Omega$ and $i \in S$ as soon as it holds *for some* $x \in \Omega$ and $i \in S$. Hence either $\varphi \equiv 0$ or all its components are strictly positive in Ω. This of course may not be true if (N5) does not hold as can be seen on examples of decoupled systems.

The assumption (i) or (ii) in the previous theorem is somewhat implicit in that it requires some knowledge of the asymptotic behavior of the solution as $t \to \infty$. One can formulate various alternative more explicit conditions in terms of the nonlinearity or the domain. The next theorem, which extends [36, Theorem 2.4] to cooperative systems, shows that the asymptotic positivity of the nonlinearity implies that if (N5) holds then (ii) is satisfied unless u converges to zero as $t \to \infty$. Other conditions can be formulated in terms of regularity and geometry of Ω (see for example Theorem 2.5 below).

Theorem 2.3. *Assume (D1), (D2), (N1), (N2), (N4). Further assume that for each* $i \in S$ *one has* $(0, 0, 0) \in \mathcal{O}_i$ *and*

$$\liminf_{t \to \infty,\, x \in \Omega} F_i(t, x, 0, 0, 0) \geq 0. \tag{2.6}$$

Let u *be a nonnegative solution of* (1.1), (1.3) *satisfying* (2.3) *and* (2.4). *Then for each* $i \in S$ *either* $\|u_i(\cdot, t)\|_{L^\infty(\Omega)} \to 0$ *or else there exists* $\varphi \in \omega(u)$ *with* $\varphi_i > 0$ *in* Ω.

Without assumptions (i), (ii), Theorem 2.1 is not valid in general, even for positive solutions of scalar equations (see [36, Example 2.3]). However, the elements of $\omega(u)$ still have some reflectional symmetry property, although the symmetry hyperplane may not be the canonical one. This is stated in the following theorem which extends [36, Theorem 2.4].

Theorem 2.4. *Assume (D1), (D2), (N1) - (N4). Let* u *be a nonnegative solution of* (1.1), (1.3) *satisfying* (2.3) *and* (2.4). *Then there exists* $\lambda_0 \in [0, \ell)$ *such that the following assertions hold for each* $z \in \omega(u)$.

(i) *For every $j \in S$ the function z_j is nonincreasing in x_1 on Ω_{λ_0} and there are $i \in S$ and a connected component U of Ω_{λ_0} such that*

$$z_i(x_1, x') = z_i(2\lambda_0 - x_1, x') \quad ((x_1, x') \in U). \tag{2.7}$$

(ii) *If (N5) is satisfied, there is a connected component U of Ω_{λ_0} such that* (2.7) *holds for each $i \in S$.*

(iii) *If (N5) is satisfied and Ω_{λ_0} is connected, then* (2.7) *holds with $U = \Omega_{\lambda_0}$ for all $i \in S$ and either $z \equiv 0$ on Ω_{λ_0} or else for each $i \in S$ the function z_i is strictly decreasing in x_1 on Ω_{λ_0}. The latter holds in the form $(z_i)_{x_1} < 0$ if $(z_i)_{x_1} \in C(\Omega_{\lambda_0})$.*

As usual, in many problems with rotational symmetry, one can use reflectional symmetries in different directions to prove the radial symmetry of solutions. We give only one such symmetry results, assuming Ω is a ball. Rotational symmetry in just some variables can be examined in a similar way. Notice that in the next theorem we do not need the assumption on the existence of a positive element of $\omega(u)$.

Assume that Ω is the unit ball centered at the origin and consider the problem

$$\left.\begin{aligned} \partial_t u_i &= F_i(t, |x|, u, |Du_i|, \Delta u_i), && (x,t) \in \Omega \times (0,\infty), \\ u_i &= 0, && (x,t) \in \partial\Omega \times (0,\infty), \end{aligned}\right\} \; i = 1, \dots, n. \tag{2.8}$$

The functions $F_i(t, r, u, \eta, \xi)$, $i \in S$, are defined on $[0,\infty) \times [0,1] \times \mathcal{B}$ where $\mathcal{B}$ is a ball in $\mathbb{R}^{n+2}$ centered at the origin and we make the following hypotheses:

$(N1)_{rad}$ For each $i \in S$ the function $F_i : [0,\infty) \times [0,1] \times \mathcal{B} \to \mathbb{R}^n$ is continuous in all variables, differentiable in ξ, and Lipschitz continuous in (u, η, ξ) uniformly with respect to $(r,t) \in [0,1] \times \mathbb{R}^+$.

$(N2)_{rad}$ There is a positive constant α_0 such that

$$\frac{\partial F_i}{\partial \xi}(t, r, u, \eta, \xi) \geq \alpha_0 \quad ((t, r, u, \eta, \xi) \in [0,\infty) \times [0,1] \times \mathcal{B},\ i \in S).$$

$(N3)_{rad}$ For each $i \in S$, F_i is nonincreasing in r.

$(N4)_{rad}$ For any $i, j \in S$ with $i \neq j$, and any $(t, r, u, \eta, \xi) \in [0, \infty) \times [0, 1] \times \mathcal{O}_i$ one has

$$\frac{\partial F_i}{\partial u_j}(t, r, u, \eta, \xi) \geq 0$$

whenever the derivative exists.

$(N5)_{rad}$ There exists $\sigma > 0$ such that for any nonempty subsets $I, J \subset S$ with $I \cap J = \emptyset$, $I \cup J = S$ there exist $i \in I$, $j \in J$ such that

$$\frac{\partial F_i}{\partial u_j}(t, r, u, \eta, \xi) \geq \sigma$$

for all $(t, r, u, \eta, \xi) \in [0, \infty) \times [0, 1] \times \mathcal{O}_i$ such the derivative exists.

Theorem 2.5. *Let Ω be the unit ball and assume that $(N1)_{rad} - (N5)_{rad}$ hold. Let u be a nonnegative solution of* (2.8) *satisfying* (2.3) *and* (2.4). *Then for any $z \in \omega(u) \setminus \{0\}$ and $i \in S$, the function z_i is radially symmetric and strictly decreasing in $r = |x|$. The latter holds in the form $(z_i)_r < 0$ if $(z_i)_r \in C(\Omega_0)$.*

3 Linear equations

The proofs of our symmetry theorems use the method of moving hyperplanes and they depend on estimates of solutions of linear equations and systems. We prepare all these estimates in this section.

Recall the following standard notation. For an open set $D \subset \mathbb{R}^N$ and for $t < T$, we denote by $\partial_P(D \times (t, T))$ the parabolic boundary of $D \times (t, T)$: $\partial_P(D \times (t, T)) := (D \times \{t\}) \cup (\partial D \times [t, T])$. For bounded sets U, U_1 in $\mathbb{R}^N$ or $\mathbb{R}^{N+1}$, the notation $U_1 \subset\subset U$ means $\bar{U}_1 \subset U$, $\operatorname{diam} U$ stands for the diameter of U, and $|U|$ for its Lebesgue measure (if it is measurable). The open ball in $\mathbb{R}^N$ centered at x with radius r is denoted by $B(x, r)$. Symbols f^+ and f^- denote the positive and negative parts of a function f: $f^\pm := (|f| \pm f)/2 \geq 0$.

We consider time dependent elliptic operators L of the form

$$L(x, t) = \sum_{k,m=1}^{N} a_{km}(x, t) \frac{\partial^2}{\partial x_k \partial x_m} + \sum_{k=1}^{N} b_k(x, t) \frac{\partial}{\partial x_k}. \tag{3.1}$$

Definition 3.1. Given an open set $U \subset \mathbb{R}^N$, an interval $\mathcal{I}$, and positive numbers α_0, β, we say that an operator L of the form (3.1) belongs to $E(\alpha_0, \beta, U, \mathcal{I})$ if its coefficients a_{km}, b_k are measurable functions defined on $U \times \mathcal{I}$ and they satisfy

$$|a_{km}(x,t)|, |b_k(x,t)| \leq \beta \quad ((x,t) \in U \times \mathcal{I},\ k,m = 1,\dots,N),$$
$$\sum_{k,m=1}^{n} a_{km}(x,t)\xi_k\xi_m \geq \alpha_0|\xi|^2 \quad ((x,t) \in U \times \mathcal{I},\ \xi \in \mathbb{R}^N).$$

Further we say that a matrix-valued function $(c^{ij})_{i,j\in S}$ belongs to $M^+(\beta, U, \mathcal{I})$ if its entries c^{ij} are measurable functions defined on $U \times \mathcal{I}$ such that

$$|c^{ij}| \leq \beta \quad (i,j \in S) \quad \text{and} \quad c^{ij} \geq 0 \quad (i,j \in S\ i \neq j).$$

Let us now recall how linear equations are obtained from (1.1) via reflections in hyperplanes.

3.1 Linearization via reflections

Assume that $\Omega \subset \mathbb{R}^N$ is a domain satisfying the symmetry hypothesis (D1) and let functions F_i satisfy (N1)-(N4). Let u be a nonnegative solution of (1.1), (1.3) satisfying (2.3). Using the notation introduced in (2.1), let $u^\lambda(x,t) = u(x^\lambda, t)$ and $w^\lambda(x,t) := u^\lambda(x,t) - u(x,t)$ for any $x \in \Omega_\lambda$, $t > 0$, and $\lambda \in [0, \ell)$. By (N3 cor), for each $i \in S$, $x \in \Omega_\lambda$, and $t > 0$, one has

$$\partial_t u_i^\lambda \geq F_i(t, x, u(x^\lambda, t), Du_i(x^\lambda, t), D^2 u_i(x^\lambda, t)).$$

Hence

$$\begin{aligned} \partial_t w_i^\lambda(x,t) &\geq F_i(t, x, u(x^\lambda, t), Du_i(x^\lambda, t), D^2 u_i(x^\lambda, t)) \\ &\qquad - F_i(t, x, u(x,t), Du_i(x,t), D^2 u_i(x,t)) \\ &= L_i^\lambda(x,t) w_i^\lambda + \sum_{j=1}^{n} c^{ij}(x,t) w_j^\lambda \quad (x,t) \in \Omega_\lambda \times (0,\infty), \end{aligned} \tag{3.2}$$

where

$$L_i^\lambda(x,t) = \sum_{k,m=1}^{N} a_{km}^i(x,t) \frac{\partial^2}{\partial x_k \partial x_m} + \sum_{k=1}^{N} b_k^i(x,t) \frac{\partial}{\partial x_k}$$

and the λ-dependent coefficients a^i_{km}, b^i_k, c^{ij} are obtained from the Hadamard formula. Specifically (omitting the argument (x,t) of u and u^λ),

$$c^{ij}(x,t) = \begin{cases} \displaystyle\int_0^1 (F_i)_{u_j}(t,x,u_1^\lambda,\dots,u_{j-1}^\lambda,u_j+s(u_j^\lambda-u_j),u_{j+1},\dots, \\ \qquad\qquad u_n, Du_i, D^2u_i)\,ds, \quad \text{if } u_j^\lambda(x,t)\neq u_j(x,t), \\ \sigma, \quad \text{if } u^\lambda(x,t)=u(x,t), \end{cases}$$

$$b^i_k(x,t) = \begin{cases} \displaystyle\int_0^1 (F_i)_{p_k}(t,x,u^\lambda,\dots,u^\lambda_{x_{k-1}},(u_i)_{x_k}+s((u_i^\lambda)_{x_k}-(u_i)_{x_k}), \\ \qquad\qquad (u_i)_{x_{k+1}},\dots,D^2u_i)\,ds, \quad \text{if } (u_i^\lambda)_{x_k}(x,t)\neq(u_i)_{x_k}(x,t), \\ 0, \quad \text{if } (u_i^\lambda)_{x_k}(x,t)=(u_i)_{x_k}(x,t), \end{cases}$$

$$a^i_{km}(x,t) = \int_0^1 (F_i)_{q_{km}}(t,x,u^\lambda,Du_i^\lambda,D^2(u_i)+s(D^2u_i^\lambda-D^2u_i))\,ds,$$

where σ is a nonnegative constant. If (N5) is not assumed, the choice of the constant σ is not relevant and we can set $\sigma=0$; if (N5) is assumed we chose σ as in (N5).

By (N1) the coefficients are well defined on $\Omega_\lambda\times(0,\infty)$ and they are measurable functions with absolute values bounded by β. This and (N2) imply that $L_i^\lambda\in E(\alpha_0,\beta,\Omega_\lambda,(0,\infty))$. Further, by (N4), we have $(c^{ij})_{i,j\in S}\in M^+(\beta,\Omega_\lambda,(0,\infty))$ and if (N5) is satisfied, then for any nonempty subsets $I,J\subset S$ with $I\cap J=\emptyset$, $I\cup J=S$, there exist $i\in I$, $j\in J$ such that

$$c^{ij}(x,t)\geq\sigma \quad ((x,t)\in\Omega_\lambda\times(0,\infty)). \tag{3.3}$$

The Dirichlet boundary condition and nonnegativity of u_i yield

$$w_i^\lambda\geq 0 \qquad \text{on} \quad \partial\Omega_\lambda\times(0,\infty),\ i\in S. \tag{3.4}$$

3.2 Estimates of solutions

We now derive several estimates for solutions of a system of inequalities such as (3.2). The results here are independent of the previous sections.

Throughout this subsection, Ω is a bounded, not necessarily symmetric, domain in $\mathbb{R}^N$ and α_0, β, σ are fixed positive numbers.

We consider the following system of parabolic inequalities

$$\partial_t w_i - L_i(x,t)w_i \geq \sum_{i,j=1}^n c^{ij}(x,t)w_j, \quad x\in U,\ t\in(\tau,T),\ i\in S, \tag{3.5}$$

where $-\infty < \tau < T \leq \infty$, U is an open subset of Ω, $L_i \in E(\alpha_0, \beta, U, (\tau, T))$ $(i \in S)$, and $(c^{ij})_{i,j\in S} \in M^+(\beta, U, (\tau, T))$. Sometimes we also assume the following condition.

(IR) There exists $\sigma > 0$ such that for any nonempty subsets $I, J \subset S$ with $I \cap J = \emptyset$, $I \cup J = S$, there exist $i \in I$, $j \in J$ such that

$$c^{ij}(x,t) \geq \sigma \quad ((x,t) \in U \times (\tau, T)). \tag{3.6}$$

We say that w is a solution of (3.5) (or that it satisfies (3.5)) if it is an element of the space $(W^{2,1}_{N+1,loc}(U \times (\tau, T)))^n$ and (3.5) is satisfied almost everywhere. If (3.5) is complemented by a system of inequalities on $\partial U \times (\tau, T)$, we also require the solution to be continuous on $\bar{U} \times (0, T)$ and to satisfy the boundary inequalities everywhere.

We will use several forms of the maximum principle, both for scalar equations and cooperative systems. For the maximum principle for strong solutions of a single equation we refer to [30]. The following is a version of the weak maximum principle for cooperative systems.

Theorem 3.2. *Let U be an open subset of Ω, $0 \leq \tau < T < \infty$, and let $L_i \in E(\alpha_0, \beta, U, (\tau, T))$ $(i \in S)$, $(c^{ij})_{i,j\in S} \in M^+(\beta, U, (\tau, T))$. Assume*

$$\sum_{j=1}^{n} c^{ij} \leq 0 \qquad in \quad D \times (\tau, T) \tag{3.7}$$

for all $i \in S$. If w is a continuous function on $\bar{U} \times [0, T)$ which is a solution of (3.5)*, then*

$$\max_{i\in S} \sup_{U\times(\tau,T)} w_i^- \leq \max_{i\in S} \sup_{\partial_P(U\times(\tau,T))} w_i^-. \tag{3.8}$$

If the right hand side of (3.8) *is 0 (that is, the functions w_i are nonnegative on the parabolic boundary), then the conclusion holds regardless of condition* (3.7).

For classical solutions the result is proved in [37, Section 3.8]. To prove it for strong solutions, one can use the arguments of [37] combined with the maximum principle for strong solutions of scalar equations.

We now prove a maximum principle for small domains. It is a generalization of Lemma 3.1 in [36] to cooperative systems. Although not needed in this paper, for future use we make one more generalization by allowing the solutions to satisfy nonzero Dirichlet boundary conditions. The maximum principle for elliptic equations on small domains was first proved in [8]. For related results and extensions to elliptic systems and parabolic equations see [4, 9, 10, 13, 20, 36].

Lemma 3.3. *Given any $q > 0$, there is a constant δ determined only by n, N, α_0, β_0, $\mathrm{diam}(\Omega)$, and q such that for any open set $U \subset \Omega$ with $|U| < \delta$ and any $0 \le \tau < T \le \infty$ the following holds. If $w \in C(\bar{U} \times [\tau, T))$ is a solution a problem (3.5), where $L_i \in E(\alpha_0, \beta, U, (\tau, T))$ ($i \in S$), $(c^{ij})_{i,j \in S} \in M^+(\beta, U, (\tau, T))$, and if*

$$w_i(x,t) \ge -\hat{\varepsilon} \quad ((x,t) \in \partial U \times (\tau, T),\ i \in S), \tag{3.9}$$

where $\hat{\varepsilon} \ge 0$ is a constant, then

$$\max_{i \in S} \|w_i^-(\cdot, t)\|_{L^\infty(U)} \le 2 \max\{\max_{i \in S} \|w_i^-(\cdot, \tau)\|_{L^\infty(U)} e^{-q(t-\tau)}, \hat{\varepsilon}\} \quad (t \in (\tau, T)).$$

In the proof we employ the following lemma of [8].

Lemma 3.4. *Given any $a_0 > 0$, $b_0 \ge 1$, there exists $\delta > 0$ determined only by a_0, b_0, N, and $\mathrm{diam}(\Omega)$ such that for any closed set $K \subset \Omega$ with $|K| \le \delta$ there exists a smooth function g on Ω such that $1 \le g \le 2$ and for any symmetric positive definite matrix (a_{ij}) with*

$$\det(a_{ij}) \ge a_0^N \tag{3.10}$$

one has

$$a_{ij} g_{x_i x_j} + b_0(|\nabla g| + g) < 0 \quad (x \in K). \tag{3.11}$$

Proof of Lemma 3.3. We claim that the assertion holds if δ is as in Lemma 3.4 with $b_0 := (n + \sqrt{N})\beta + q$, $a_0 := \alpha_0$. To prove this, let $U \subset \Omega$ and w satisfy the hypotheses of Lemma 3.3. Without loss of generality we may assume that $|\bar{U}| < \delta$; otherwise we first prove the results for each open set $U_1 \subset\subset U$ and then use an approximation argument (or alternatively we can proceed similarly as in the proof of Lemma 3.1 in [36]). Let g be as in the conclusion of Lemma 3.4 with $K = \bar{U}$.

Denote by a^i_{km}, b^i_k the coefficients of L_i:

$$L_i(x,t) = \sum_{k,m=1}^N a^i_{km}(x,t) \frac{\partial^2}{\partial x_k \partial x_m} + \sum_{k=1}^N b^i_k(x,t) \frac{\partial}{\partial x_k}.$$

For each $i \in S$ set $z_i := w_i/g$. Then $z_i(\cdot, t) \geq -\hat{\varepsilon}$ on ∂U for all $t \in (\tau, T)$ and a simple computation shows that $z = (z_1, \ldots, z_n)$ satisfies

$$(z_i)_t - \tilde{L}_i(x,t)z_i \geq \sum_{\substack{j=1 \\ j \neq i}}^{N} c^{ij}(x,t)z_j + \tilde{c}^{ii}(x,t)z_i \quad (x,t) \in U \times (\tau, T),\ i \in S, \tag{3.12}$$

where

$$\tilde{L}_i(x,t) = L_i(x,t) - \frac{2}{g(x)} \sum_{k,m=1}^{N} a^i_{km}(x,t) g_{x_k}(x) \frac{\partial}{\partial x_m}$$

and

$$\begin{aligned} \tilde{c}^{ii} &= \frac{1}{g} L_i g + c^{ii} \\ &\leq \frac{1}{g} \left(\sum_{k,m=1}^{N} a^i_{km} g_{x_k x_m} + \sqrt{N} \beta (|Dg| + g) \right) \\ &= \frac{1}{g} \left(\sum_{k,m=1}^{N} a^i_{km} g_{x_k x_m} + (b_0 - q - n\beta)(|Dg| + g) \right). \end{aligned}$$

By (3.11), we have

$$\tilde{c}^{ii} < -q - n\beta \quad \text{on} \quad U \times (\tau, T) \quad (i \in S).$$

We further transform (3.12) substituting $z_i(x,t) = e^{-qt}\hat{z}_i(x,t)$. Then $\tilde{z}_i$ satisfies inequalities (3.12) with $\tilde{c}^{ii}$ replaced by $\hat{c}^{ii} = \tilde{c}^{ii} + q < -n\beta$. This implies

$$\hat{c}^{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} c^{ij} < 0 \qquad \text{in} \quad U \times (\tau, T), i \in S.$$

We also have $\hat{z}_i(x,s) \geq -\hat{\varepsilon}e^{qt}$ for any $i \in S$, $x \in \partial U$, and $s \in (\tau, t)$. Hence Theorem 3.2 implies

$$\max_{i \in S} \|\hat{z}_i^-(\cdot, t)\|_{L^\infty(U)} \leq \max\{\max_{i \in S} \|\hat{z}_i^-(\cdot, \tau)\|_{L^\infty(U)}, \hat{\varepsilon}e^{qt}\} \quad (t \in (\tau, T)),$$

or, equivalently,

$$\max_{i \in S} \|z_i^-(\cdot, t)\|_{L^\infty(U)} \leq \max\{\max_{i \in S} \|z_i^-(\cdot, \tau)\|_{L^\infty(U)} e^{q(\tau - t)}, \hat{\varepsilon}\} \quad (t \in (\tau, T)).$$

Finally, since $1 \le g \le 2$, the substitution $z_i = w_i/g$ yields the desired estimate

$$\max_{i\in S} \|w_i^-(\cdot,t)\|_{L^\infty(U)} \le 2\max\{\max_{i\in S}\|w_i^-(\cdot,\tau)\|_{L^\infty(U)} e^{q(\tau-t)}, \hat{\varepsilon}\} \quad (t\in(\tau,T)).$$

□

The next lemma is proved in [36, Lemma 3.3].

Lemma 3.5. *For any $r>0$ there exist a constant $\gamma>0$ depending only on N, r, α_0, β, and a smooth function h_r on $B(0,r)$ with*

$$h_r(x) > 0 \quad (x \in B(0,r)), \qquad h_r(x) = 0 \quad (x\in \partial B(0,r)),$$

such that the following holds. For any $x_0 \in \Omega$ with $U := B(x_0, r) \subset \Omega$, any $L \in E(\alpha_0, \beta, U, (0,\infty))$, and any $c \in L^\infty(U\times(0,\infty))$ with $\|c\|_{L^\infty(U\times(0,\infty))} \le \beta$, the function $\phi(x,t) = e^{-\gamma t} h_r(x-x_0)$ satisfies

$$\begin{array}{rcll} \partial_t\phi - L(x,t)\phi - c(x,t)\phi & < & 0, & (x,t)\in B(x_0,r)\times(0,\infty), \\ \phi & = & 0 & (x,t)\in \partial B(x_0,r)\times(0,\infty). \end{array} \tag{3.13}$$

We now need to introduce further notation. For an open bounded subset Q of $\mathbb{R}^{n+1}$, a bounded continuous function $f: Q\to\mathbb{R}$, and $p>0$, we set

$$[f]_{p,Q} = \left(\frac{1}{|Q|}\int_Q |f|^p dx\, dt\right)^{\frac{1}{p}}.$$

Also

$$[f]_{\infty,Q} = \sup_Q |f|.$$

Lemma 3.6. *Given $d>0$, $\varepsilon>0$, $\theta>0$, there are positive constants κ, κ_1, and p, determined only by n, N, $\operatorname{diam}\Omega$, α_0, β, d, ε and θ, such that the following statement holds. Assume that $\tau\in\mathbb{R}$; D and U are domains in Ω with $D \subset\subset U$, dist$(\bar D, \partial U)\ge d$, and $|D|>\varepsilon$; $L_i \in E(\alpha_0,\beta,U,(\tau,\tau+4\theta))$ for all $i\in S$; and $(c^{ij})_{i,j\in S} \in M^+(\beta, U, (\tau,\tau+4\theta))$. If $v=(v_1,\dots,v_n)\in (C(\bar U\times[\tau,\tau+4\theta]))^n$ is a solution of (3.5), then for all $i\in S$*

$$\inf_{D\times(\tau+3\theta,\tau+4\theta)} v_i(x,t) \ge \kappa [v_i^+]_{p, D\times(\tau+\theta,\tau+2\theta)} - \kappa_1 \max_{j\in S} \sup_{\partial_P(U\times(\tau,\tau+4\theta))} v_j^-.$$

Remark 3.7. Lemma 3.6 applied to a nonnegative solution v of (3.5), gives a weak component-wise Harnack inequality for v:

$$\inf_{D\times(\tau+3\theta,\tau+4\theta)} v_i(x,t) \geq \kappa [v_i]_{p,D\times(\tau+\theta,\tau+2\theta)}.$$

It is clear from the proof below that in this case the continuity of v up to the boundary of $U\times(\tau,T)$ is not needed. Under the irreducibility assumption (IR), the full Harnack inequality for nonnegative solutions is given in Theorem 3.9 below. For elliptic cooperative systems similar results have been proved in [3, 14, 12]. The most general ones are those in [12], where viscosity solutions are considered. We only deal with strong solutions and the assumptions in our Harnack type results are stronger than the parabolic analogs of the assumptions in [12] (this is more than satisfactory for our applications to classical solutions of nonlinear equations). For parabolic systems in the divergence form a Harnack type result is given in [31].

For a single equation, Lemma 3.6 is proved in a more general form in [36, Lemma 3.5]. We now state a version of that result as we need it in the proof of Lemma 3.6 and elsewhere in this paper. The lemma is an extension of Krylov-Safonov Harnack inequality [28] and its weak version for supersolutions [24, 30].

Lemma 3.8. *Given $d>0$, $\varepsilon>0$, $\theta>0$, there are positive constants κ, κ_2, p, determined only by N, $\operatorname{diam}\Omega$, α_0, β, d, ε and θ, such that the following statement holds. Assume that $\tau\in\mathbb{R}$; D and U are domains in Ω with $D\subset\subset U$, $dist(\bar{D},\partial U)\geq d$, and $|D|>\varepsilon$; $L\in E(\alpha_0,\beta,U,(\tau,\tau+4\theta))$; $c\in L^\infty(U\times(\tau,\tau+4\theta))$ satisfies $m:=|c|_{L^\infty(U\times(\tau,\tau+4\theta))}\leq\beta$; and $g\in L^{N+1}(U\times(\tau,\tau+4\theta))$. If $v\in C(\bar{U}\times[\tau,\tau+4\theta])$ is a solution of*

$$v_t - L(x,t)v \geq c(x,t)v + g(x,t) \quad (x,t)\in U\times(\tau,\tau+4\theta), \tag{3.14}$$

then

$$\inf_{D\times(\tau+3\theta,\tau+4\theta)} v(x,t) \geq \kappa[v^+]_{p,(D\times(\tau+\theta,\tau+2\theta))} - \kappa_1\|g\|_{L^{N+1}(U\times(\tau,\tau+4\theta))} - \sup_{\partial_P(U\times(\tau,\tau+4\theta))} e^{m\theta}v^-.$$

If (3.14) *is an equation, rather than inequality, then the conclusion holds with $p=\infty$ and with κ, κ_2 independent of ε.*

We remark that in the statement regarding supersolutions, one can replace g in (3.14) (and in the conclusion) with g^-.

Proof of Lemma 3.6. For $i \in S$ let

$$g_i(x,t) := \sum_{\substack{j=1 \\ j\neq i}}^{n} c^{ij}(x,t) v_j^- .$$

Obviously, $g_i \in L^\infty(U \times (\tau, \tau + 4\theta))$. Since $c^{ij} \geq 0$ for $i \neq j$, v_i satisfies

$$(v_i)_t - L_i(x,t)v_i - c^{ii}(x,t)v_i \geq g_i(x,t), \quad (x,t) \in U \times (\tau, \tau + 4\theta).$$

Therefore

$$\inf_{D\times(\tau+3\theta,\tau+4\theta)} v_i(x,t) \geq \kappa[v_i^+]_{p,D\times(\tau+\theta,\tau+2\theta)} - \kappa_2 \|g_i\|_{L^{N+1}(U\times(\tau,\tau+4\theta))} - \sup_{\partial_P(U\times(\tau,\tau+4\theta))} e^{4M\theta} v_i^-, \quad (3.15)$$

where κ, κ_2, and p are as in Lemma 3.8 and $M = \max_{i,j\in S} \sup_{U\times(\tau,\tau+4\theta)} c^{ij}(x,t)$. Now

$$\begin{aligned} \|g_i\|_{L^{N+1}(U\times(\tau,\tau+4\theta))} &\leq \tilde\kappa_2 \|g_i\|_{L^\infty(U\times(\tau,\tau+4\theta))} \\ &\leq \tilde\kappa_2 \beta n \max_{j\in S} \|v_j^-\|_{L^\infty(U\times(\tau,\tau+4\theta))}, \end{aligned} \quad (3.16)$$

where $\tilde\kappa_2$ depends only on $\mathrm{diam}(\Omega)$ and N. Next, the function $\tilde v := e^{nMt} v$ satisfies the inequalities

$$(\tilde v_i)_t - L_i(x,t)\tilde v_i \geq \sum_{j=1}^{n} c^{ij}(x,t)\tilde v_j - nM\tilde v_i, \quad (x,t) \in U \times (\tau, \tau+4\theta),\ i \in S,$$

where, by the definition of M,

$$\sum_{j=1}^{n} c^{ij}(x,t) - nM \leq 0 \quad ((x,t) \in U \times (\tau, \tau + 4\theta),\ i \in S).$$

Hence, Theorem 3.2 implies

$$\max_{i\in S} \sup_{U\times(\tau,\tau+4\theta)} \tilde v_i^- \leq \max_{i\in S} \sup_{\partial_P(U\times(\tau,\tau+4\theta))} \tilde v_i^- .$$

Consequently, by (3.16),

$$\begin{aligned}\|g_i\|_{L^{N+1}(U\times(\tau,\tau+4\theta))} &\leq \tilde{\kappa}_2 n\beta e^{-nM\tau}\max_{i\in S}\sup_{\partial_P(U\times(\tau,\tau+4\theta))}\tilde{v}_i^- \\ &\leq \tilde{\kappa}_2 n\beta e^{4nM\theta}\max_{i\in S}\sup_{\partial_P(U\times(\tau,\tau+4\theta))} v_i^- .\end{aligned}$$

Substituting this into (3.15), we obtain the estimate stated in Lemma 3.6. □

The last result of this section is a stronger version of the Harnack inequality for irreducible systems.

Theorem 3.9. *Given $d>0$, $\varepsilon>0$, $\theta>0$, there are positive constants $\bar{\kappa}$ and p, determined only by n, N, $\operatorname{diam}\Omega$, α_0, β, σ, d, ε and θ, such that the following statement holds. Assume that*

(A) *D and U are domains in Ω with $D\subset\subset U$, $dist(\bar{D},\partial U)\geq d$, and $|D|>\varepsilon$; $L_i\in E(\alpha_0,\beta,U,(\tau,\tau+4\theta))$ for all $i\in S$; $(c^{ij})_{i,j\in S}\in M^+(\beta,U,(\tau,\tau+4\theta))$ is such that (IR) holds; and $v=(v_1,\dots,v_n)\in(C(\bar{U}\times[\tau,\tau+4\theta]))^n$ is a nonnegative solution of* (3.5).

Then for all $i\in S$

$$\inf_{D\times(\tau+\frac{7}{2}\theta,\tau+4\theta)} v_i(x,t)\geq\bar{\kappa}\max_{j\in S}[v_j]_{p,D\times(\tau+\theta,\tau+2\theta)}. \tag{3.17}$$

If all inequalities in (3.5) *are replaced by equations, then the conclusion holds with $p=\infty$ and with $\bar{\kappa}$ independent of ε.*

Proof. Given $d>0$, $\varepsilon>0$, $\theta>0$, we first fix p and κ such that the statement of Lemma 3.6 is valid with d replaced by $d/(2n)$. These constants p and κ depend only on the indicated quantities.

Assume (A) is satisfied. Relabeling the components of v, we may without loss of generality assume that

$$[v_1]_{p,D\times(\tau+\theta,\tau+2\theta)}=K_0:=\max_{j\in S}[v_j]_{p,D\times(\tau+\theta,\tau+2\theta)}.$$

We may also assume that $c^{ii}\geq 0$ for all $i\in S$. Indeed, these relations are achieved by the substitution $v\to e^{-\beta t}v$, which clearly does not affect the validity of the statement.

For each $k \in S$ denote $\tau^k = \tau + \frac{7}{2}\theta - \frac{\theta}{2^k}$ and fix a sequence of domains $\{U_k\}_{k=1}^n$ such that $D \subset\subset U_{k+1} \subset\subset U_k \subset\subset U$ and $\operatorname{dist}(\bar{U}_{k+1}, \partial U_k) \geq \frac{d}{2n}$ for $k = 1, \ldots, n-1$.

We now use an induction argument. In the first step we apply Lemma 3.6 (see also Remark 3.7), with d replaced by $d/(2n)$, to the sets $U_1 \subset U$. Note that the application of Lemma 3.6 is legitimate by the choice of U_1. This gives

$$\inf_{U_1 \times (\tau_1, \tau+4\theta)} v_1(x,t) = \inf_{U_1 \times (\tau+3\theta, \tau+4\theta)} v_1(x,t) \geq \kappa [v_1]_{p, U_1 \times (\tau+\theta, \tau+2\theta)} \geq \kappa_1 K_0.$$

Here $\kappa_1 = \kappa \varepsilon^p / |B(0, \operatorname{diam}\Omega)|^p$ and the last inequality follows from the relations

$$\begin{aligned}
[v_1]_{p, U_1 \times (\tau+\theta, \tau+2\theta)} &\geq \left(\frac{|D|}{|U_1|}\right)^p [v_1]_{p, D \times (\tau+\theta, \tau+2\theta)} \\
&\geq \left(\frac{\varepsilon}{|\Omega|}\right)^p K_0 \geq \frac{\varepsilon^p}{|B(0, \operatorname{diam}\Omega)|^p} K_0.
\end{aligned}$$

Next assume that for some $k \in S$ there is a subset S_k of S with k elements such that

$$\inf_{U_k \times (\tau_k, \tau+4\theta)} v_j(x,t) \geq \kappa_k K_0 \quad (j \in S_k), \tag{3.18}$$

where κ_k is a constant depending only on the indicated quantities. If $k = n$, then the theorem is already proved: (3.18) and the relations $D \subset U_k$, $\tau_k < \tau + 7/2\theta$ give (3.17) with $\bar{\kappa} = \kappa_n$. We proceed assuming $1 \leq k < n$. By (IR), there exist $j \in S_k$ and $i \in S \setminus S_k$ such that $c^{ij} \geq \sigma$ in $U \times (\tau, \tau + 4\theta)$. Then, since $(c^{ij})_{i,j \in S} \in M^+(\beta, U, (\tau, \tau+4\theta))$, $c^{ii} \geq 0$, and v is a nonnegative solution,

$$\begin{aligned}
(v_i)_t - L_i(x,t)v_i &\geq \sum_{k=1}^n c^{ik}(x,t) v_k \geq c^{ij}(x,t) v_j \\
&\geq \sigma \kappa_k K_0 \qquad ((x,t) \in U^k \times (\tau_k, \tau+4\theta)).
\end{aligned}$$

Fix an arbitrary point $x_0 \in U_{k+1}$ and set $\rho = d/(2n)$. Since $\operatorname{dist}(\bar{U}_{k+1}, \partial U_k) \geq d/2n$, we have $B(x_0, \rho) \subset U_k$. Define a radial function $\phi : B(x_0, \rho) \to \mathbb{R}$ by $\phi(x,t) = \eta(\frac{|x-x_0|}{\rho}, t)$, where

$$\eta(r,t) = \begin{cases} (t-\tau_k)V & \text{if } r \leq \frac{1}{2},\ t \in [\tau_k, \tau+4\theta], \\ (t-\tau_k)V(1-(2r-1)^3) & \text{if } \frac{1}{2} < r \leq 1,\ t \in [\tau_k, \tau+4\theta], \end{cases}$$

and V is a positive constant to be specified below. Clearly $\phi \in C^{2,1}(\bar{B}(x_0,\rho)\times [\tau_k,\tau+4\theta])$, $\phi(x,t)=0$ for $|x|=\rho$ and $t\in[\tau_k,\tau+4\theta]$, $\phi(\cdot,\tau_k)=0$ in $B(x_0,\rho)$, and

$$\phi_t(x,t)-L_i(x,t)\phi(x,t)=V \quad (x\in B(x_0,\frac{\rho}{2}),\ t\in[\tau_k,\tau+4\theta]).$$

Further, it is clear that there is a constant $K\geq 1$ depending only on N, β, θ, d, and n such that

$$\begin{aligned}\phi_t(x,t)-L_i(x,t)\phi(x,t)\leq |\phi_t(x,t)-L_i(x,t)\phi(x,t)|\leq KV \\ ((x,t)\in B(x_0,\rho)\setminus B(x_0,\frac{\rho}{2}))\times[\tau_k,\tau+4\theta]).\end{aligned}$$

Hence, choosing $V:=\sigma\kappa_k K_0/K\leq \sigma\kappa_k K_0$, we see that v_i and ϕ are, respectively, a supersolution and subsolution of the same scalar equation on $B(x_0,\rho)\times(\tau_k,\tau+4\theta)$. Moreover, on $\partial_P(B(x_0,\rho)\times(\tau_k,\tau+4\theta))$ we have $v_i\geq 0=\phi$. The maximum principle therefore implies

$$v(x_0,t)\geq\phi(x_0,t)\geq\frac{\sigma\kappa_k K_0}{K}(\tau_{k+1}-\tau_k)\geq\kappa_{k+1}K_0 \quad (t\in(\tau_{k+1},\tau+4\theta)),$$

where

$$\kappa_{k+1}=\frac{\sigma\kappa_k\theta}{4K2^{k+1}}.$$

Since $x_0\in U_{k+1}$ was arbitrary, we obtain (3.18) with k replaced by $k+1$ and with $S_{k+1}=S_k\cup\{i\}$. This completes the induction argument showing that after a finite number of steps we establish the validity of (3.18) with $k=n$, hence of (3.17), as noted above.

To prove the last statement of the theorem we combine, as usual, estimate (3.17) with a local maximum principle, as formulated in Lemma 3.10 below. Assume that all inequalities in (3.5) are replaced by equations and let $d>0$ and $\theta>0$ be given. Assume that (A) holds with condition $|D|>\epsilon$ deleted.

Let $\tilde{D}$ be a domain with the following properties:

$$D\subset\subset\tilde{D}\subset\subset U,\quad \operatorname{dist}(D,\partial\tilde{D})>\frac{d}{4},\quad \operatorname{dist}(\tilde{D},\partial U)>\frac{d}{4}.$$

Set $\epsilon:=|B(0,d/4)|$ and note that $|\tilde{D}|>\epsilon$. With this ϵ and with d replaced by $d/4$, we apply the already proved statement of the theorem. Hence we obtain (3.17) with D replaced by $\tilde{D}$ and with some constants p and $\bar{\kappa}$ determined only by the indicated quantities, which now refer to n, N, $\operatorname{diam}\Omega$, α_0, β, σ, d, and θ (not ϵ). Hence

$$\inf_{D\times(\tau+\frac{7}{2}\theta,\tau+4\theta)} v_i(x,t) \geq \inf_{\tilde{D}\times(\tau+\frac{7}{2}\theta,\tau+4\theta)} v_i(x,t) \geq \bar{\kappa}\max_{j\in S}[v_j]_{p,\tilde{D}\times(\tau+\theta,\tau+2\theta)}.$$

The proof will be completed if we show that the last term can be estimated below by CK_1, where

$$K_1 := \max_{j\in S}\sup_{D\times(\tau+\theta,\tau+2\theta))} v_j.$$

Here C is a constant depending only on the indicated quantities and such are the constants $C_1, C_2, \dots$ in the forthcoming estimates.

Obviously, $K_1 = v_m(x_0,t_0)$ for some $m \in S$ and $(x_0,t_0) \in \bar{D}\times[\tau+\theta,\tau+2\theta]$. Assume first that $t_0 > \tau + 5\theta/4$. Set $\delta = \min\{d/4, \sqrt{\theta}/2\}$. Lemma 3.10 gives

$$K_1 = v_m(x_0,t_0) \leq C_1\max_{j\in S}[v_j]_{p,B(x_0,\delta)\times(t_0-\delta^2,t_0)} \leq C_2\max_{j\in S}[v_j]_{p,\tilde{D}\times(\tau+\theta,\tau+2\theta)}.$$

Thus we have proved the desired estimate under the assumption $t_0 > 5\tau/4$. In other words, we have proved the estimate

$$\inf_{D\times(\tau+\frac{7}{2}\theta,\tau+4\theta)} v_i(x,t) \geq C_3\max_{j\in S}\|v_j\|_{L^\infty(D\times(\tau+\frac{5\theta}{4},\tau+2\theta))} \quad (i\in S).$$

Applying this result with τ replaced by $\tau + 7\theta/12$ and θ by $\theta/3$, we next obtain

$$C_4\max_{j\in S}\|v_j\|_{L^\infty(D\times(\tau+\theta,\tau+\frac{5\theta}{4})} \leq \inf_{D\times(\tau+\frac{7}{4}\theta,\tau+\frac{23}{12}\theta)} v_i(x,t) \leq \sup_{D\times(\tau+\frac{5}{4}\theta,\tau+2\theta)} v_i(x,t).$$

Combining this with the previous estimate, we conclude that (3.17) holds with $p=\infty$ and some constant $\bar{\kappa}$ depending only on the indicated quantities. □

Lemma 3.10. *For some $(x_0,t_0) \in \mathbb{R}^{N+1}$ and $\delta > 0$ assume that*

$$L_i \in E(\alpha_0,\beta,B(x_0,\delta),(t_0-\delta^2,t_0)) \quad (i\in S),$$

$(c^{ij})_{i,j\in S} \in M^+(\beta,B(x_0,\delta),(t_0-\delta^2,t_0))$, and v is a solution of the system (3.5) on $U\times(\tau,T) = B(x_0,\delta)\times(t_0-\delta^2,t_0))$ with the inequality sign reversed (replaced by "$\leq$"). Then for each $p>0$ and $\rho\in(0,1)$ one has

$$\max_{j\in S}\sup_{B(x_0,\rho\delta)\times(t_0-(\rho\delta)^2,t_0)} v_j \leq C_0\max_{j\in S}[v_j]_{p,B(x_0,\delta)\times(t_0-\delta^2,t_0)},$$

where C_0 is a constant determined only by δ, ρ, p, n, N, α_0, β.

The proof of this lemma can be carried out in a similar way as in the scalar case, see [30, Theorem 7.21]. Since (3.5) is coupled only in the zero-oder terms, the adaptation of the proof in [30] is straightforward and is omitted. We remark that the cooperativity condition $c_{ij} \geq 0$ is not needed and can be omitted in the assumptions of this lemma.

4 Proofs of the symmetry results

In this section the assumptions are as in Section 2. Specifically, Ω is a bounded domain in $\mathbb{R}^N$, satisfying (D1), (D2), the nonlinearity F satisfies (N1) - (N4) and at some places, where explicitly stated, also (N5). We consider a solution u of (1.1), (1.3) satisfying (2.3) and (2.4).

We use the notation introduced at the beginning of Section 2 and the following one. For any function g on Ω, scalar or vector valued, and any $\lambda \in [0, \ell)$ we let

$$V_\lambda g(x) := g(x^\lambda) - g(x), \quad (x \in \Omega_\lambda).$$

We also set

$$w^\lambda(x,t) := V_\lambda u(x,t) = u(x^\lambda, t) - u(x,t) \quad (x \in \Omega_\lambda,\, t \geq 0).$$

As shown in Subsection 3.1, the function w^λ solves a linear problem (3.2), (3.4), with $L_i \in E(\alpha_0, \beta, \Omega_\lambda, (0,\infty))$, $(c^{ij})_{i,j \in S} \in M^+(\beta, \Omega_\lambda, (0,\infty))$. If (N5) is satisfied, then also the irreducibility condition (IR) holds with $U = \Omega_\lambda$, $\tau = 0$, $T = \infty$. Hence the results of Subsection 3.2 are applicable to w^λ. We use this observation below, often without notice.

We carry out the process of moving hyperplanes in the following way. Starting from $\lambda = \ell$ we move λ to the left as long as the following property is preserved

$$\lim_{t\to\infty} \max_{i \in S} \|(w_i^\lambda(\cdot,t))^-\|_{L^\infty(\Omega_\lambda)} = 0. \tag{4.1}$$

We show below that the process can get started and then examine the limit of the process given by

$$\lambda_0 := \inf\{\mu > 0 : \lim_{t\to\infty} \max_{i\in S} \|(w_i^\lambda(\cdot,t))^-\|_{L^\infty(\Omega_\lambda)} = 0 \text{ for each } \lambda \in [\mu, \ell)\ \}. \tag{4.2}$$

Remark 4.1. Note that, by compactness of $\{u(\cdot,t) : t \geq 0\}$ in E, (4.1) is equivalent to the following property: for each $z \in \omega(u)$ and $i \in S$ one has

$V_\lambda z_i \geq 0$ in Ω_λ. By the definition of λ_0 and continuity of the functions in $\omega(u)$, we have

$$V_{\lambda_0} z_i(x) \geq 0 \quad (x \in \Omega_{\lambda_0},\ z \in \omega(u),\ i \in S,\ \lambda \in [\lambda_0, \ell)). \tag{4.3}$$

Also note that the function z_i is nonincreasing in x_1 in Ω_{λ_0}. Indeed, if $x_1 > \tilde{x}_1$ and $(x_1, x'), (\tilde{x}_1, x') \in \Omega_{\lambda_0}$, then $V_\lambda z_i \geq 0$ with $\lambda = (x_1 + \tilde{x}_1)/2 > \lambda_0$ gives $z_i(x_1, x') \geq z_i(\tilde{x}_1, x')$.

Lemma 4.2. *We have $\lambda_0 < \ell$. Moreover $|\Omega_{\lambda_0}| \geq \delta$ where δ is a constant depending only on $\alpha_0, \beta, \Omega, n, N$.*

Proof. Let $\delta > 0$ be as in Lemma 3.3 with $q = 1$. If $\lambda < \ell$ is such that $|\Omega_\lambda| < \delta$, then Lemma 3.3 with $w = w^\lambda$, $\hat{\varepsilon} = 0$, $\tau = 0$ and $T = \infty$ gives

$$\max_{i \in S} \|(w_i^\lambda(t))^-\|_{L^\infty(\Omega_\lambda)} \leq Ce^{-t} \to 0 \text{ as } t \to \infty.$$

This and the definition of λ_0 imply that $|\Omega_{\lambda_0}| \geq \delta$. Since $|\Omega_\lambda| < \delta$ for $\lambda \approx \ell$, we have $\lambda_0 < \ell$. □

Lemma 4.3. *For any $z \in \omega(u)$, $\lambda \in [\lambda_0, \ell)$, and any connected component U of Ω_λ the following statements hold.*

- *(i) For each $i \in S$ either $V_\lambda z_i \equiv 0$ or $V_\lambda z_i > 0$ in U.*
- *(ii) If (N5) holds and $V_\lambda z_i \not\equiv 0$ for some $i \in S$, then $V_\lambda z_j > 0$ in U for each $j \in S$.*

Proof. Fix any $\lambda \in [\lambda_0, \ell)$ and $z \in \omega(u)$, and let U be a connected component of Ω_λ. Assume that $V_\lambda z_i \not\equiv 0$ on U for some and $i \in S$. Since $V_\lambda z_i$ is continuous and nonnegative, we have

$$V_\lambda z_i > 4r_0 \quad \text{in } \bar{B}_0$$

for some open ball $B_0 \subset U$ and $r_0 > 0$. Choose an increasing sequence $t_k \to \infty$ such that $u(\cdot, t_k) \to z$ in E. Then $w^\lambda(\cdot, t_k) \to V_\lambda z$, hence for a large enough k_0 we have

$$w_i^\lambda(x, t_k) > 2r_0 \quad (x \in \bar{B}_0,\ k > k_0).$$

By the equicontinuity property (see (2.4)), there is $\vartheta > 0$ independent of k such that

$$w_i^\lambda(x, t) > r_0 \quad ((x, t) \in \bar{B}_0 \times [t_k - 4\vartheta, t_k],\ k > k_0). \tag{4.4}$$

Take now an arbitrary domain $D \subset\subset U$ with $B_0 \subset\subset D$. In view of (4.1) and (4.4), an application of Lemma 3.6 (with $v = w^\lambda$, $\tau = t_k - 4\vartheta$, $\theta = \vartheta$ and k sufficiently large) shows that there is $r_1 > 0$ such that for $j = i$ we have

$$w_j^\lambda(\cdot, t_k) > r_1 \quad \text{in} \quad \bar{D}. \tag{4.5}$$

Letting $k \to \infty$, we obtain

$$V_\lambda z_j \geq r_1 \qquad \text{in} \quad \bar{D}.$$

Since the domain D was arbitrary, we have $V_\lambda z_j > 0$ in U for $j = i$. This proves (i). If (N5) holds then we can also apply the full Harnack inequality, Theorem 3.9, to $v = w^\lambda$. In this case, (4.4) implies (4.5) for each $j \in S$ and the above arguments prove (ii). □

In the next key lemma we consider the possibility $\lambda_0 > 0$. We show that it implies that each $z \in \omega(u)$ has a partial reflectional symmetry around H_{λ_0}. This will be needed in the proof of Theorem 2.4 and also in the proof of Theorem 2.1 (where the possibility $\lambda_0 > 0$ is ruled out).

Lemma 4.4. *If $\lambda_0 > 0$, then for each $z \in \omega(u)$ there exist $i \in S$ and a connected component U of Ω_{λ_0} such that $V_{\lambda_0} z_i \equiv 0$ in U.*

Proof. The proof is by contradiction. Assume that the statement is not true. Then, by Lemma 4.3, there exists $\tilde{z} \in \omega(u)$ such that

$$V_{\lambda_0} \tilde{z}_i(x) > 0 \quad (x \in \Omega_{\lambda_0},\, i \in S). \tag{4.6}$$

We show that this implies the existence of $\varepsilon_0 > 0$ such that (4.1) holds for all $\lambda \in (\lambda_0 - \varepsilon_0, \lambda_0]$, which is a contradiction to the definition of λ_0. We follow a similar scheme of arguing as in the proof proof of Lemma 4.2 in [36]. First we show that for $\lambda \approx \lambda_0$, w^λ is positive in a bounded cylinder $D \times [\bar{t}, \bar{t} + \theta]$ which has a "small" complement in $\Omega_\lambda \times [\bar{t}, \bar{t} + \theta]$ and the functions w_i^λ have small negative parts in the complement. We then show that this implies (4.1).

In what follows we assume that (4.6) holds.

Denote by $(t_k)_{k \in \mathbb{N}}$ an increasing sequence converging to ∞ for which $u(\cdot, t_k) \to \tilde{z}$ in E.

For the remainder of the proof, we fix positive constants r_0, γ, and δ as follows. First we choose $r_0 > 0$ such that each connected component of Ω_{λ_0} contains a closed ball of radius r_0. Such a choice is possible as Ω_{λ_0} has only finitely many connected components by (D2). Corresponding to $r = r_0$ (and the constants α_0, β from (N1), (N2)), we take γ as in Lemma 3.5. Finally, let $\delta > 0$ be such that the conclusion of Lemma 3.3 holds for $k = \gamma + 1$.

Now let $D \subset\subset \Omega_{\lambda_0}$ be an open set such that $|\Omega_{\lambda_0} \setminus D| < \delta/2$ and such that the intersection of D with any connected component of Ω_{λ_0} is a domain containing a ball of radius r_0. In particular, D has the same number of connected components as Ω_{λ_0}. Let $\varepsilon_1 > 0$ be so small that $|\Omega_{\lambda_0 - \varepsilon_1} \setminus \Omega_{\lambda_0}| < \delta/2$, hence also

$$|\Omega_\lambda \setminus D| < \delta \quad (\lambda \in (\lambda_0 - \varepsilon_1, \lambda_0]). \tag{4.7}$$

Using (4.6) and the equicontinuity of w^{λ_0} one shows easily (cp. the proof of (4.4) in the proof of Lemma 4.3) that there exist positive constants θ, d_1, and k_1 such that for each $i \in S$

$$w_i^{\lambda_0}(x,t) > 2d_1 \quad (x \in \bar{D},\ t \in [t_k, t_k + 4\theta], \quad k > k_1). \tag{4.8}$$

By the equicontinuity of the function u we have

$$\sup_{\substack{D \times [t_k, t_k + 4\theta] \\ j \in S}} |w_j^\lambda - w_j^{\lambda_0}| \to 0 \tag{4.9}$$

as $\lambda \to \lambda_0$, uniformly with respect to k. This and (4.8) imply that, possibly with a smaller $\varepsilon_1 > 0$, for each $k > k_1$

$$w_i^\lambda(x,t) > d_1 \quad (i \in S,\ x \in \bar{D},\ t \in [t_k, t_k + 4\theta],\ \lambda \in (\lambda_0 - \varepsilon_1, \lambda_0]). \tag{4.10}$$

Thus we have established the positivity of w^λ in the bounded cylinder $\bar{D} \times [t_k, t_k + 4\theta]$.

Our next aim is to show that the functions $w_i^\lambda(\cdot, t_k)$ have small negative parts. Namely, we claim that given any $\varsigma > 0$ there are $k_2 \geq k_1 > 0$ and $\varepsilon_2 \in (0, \varepsilon_1]$ such that for any $\lambda \in (\lambda_0 - \varepsilon_2, \lambda_0]$

$$\|(w_i^\lambda)^-(\cdot, t_k)\|_{L^\infty(\Omega_\lambda)} \leq \varsigma \quad (i \in S,\ k > k_2). \tag{4.11}$$

The arguments here are very similar to those used in the proof of estimate (4.10) in [36], we just recall them briefly. By (4.3), estimate (4.11) holds for $\lambda = \lambda_0$ with ς replaced by $\varsigma/2$ if k_2 is large enough. Therefore, by the equicontinuity of u, estimate (4.11) with Ω_λ replaced by Ω_{λ_0} also holds for all $\lambda \approx \lambda_0$. Next one shows, using the equicontinuity of u and Dirichlet boundary conditions, that there exists a neighborhood $\mathcal{E}$ of $\partial\Omega$, independent of λ, such that (4.11) holds with Ω_λ replaced by $\Omega_\lambda \cap \mathcal{E}$. Finally, the remaining set $\bar{\Omega}_\lambda \setminus (\mathcal{E} \cup \Omega_{\lambda_0})$ is contained in an arbitrarily small neighborhood G_0 of $H_{\lambda_0} \cap \Omega_{\lambda_0} \setminus \mathcal{E}$ if $\lambda \approx \lambda_0$. Since $w^{\lambda_0}(\cdot, t)$ vanishes on $H_{\lambda_0} \cap \Omega_{\lambda_0} \setminus \mathcal{E} \subset\subset \Omega$, using the equicontinuity we can choose G_0 so that $G_0 \subset\subset \Omega$ and (4.11) holds for $\lambda = \lambda_0$ with Ω_λ replaced by G_0 and with ς replaced by $\varsigma/2$. Then, using the equicontinuity one more time, we obtain that (4.11) holds with Ω_λ replaced by G_0 for any $\lambda \approx \lambda_0$. Combining all these estimates we conclude that (4.11) holds for all $\lambda \in (\lambda_0 - \varepsilon_2, \lambda_0]$ if ε_2 is small enough.

Our final goal is to prove that (4.10) and (4.11) with a sufficiently small ς imply that if $k > k_2$, then

$$w_i^\lambda(x,t) > 0 \quad (i \in S,\ x \in \bar{D},\ t \in [t_k, \infty),\ \lambda \in (\lambda_0 - \varepsilon_2, \lambda_0]). \tag{4.12}$$

Observe that (4.12), in conjunction with $w_i^\lambda \geq 0$ on $\partial\Omega_\lambda \times (0,\infty)$, gives $w_i^\lambda \geq 0$ on $\partial(\Omega_\lambda \setminus D) \times (t_k, \infty)$. Since $|\Omega_\lambda \setminus D| < \delta$, Lemma 3.3 and our choice of δ imply that for any $\lambda \in (\lambda_0 - \varepsilon_2, \lambda_0]$

$$\lim_{t\to\infty} \max_{i \in S} \|(w_i^\lambda(\cdot,t))^-\|_{L^\infty(\Omega_\lambda)} = \lim_{t\to\infty} \max_{i\in S} \|(w_i^\lambda(\cdot,t))^-\|_{L^\infty(\Omega_\lambda \setminus D)} = 0.$$

Thus having proved (4.12), we will have the desired contradiction and the proof of Lemma 4.4 will be complete.

To derive (4.12) we assume that (4.11) holds, ς being a sufficiently small constant as specified below, see (4.17). Fix any $k > k_2$. Let T be the maximal element of $(t_k, \infty]$ such that

$$w_i^\lambda(x,t) > 0 \quad (i \in S,\ x \in \bar{D},\ t \in (t_k, T),\ \lambda \in (\lambda_0 - \varepsilon_2, \lambda_0]). \tag{4.13}$$

By (4.10), $T > t_k + 4\theta$. We need to prove that $T = \infty$. Assume $T < \infty$. Then there exist $\lambda \in (\lambda_0 - \varepsilon_2, \lambda_0]$, $\bar{x} \in \partial D$, and $i_0 \in S$ such that $w_{i_0}^\lambda(\bar{x}, t) = 0$. We show that this leads to a contradiction by estimating $w_{i_0}^\lambda$ from below using Lemma 3.6. For that we first estimate the functions $(w_i^\lambda)^-$, $i \in S$, from above. We have $w_i^\lambda \geq 0$ on $\partial(\Omega_\lambda \setminus D) \times (t_k, T)$ and $|\Omega_\lambda \setminus D| < \delta$. Hence (4.13), Lemma 3.3, and the choice of δ give

$$\begin{aligned} \|(w_i^\lambda)^-(\cdot,t)\|_{L^\infty(\Omega_\lambda \setminus D)} &\leq 2e^{-(\gamma+1)(t-t_k)} \max_{j\in S} \|(w_j^\lambda)^-(\cdot,t_k)\|_{L^\infty(\Omega_\lambda\setminus D)} \\ &\leq 2e^{-(\gamma+1)(t-t_k)} \varsigma \quad (t \in [t_k, T],\ i \in S), \end{aligned} \tag{4.14}$$

where ς is as in (4.11).

Next we estimate $w_{i_0}^\lambda$ from below on a ball in D. Let D_0 be a connected component of D such that $\bar{x} \in \partial D_0$. By the manner in which D was chosen, D_0 contains a ball $B_0 = B(x_0, r_0)$ of radius r_0. Recall that γ was defined so that the statement of Lemma 3.5 holds with $r = r_0$. Let h_{r_0} be as in that statement. Then for each $i \in S$ the function $\phi(x,t) = e^{-\gamma t} h_r(x - x_0)$ satisfies

$$\begin{aligned} \partial_t \phi - L_i^\lambda(x,t)\phi &< c^{ii}(x,t)\phi, \qquad (x,t) \in B_0 \times (t_k, \infty), \\ \phi &= 0 \qquad\qquad\qquad (x,t) \in \partial B_0 \times (t_k, \infty), \end{aligned}$$

where $L_i^\lambda \in E(\alpha_0, \beta, \Omega_\lambda, (0,\infty))$ and $(c^{ij})_{i,j \in S} \in M^+(\beta, \Omega_\lambda, (0,\infty))$ are as in (3.2). Since $c^{ij} \geq 0$ for $i \neq j$, we also have

$$\partial_t \phi - L_i(x,t)\phi < \sum_{j=1}^n c^{ij}(x,t)\phi \quad (x,t) \in B_0 \times (t_k, \infty),\ i \in S.$$

We view this as a system of inequalities for the vector function $(\phi, \dots, \phi)$; it has the opposite inequality signs than the system satisfied by w^λ. For $t = t_k$ we have by (4.10)

$$w_i^\lambda(x, t_k) > d_1 \geq d_1 \frac{\phi(x, t_k)}{\|\phi(\cdot, t_k)\|_{L^\infty(B_0)}} \quad (x \in B_0,\ i \in S).$$

Also, as $B_0 \subset D$, we have $w_j \geq 0 = \phi$ on $\partial B_0 \times (t_k, T)$. These relations justify an application of Theorem 3.2 to $w^\lambda - d_1(\phi, \dots, \phi)/\|\phi(\cdot, t_k))\|_{L^\infty(B_0)}$ and we conclude that

$$w_{i_0}^\lambda(x,t) \geq d_1 \frac{\phi(x,t)}{\|\phi(\cdot, t_k)\|_{L^\infty(B_0)}} = d_1 e^{-\gamma(t - t_k)} \frac{h_{r_0}(x - x_0)}{\|h_{r_0}(\cdot - x_0)\|_{L^\infty(B_0)}} \qquad ((x,t) \in B_0 \times (t_k, T)). \quad (4.15)$$

Equipped with (4.14), (4.15), we are ready to use Lemma 3.6 in order to estimate $w_{i_0}^\lambda$ from below everywhere in $D_0 \times (t_k, T)$. We apply the lemma on the interval $(T - 4\theta, T)$ and the sets $D_0 \subset\subset \Omega_\lambda$, noting that $(w_{i_0}^\lambda)^+ = w_{i_0}^\lambda$ in $D_0 \times (T - 4\theta, T)$ and $\operatorname{dist}(\bar{D}_0, \Omega_\lambda) \geq d := \operatorname{dist}(\bar{D}, \Omega_{\lambda_0})$. This gives

$$w_{i_0}^\lambda(x,T) \geq \kappa [w_{i_0}^\lambda]_{p, D_0 \times (T-3\theta, T-2\theta)} - \kappa_1 \max_{j \in S} \sup_{\partial_P(\Omega_\lambda \times (T-4\theta, T))} (w_j^\lambda)^- \quad (x \in D_0),$$

where κ, κ_1, and p are constants depending only on d, n, N, $\operatorname{diam}\Omega$, α_0, β, θ and r_0. By (4.15), (4.14), we therefore have

$$w_{i_0}^\lambda(x,T) \geq \kappa e^{-\gamma(T - t_k)} G(p, r_0) - 2\kappa_1 e^{-(\gamma+1)(T - 4\theta - t_k)} \quad (x \in D_0),$$

where

$$G(p, r_0) := \frac{1}{\|h_{r_0}\|_{L^\infty(B_0)}} \left(\frac{1}{|D_0|} \int_{B_0} h_r^p \right)^{\frac{1}{p}} > 0.$$

Consequently

$$\inf_{x \in D_0} w_{i_0}^\lambda(x, T) \geq e^{-(\gamma+1)(T-t_k)} (\kappa G(p, r_0) - \kappa_1 \varsigma). \tag{4.16}$$

We now specify the choice of ς in (4.14):

$$\varsigma := G(p, r_0)/2\kappa_1 \tag{4.17}$$

This is legitimate, as the number is independent of k and λ. Then (4.16) and the continuity of w^λ imply that $w_{i_0}^\lambda(\cdot, T) > 0$ on ∂D_0 contradicting $w_{i_0}^\lambda(\bar{x}, T) = 0$. This contradiction shows that the maximal T for which (4.13) holds is equal ∞, that is, (4.12) must hold. As remarked above, this completes the proof of Lemma 4.4. □

The following lemma addresses the strict monotonicity of the functions in $\omega(z)$. The proof is a straightforward modification of the corresponding lemma in a scalar case, see [36, Lemma 4.6], and is omitted.

Lemma 4.5. *Assume that Ω_{λ_0} is connected. Then for any $z \in \omega(u)$ and any $i \in S$, either $z_i \equiv 0$ on Ω_{λ_0} or else $z_i > 0$ in Ω_{λ_0} and z_i is strictly decreasing in x_1 in Ω_{λ_0}. The latter holds in the form $(z_j)_{x_1} < 0$ if $(z_j)_{x_1} \in C(\Omega_{\lambda_0})$ for some $j \in S$.*

We need one more lemma for the proof of our symmetry results. Its assumption is identical to hypothesis (i) of Theorem 2.1.

Lemma 4.6. *Assume that there exists $\varphi = (\varphi_1, \dots, \varphi_n) \in \omega(u)$ such that $\varphi_i > 0$ in Ω for all $i \in \{1, \dots, n\}$. Then $\lambda_0 = 0$.*

Proof. Assume $\lambda_0 > 0$. By Lemma 4.4, there is $i \in S$ such that $V_{\lambda_0}\varphi_i \equiv 0$ on some connected component of Ω_{λ_0}. In view of Dirichlet boundary condition, this clearly implies that φ_i vanishes somewhere in Ω a contradiction. □

Now we are ready to prove our symmetry theorems.

Proof of Theorems 2.1 and 2.4 . We claim that the statements (i)-(iii) of Theorem 2.4 hold with λ_0 defined in (4.2). First assume $\lambda_0 > 0$. Then statement (i) follows directly from Remark 4.1 and Lemmas 4.3, 4.4; statement (ii) follows from (i) and Lemma 4.3; and statement (iii) follows from (ii) and Lemma 4.5.

Now we consider the case $\lambda_0 = 0$. Note that $\Omega_{\lambda_0} = \Omega_0$ is connected by (D1). By Lemma 4.5, for each $z \in \omega(u)$ and $i \in S$ either $z_i \equiv 0$ (and then of course $V_0 z_i \equiv 0$) or $z_i > 0$ in Ω_0. As $V_0 z_i \geq 0$ in Ω_0 by (4.3), the relation $z_i > 0$ in Ω_0 implies $z_i > 0$ in Ω. We conclude that for each $z \in \omega(u)$

$$\text{either } z_i \equiv 0 \text{ in } \Omega \text{ or } z_i > 0 \text{ in } \Omega \quad (i \in S). \tag{4.18}$$

In case (N5) holds, a stronger version of this statement follows from Lemma 4.3(ii):

$$\text{either } z \equiv 0 \text{ in } \Omega \text{ or } z_i > 0 \text{ in } \Omega \text{ for each } i \in S. \tag{4.19}$$

Now observe that statement (i) of Theorem 2.4 is trivially satisfied for any $z \in \omega(u)$ such that $z_i \equiv 0$ for some $i \in S$. Similarly, if (N5) holds, then statements (ii) and (iii) are satisfied for any such z, for it has to satisfy $z \equiv 0$ by (4.19). Thus, in view of (4.18), we only need to consider the case that there exists $\varphi \in \omega(u)$ with $\varphi_i > 0$ for all $i \in S$, that is, hypothesis (i) of Theorem 2.1 is satisfied. Because of this and Lemma 4.6, the remaining part of the proof is common to Theorems 2.4 and 2.1. We prove that hypothesis (i) implies

$$V_0 z_i \equiv 0 \quad (z \in \omega(u),\ i \in S). \tag{4.20}$$

This will complete the proof of the symmetry statements of Theorems 2.4 and 2.1 (note that hypothesis (ii) Theorem 2.1 implies hypothesis (i) by (4.19)). The strict monotonicity statements follow from Lemma 4.5 as above.

To prove (4.20), we apply the results of this section to the solution $\tilde{u}(x_1, x', t) = u(-x_1, x', t)$ in place of u ($\tilde{u}$ is indeed a solution as F is even in x_1). Denote by $\tilde{\lambda}_0$ the corresponding number defined as in (4.2) with u replaced by $\tilde{u}$. Since $\tilde{u}$ has a strictly positive element $\tilde{\varphi}$ in its ω-limit set, Lemma 4.6 gives $\tilde{\lambda}_0 = 0$. This clearly implies (4.20). $\square$

Proof of Theorem 2.3. For each $i \in S$ we have, similarly as in Subsection 3.1,

$$\begin{aligned}(u_i)_t &= F_i(t, x, u, Du_i, D^2 u_i) - F_i(t, x, 0, 0, 0) + F_i(t, x, 0, 0, 0) \\ &= L_i(x, t) u_i + \sum_{j=1}^{n} c^{ij}(x, t) u_j + F_i(t, x, 0, 0, 0) \quad (x \in \Omega,\ t > 0)\end{aligned}$$

for suitable $L_i \in E(\alpha_0, \beta, \Omega, (0, \infty))$ and $(c^{ij})_{i,j \in S} \in M^+(\beta, \Omega_\lambda, (0, \infty))$. Setting $g_i(x,t) := -F_i^-(t, x, 0, 0, 0)$ and using the relations $c^{ij} \geq 0$ for $i \neq j$, we obtain the following scalar inequality for each $i \in S$

$$(u_i)_t \geq L_i(x,t)u_i + c^{ii}(x,t)u_i + g_i(x,t) \quad (x \in \Omega,\ t > 0).$$

By (2.6), $\sup_{x \in \Omega} g_i(x,t) \to 0$ as $t \to \infty$.

Such scalar inequalities are considered in the proof of Theorem 2.5 of [36]. The arguments used there can be repeated for each i to obtain the conclusion of Theorem 2.3.

□

Proof of Theorem 2.5. Let λ_0 be as in (4.2). It is sufficient to prove that under the assumptions of Theorem 2.5 we have $\lambda_0 = 0$. Indeed, as in the proof of Theorems 2.1 and 2.4, $\lambda_0 = 0$ gives the reflectional symmetry and the strict monotonicity in $x_1 > 0$ of z_i for each $z \in \omega(u) \setminus \{0\}$ and $i \in S$. As problem (1.1), (1.3) is invariant under rotations, we can apply this to any of the solutions $\tilde{u}(x,t) = u(Rx,t)$, $R \in O(n)$. This gives the conclusion of Theorem 2.5.

The proof is by contradiction. Assume $\lambda_0 > 0$. Then by (4.2) (see also Remark 4.1) and the compactness of $\omega(z)$ in E, there exist $z \in \omega(u)$, $i \in S$, and a sequence $\lambda_k \nearrow \lambda_0$ such that $(V_{\lambda_k} z_i)^- \not\equiv 0$ in Ω_{λ_k} for $k = 1, 2, \ldots$. At the same time we have $V_{\lambda_0} z_i \equiv 0$, by Lemmas 4.3(ii) and 4.4.

We know by Lemma 4.5 that either $z_i \equiv 0$ in Ω_{λ_0} or $z_i > 0$ in Ω_{λ_0}.

First assume $z_i > 0$ in Ω_{λ_0}. Since $V_{\lambda_0} z_i \equiv 0$ and $z_i = 0$ on $\partial\Omega$, z_i vanishes on $\partial\Omega'_{\lambda_0} \setminus H_{\lambda_0}$ (recall that Ω'_λ is the reflection of Ω_λ in H_λ). It follows that if $x_0 \in H_{\lambda_0} \cap \partial\Omega$ and U is any neighborhood of x_0 in $\bar{\Omega}$ then z_i vanishes somewhere in $\Omega \cap U$ while at the same time $z_i \not\equiv 0$ in U. Using a rotation R which takes such a point x_0 to $(1, 0, \ldots, 0)$ and considering the solution $\tilde{u}(x,t) = u(Rx,t)$, we clearly obtain a contradiction to Lemma 4.5.

Next assume $z_i \equiv 0$ in Ω_{λ_0}. As $V_{\lambda_0} z_i \equiv 0$, we have $z_i \equiv 0$ in $\bar{\Omega}_{\lambda_0} \cup \Omega'_{\lambda_0}$. If there exists $x_0 \in H_{\lambda_0} \cap \partial\Omega$ such that for each neighborhood U of x_0 in $\bar{\Omega}$ one has $z_i \not\equiv 0$, then we obtain a contradiction as in the previous case. Otherwise, $z_i \equiv 0$ on some neighborhood $\mathcal{N}$ of $H_{\lambda_0} \cap \partial\Omega$. Now, for $\lambda < \lambda_0$ sufficiently close to λ_0 one has $\Omega_\lambda \subset \bar{\Omega}_{\lambda_0} \cup \Omega'_{\lambda_0} \cup \mathcal{N}$. Therefore $z_i \equiv 0$ in Ω_λ and consequently $V_\lambda z_i \geq 0$ in Ω_λ for all $\lambda \approx \lambda_0$. This is a contradiction to the existence of the sequence λ_k.

Thus in either case, $\lambda_0 > 0$ leads to a contradiction. This proves that $\lambda_0 = 0$ as needed. □

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